RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 76 (1986), p. 45-58

<http://www.numdam.org/item?id=RSMUP_1986__76__45_0>

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REND. SEM. MAT. UNIV. PADOVA, Vol. 76 (1986)

Direct Products of Linearly Compact Primary Rings.

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SUMMARY - Numakura [4], [5] gave criterions for compact rings to be direct products of primary rings. In this note we extend these results to linearly compact rings. As a consequence we get a characterization of those rings which are direct products of division rings, of uni-serial rings (i.e. such artinian rings in which the P^k ($k = 1, 2, \ldots$) are all the one-sided ideals where P denotes the unique maximal ideal) and of rings of complete discrete valuations on division rings.

1. All rings considered will be associative rings with identity. In what follows R denotes always a ring and $\mathfrak{J}$ is its Jacobson radical (briefly: radical). A ring R is said to be *topologically artinian* or *topologically noetherian* if it is the inverse limit of artinian or noetherian left R-modules endowed with the inverse limit topology. A topological ring R is called *linearly compact* if open left ideals form a base for neighbourhoods of zero and every finitely solvable system of congruences $x \equiv x_k (\operatorname{mod} L_k)$ where the L_k are closed left ideals, is solvable. Remark that every topologically artinian ring is linearly compact (see [2] satz 4) and every compact ring is topologically artinian (see [4] Lemma 5).

By a classical result of Artin every artinian primary ring is a matrix ring over a local ring. In [3] Leptin proved that every topologically artinian primary ring is a matrix ring (not necessarily of finite size) over a local ring. These results can be considered as a generalization

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of Artin-Wedderburn Theorem on simple artinian rings. Throughout this note a primary ring is understood in a more general sense than what is usual. We call a ring *primary* if the factor by its radical is the endomorphism ring of a vector space over a division ring. Now our purpose is to characterize direct products of linearly compact rings of some kind.

Before turning to the main results we need some preparations.

For every subset A of a topological space we denote by $\bar{A}$ its topological closure. Define for every ideal I of a topological ring the following ideals

$${}_1I = \bar{I}\,,\quad {}_{\mu+1}I = \overline{I\cdot {}_\mu I},\quad {}_\lambda I = \bigcap_{\mu<\lambda} {}_\mu I \qquad \text{if } \lambda \text{ is a limit ordinal},$$

$$I = \bar{I},\quad \underset{\mu+1}{I} = \overline{\underset{\mu}{I}\cdot \underset{\mu}{I}},\quad \underset{\lambda}{I} = \bigcap_{\mu<\lambda} \underset{\lambda}{I} \qquad \text{if } \lambda \text{ is a limit ordinal},$$

$$I_1 = \bar{I},\quad I_{\mu+1} = \overline{I_\mu\cdot I},\quad I_\lambda = \bigcap_{\mu<\lambda} I_\mu \qquad \text{if } \lambda \text{ is a limit ordinal}.$$

Then there is an ordinal η for which ${}_\xi I = {}_\eta I$, $\underset{\xi}{I} = \underset{\eta}{I}$, $I_\xi = I_\eta$ for all $\xi \geqslant \eta$. These ${}_\eta I$, $\underset{\eta}{I}$, I_η will be denoted by ${}_*I$, $\underset{*}{I}$, I_* and $\overline{I\cdot {}_*I} = {}_*I$, $\overline{\underset{*}{I}^2} = \underset{*}{I}$, $\overline{I_*\cdot I} = I_*$ hold clearly. If ${}_*I = 0$ ($I_* = 0$), then I is called *$l(r$ —$)$ transfinitely nilpotent.* If $\underset{*}{I} = 0$, then I is said to be *transfinitely nilpotent.*

PROPOSITION 1 ([2] (3)). *If the set $\{L_\alpha\}$ of closed left ideals in a linearly compact ring has the finite intersection property, then*

$$A + \bigcap L_\alpha = \bigcap (A + L_\alpha)$$

holds for every closed left ideal A.

PROPOSITION 2 ([2] Lemma). *For any two (one-sided) ideals A and B of a topological ring we have*

$$\overline{AB} = \overline{\bar{A}B} = \overline{A\bar{B}} = \overline{\bar{A}\bar{B}}\,.$$

PROPOSITION 3 ([2] Satz 9). *The radical of a topologically artinian ring is r-transfinitely nilpotent.*

PROPOSITION 4 ([2] (4)). *If A and B are closed (left) ideals in a linearly compact ring, then $A + B$ is closed.*

PROPOSITION 5 ([2] Satz 4). *Every continuous isomorphism between topologically artinian rings is a topological isomorphism.*

PROPOSITION 6 ([3] Satz 1). *Every linearly compact module over a semisimple linearly compact ring is a direct product of simple modules.*

PROPOSITION 7. *Let A and B be closed (left) ideals in a linearly compact R. If $A + B = R$ holds, then we have $A_\xi + B_\xi = R$ for every ordinal $\xi \geqslant 1$.*

PROOF. We prove the assertion by transfinite induction. By the assumption we have $A_1 + B_1 = A + B = R$. Suppose that $A_\xi + B_\xi = R$, then there are elements $a \in A_\xi$, $b \in B_\xi$ such that $a + b = 1$ and so $a^2 = 1 - 2b + b^2$. By $2b - b^2 \in B_\xi$ we have $A_{\xi+1} + B_\xi = R$. This implies the existence of elements $a^* \in A_{\xi+1}$ and $b^* \in B_\xi$ such that $a^* + b^* = 1$. This shows that $(b^*)^2 = 1 - 2a^* + (a^*)^2 = 1 - c$ with $c = 2a^* - (a^*)^2 \in A_{\xi+1}$ from which we obtain $A_{\xi+1} + B_{\xi+1} = R$.

If we have now the equality $A_\xi + B_\xi = R$ for every ordinal $\xi < \lambda$ where λ is a limit ordinal, then for each $\mu < \lambda$ it is true that

$$A_\xi + B_\mu = A_\xi + B_\xi + B_\mu = R \quad \text{if } \lambda > \xi \geqslant \mu$$

and

$$A_\xi + B_\mu = A_\xi + A_\mu + B_\mu = R \quad \text{if } \lambda > \mu > \xi$$

that is, $A_\xi + B_\mu = R$ holds for every ξ, $\mu < \lambda$.

Now we have by Proposition 1 for every fixed $\mu < \lambda$

$$R = \bigcap_{\xi<\lambda} (A_\xi + B_\mu) = \bigcap_{\xi<\lambda} A_\xi + B_\mu = A_\lambda + B_\mu$$

which implies

$$R = \bigcap_{\mu<\lambda} (A_\lambda + B_\mu) = A_\lambda + \bigcap_{\mu<\lambda} B_\mu = A_\lambda + B_\lambda .$$

This completes the proof.

PROPOSITION 8. *For any closed (left) ideals A and B satisfying $A + B = R$ in a linearly compact ring R. $AB = BA$ implies $AB = A \cap B = \overline{AB}$ and $\overline{AB} = \overline{BA}$ implies $\overline{AB} = A \cap B$.*

PROOF.

$$A \cap B = (A \cap B)R = (A \cap B)(A + B) \subseteq \overline{AB}$$

implying $AB = A \cap B = \overline{AB}$ in the case $AB = BA$.

$$A \cap B = (A \cap B)R = (A \cap B)(A + B) \subseteq \overline{AB + BA} \subseteq AB + BA = \overline{AB}$$

implying $A \cap B = \overline{AB}$ in the case $\overline{AB} = \overline{BA}$.

2. A topological ring R is called of *finite type* if the set of all isomorphism classes of simple submodules in all factors of any finitely generated discrete R-module is finite. Consider now a topologically artinian ring R. Let $\{P_i | i \in I\}$ be the set of all maximal closed ideals of R where I is the index set and $P_i \neq P_j$ for $i \neq j$, $i, j \in I$.

In case $i \neq j$ by Proposition 4 the ideal $P_i + P_j$ is closed in R and hence we get $P_i + P_j = R$. Therefore we obtain $(P_i)_\xi + (P_j)_\xi = R$ for every ordinal ξ by Proposition 7. Thus $(P_i)_* + (P_j)_* = R$ holds. Henceforth for any finite set $\{(P_1)_*, \ldots, (P_n)_*\}$ we have by the Chinese Remainder Theorem

$$(P_i)_* + \bigcap_{k \neq i} (P_k)_* = R, \qquad k = 1, \ldots, n.$$

PROPOSITION 9. *Let R be a topologically artinian ring of finite type. If $\overline{P_i P_j} = \overline{P_j P_i}$, $\overline{P_i (P_j)_\lambda} = \overline{(P_j)_\lambda P_i}$, $\overline{(P_j)_\lambda (P_i)_\xi} = \overline{(P_i)_\xi (P_j)_\lambda}$ hold for any $i \neq j$, $i, j \in I$ and any limit ordinals ξ, λ then $\bigcap_i (P_i)_* = 0$.*

PROOF. First we obtain $\overline{(P_i)_\xi (P_j)_\eta} = \overline{(P_j)_\eta (P_i)_\xi}$ for any $i \neq j$, $i, j \in I$ and any ordinals ξ and η. We proced by double transfinite induction. Suppose that for some ξ and some fixed η, we have $\overline{(P_i)_\chi (P_j)_\eta} = \overline{(P_j)_\eta (P_i)_\chi}$ for every $\chi \leqslant \xi$. Then we have by Proposition 2 and the associativity of the multiplication of ideals

$$\overline{(P_i)_{\xi+1} (P_j)_\eta} = \overline{(\overline{(P_i)_\xi P_i})\, (P_j)_\eta} = \overline{(P_i)_\xi\, (\overline{P_i (P_j)_\eta})} =$$
$$= \overline{(P_i)_\xi\, (\overline{(P_j)_\eta P_i})} = \overline{(\overline{(P_i)_\xi (P_j)_\eta})\, P_i} = \overline{(\overline{(P_j)_\eta (P_i)_\xi})\, P_i} =$$
$$= \overline{(P_j)_\eta\, (\overline{(P_i)_\xi P_i})} = \overline{(P_j)_\eta (P_i)_{\xi+1}}$$

if now $(P_i)_\xi (P_j)_\eta = (P_j)_\eta (P_i)_\xi$ holds for some fixed η and every $\xi < \lambda$ where λ is a limit ordinal, then there exists a limit ordinal τ such that $\eta = \tau + n$ for some natural number n, and hence we have again by Proposition 2 and the associativity of the multiplication of ideals

$$\overline{(P_i)_\lambda (P)_\eta} = \overline{\left(\overline{(P_i)_\lambda (P_j)_\tau}\right) P_j^n} = \overline{\left((P_j)_\tau (P_i)_\lambda\right) P_j^n} = \overline{(P_j)_\tau \left(\overline{(P_i)_\lambda P_j^n}\right)} =$$

$$= \overline{(P_j)_\tau \left(P_j^n (P_i)_\lambda\right)} = \overline{\left((P_j)_\tau P_j^n\right) (P_i)_\lambda} = \overline{(P_j)_\eta (P_i)_\lambda}\,.$$

Finally we let η vary and complete the proof in the same way.

Assume now indirectly that $\bigcap_i (P_i)_* \neq 0$. Then there is an element $c \in R$ and an open left ideal L with $c \in \bigcap_i (P_i)_*$ and $c \notin L$. Since R is of finite type, there are finitely many maximal closed ideals P_i, say $P_1, \ldots, P_n$ such that every simple submodule of each factor of R/L is annihilated by some one of them. Consider now the submodule M of R/L consisting of those elements which are annihilated by $(P_1)_* \ldots (P_n)_*$. We claim $M = R/L$. In fact, if $M \neq R/L$, then there exists an element $m \in R/L$ such that $(Rm + M)/M$ is simple, because R/L is artinian. Therefore $(Rm + M)/M$ is annihilated by some P_i, say P_1. We obtain now

$$[(P_1)_* \ldots (P_n)_*]m = [(P_1)_* P_1 \ldots (P_n)_*]m =$$

$$= [(P_1)_* \ldots (P_n)_*](P_1 m) \subseteq [(P_1)_* \ldots (P_m)_*]M = 0$$

and consequently $m \in M$, which is a contradiction. Thus $M = R/L$ holds. This implies by $c \in \bigcap_i (P_i)_* \subseteq (P_1)_* \ldots (P_n)_*$ that $c \cdot R/L = 0$, *i.e.* $c \in L$, a contradiction. Therefore $\bigcap_i (P_i)_* = 0$ and the proof is complete.

As $(P_i)_*$ is a closed ideal of the topologically artinian ring R, the factor $R_i = R/(P_i)_*$ is a topologically artinian ring, too. We denote by $\tilde{R}$ the direct product of the R_i, $i \in I$. Then $\tilde{R}$ endowed with the product topology is a topologically artinian ring. R is topologically isomorphic to $\tilde{R}$ by the following proposition.

PROPOSITION 10. *If A_i $(i \in I)$ are closed ideals in a topologically artinian ring R such that $\cap A_i = 0$ and $A_i + A_j = 0$ hold for all $i \neq j$; $i, j \in I$, then R is topologically isomorphic to the direct product $\prod R/A_i$.*

PROOF. Taking the mapping $\varphi: R \to \prod R/A_i$ define by $\varphi(x) = (\dots, x_i, \dots)$ with $x_i = x + (A_i) \in R/A_i$ for every $i \in I$, we have a homomorphism from R into $\prod R/A_i$. Further $\varphi(x) = 0$ implies $x_i = 0$ for each $i \in I$, *i.e.* $x \in \bigcap_{i \in I} A_i = 0$, hence $x = 0$. Thus φ is an injective homomorphism, and it is clear that φ is continuous.

Finally, let $(\dots, \tilde{x}_i, \dots)$ be any element of $\prod R/A_i$, then $M_i = \varphi_i^{-1}(\tilde{x}_i)$ is a coset of the ideal A_i in R where $\varphi_i: R \to R/A_i$ is the natural projection of R onto R/A_i. Hence we can express M_i as $x_i + A_i$, $x_i \in M_i$. Taking any finite number of $M_i - s$, say $M_1, \dots, M_n$, we have $\bigcap_{k=1} M_k \neq \emptyset$ by the Chinese Remainder Theorem, hence $\bigcap_{i \in} M_i \neq \emptyset$, since R is topopogically artinian. Choosing an element $x \in \bigcap_{i \in I} M_i$, it is obvious that $\varphi(x) = (\dots, \tilde{x}_i, \dots)$. This implies that φ is a continuous isomorphism. Since R is topologically artinian, by Proposition 5 φ is a homeomorphism. This completes the proof of the assertion that R is topologically isomorphic to $\prod R/A_i$.

PROPOSITION 11. *The rings R_i, $i \in I$ are primary rings.*

PROOF. Let $\tilde{P}_i = P_i/(P_i)_*$. Then $R_i/\tilde{P}_i \simeq R/P_i$ which is the endomorphism ring of a vector space over a division ring. Since the radical of R_i is the intersection of all maximal closed ideals which are exactly the images of maximal closed ideals of R by the natural homomorphism $R \to R_i$, $\tilde{P}_i$ is clearly the radical of R_i. This means that the rings R_i $(i \in I)$ are primary rings.

The above considerations yield the following

THEOREM 12. *Let R be a topologically artinian ring. The following conditions are equivalent*

1) *R is a direct product of topologically artinian primary rings R_i, $i \in I$.*

2) *Let $\{P_i | i \in I\}$ be the set of all maximal closed ideals in R then the equalities $P_iP_j = P_jP_i$, $P_i(P_j)_\lambda = (P_j)_\lambda P_i$, $(P_i)_\mu((P_j)_\lambda = (P_j)_\lambda(P_i)_\mu$ hold for all $i \neq j$, i, $j \in I$ and limit ordinals μ, λ and R is of finite type.*

3) *The equalities $\overline{P_iP_j} = \overline{P_jP_i}$, $\overline{P_i(P_j)_\lambda} = \overline{(P_j)_\lambda P_i}$, $\overline{(P_i)_\mu(P_j)_\lambda} = \overline{(P_j)_\lambda(P_i)_\mu}$ hold for all $i \neq j$, i, $j \in I$ and limit ordinals μ, λ, and R is of finite type.*

4) *Any ideal K of R such that $K = \overline{K^2}$, is a unital ring.*

PROOF. $1 \Rightarrow 2$. This implication is trivial, since maximal closed ideals in R are the products of a maximal closed ideal in one component with the other components. For any discrete, finitely generated module, let $\{x_1, \ldots, x_n\}$ be a generator set of M. The annihilator $\operatorname{ann}_R x_i$ of x_i is an open left ideal of R, and hence it contains almost all R_i. Therefore $\operatorname{ann}_R (x_1, \ldots, x_n)$ contains almost all R_i. This shows that almost all R_i is contained in the annihilator $\operatorname{ann}_R M$ of M and henceforth M can be considered as a discrete module over a finite direct sum of primary rings R_i, say $R_1 \times \ldots \times R_n$: Since the set of isomorphism classes of discrete simple factors of $R_1 \times \ldots \times R_n$ is clearly finite, M is obviously of finite type, i.e. R is of finite type.

$2 \Rightarrow 3$. This implication is also trivial, since $A = B$ implies $\bar{A} = \bar{B}$.

$3 \Rightarrow 1$. This implication is the consequence of the fact that R and $\tilde{R}$ are topologically isomorphic.

$1 \Rightarrow 4$. Let φ_i denote the natural projection of R onto R_i for each $i \in I$. For $K_i = \varphi_i(K)$ it follows by [6] Lemma 6.1 that K is the direct product of the K_i, $i \in I$. Since $K = \overline{K^2}$, we have $K_i = \overline{K_i^2}$, and hence either $K_i = 0$ or $K_i = R_i$, because R_i is a primary ring and its radical is r-transfinitely nilpotent. This shows that K is a unital ring.

$4 \Rightarrow 1$. From the assumption it follows by $\underset{*}{P}_i^2 = \underset{*}{P}_i$ that $\underset{*}{P}_i$ has an identity e_i. It is easy to see that $R_i = R/\underset{*}{P}_i$ is a primary ring. To come to the end of the proof of the implication $4 \Rightarrow 1$ we show $\bigcap_{i \in I} \underset{*}{P}_i = 0$. For this aim let x be an arbitrary element in $\bigcap_{i \in I} \underset{*}{P}_i$. Then we have $x = xe_i$ for each $i \in I$ and $x \in \bigcap_{i \in I} P_i = J$. Assume that x is an element of J_λ. By $(J_\lambda/J_{\lambda+1})\, J = 0$ we can consider $J_\lambda/J_{\lambda+1}$ as a right R/J-module. Since $x = xe_j$, we have $\bar{x}(\bar{1} - \bar{e}_i) = 0$ and hence $\bar{x} \cdot 1 = 0$ where $\bar{a}$ denotes the image of $a \in J_\lambda$ in $J_\lambda/J_{\lambda+1}$. This implies that x belongs to $J_{\lambda+1}$. On the other hand the radical J is r-transfinitely nilpotent so we have $x = 0$. Similarly, to proposition 7 it is routine to verify that $\underset{\xi}{P}_i + \underset{\xi}{P}_j = R$ for every ordinal ξ and $i \neq j$. Therefore we have $\underset{*}{P}_i + \underset{*}{P}_j = R$ for all $i \neq j$ and thus by Proposition 10 R is topologically isomorphic to the product of the primary rings R_i and this completes the proof of Theorem 12.

If $\bigcap \bar{J}^n = 0$ holds, we have the following.

COROLLARY 13. *Let R be a topologically artinian ring satisfying $\bigcap J^n = 0$. The following assertions are equivalent*

1) *R is a direct product of topologically artinian primary rings.*

2) *$P_iP_j = P_jP_i$ where $\{P_i : i \in I\}$ is the set of maximal closed ideals of R.*

3) *$Q_iQ_j = Q_jQ_i$ where $Q_i = \bigcap P_i^n$.*

4) *Every ideal K of R with $K = \overline{K^2}$ is a unital ring.*

5) *$\bigcap Q_i = 0$.*

First we prove the following.

PROPOSITION 14. *Let R be a topologically artinian ring satisfying $\bigcap \overline{J^n} = 0$. For any open left ideal L there are (not necessarily different) maximal closed ideals $P_1, \ldots, P_n$ with $P_n \ldots P_1 \subseteq L$.*

PROOF. Since L is open, the left R-module R/L is artinian. Because $L = L + \left(\bigcap_n \overline{J^n}\right) = \bigcap_n (L + J^n)$ holds, there is a natural number t with $J^{t+k} + L = J^t + L$ for every non-negative integer k. Hence L contains J^t. The artinian module $R/(J + L)$ can be considered as a left R/J-module, and then by Proposition 6 it is a finite direct sum of simple modules, *i.e.* it is noetherian. Consider the artinian R-module $(J + L)/(J^2 + L)$. By $J[(J + L)/(J^2 + L)] = 0$ it can be considered as an R/J-module and hence by Proposition 6 it is a finite direct sum of simple modules. Thus it is noetherian. Iterating this process in t steps we get that $(J^{t-1} + L)/(J^t + L) = (J^{t-1} + L)/L$ is noetherian. Thus R/L is noetherian, *i.e.*, it has a composition series $R/L = M_1 \supset M_2 \supset \ldots \supset M_n \supset 0$ where M_i/M_{i+1} is simple. Let P_k be the annihilator ideal of M_k/M_{k+1} $(k = 1, \ldots, n)$, then the P_k are (not necessarily different) maximal closed ideals and $(P_n \ldots P_1)R/L = 0$, thus $P_n \ldots P_1 \subset L$. This completes the proof of Proposition 14.

PROOF OF COROLLARY 13. By Theorem 12 we have $1 \Leftrightarrow 4$ and $1 \Rightarrow 1$, $1 \Rightarrow 3$. Next we show $2 \Rightarrow 5$. Suppose that $P_iP_j = P_jP_i$ holds for any $i, j \in I$. If there were an element $0 \neq c \in \bigcap_{i \in I} Q_i$, then we should have an open left ideal L with $c \notin L$. By Proposition 14 there are maximal closed ideals $P_1, \ldots, P_n$ with $P_1 \ldots P_n \subseteq L$. This shows by Propositions 7 and 8 that

$$c \in \bigcap_{i \in I} Q_i \subseteq \bigcap_{i=1}^{n} (P_i^n) \subseteq P_1 \dots P_n \subseteq L$$

which contradicts $c \notin L$.

Similarly we can see that $3 \Rightarrow 5$.

Finally, suppose $\bigcap_{i \in I} Q_i = 0$. Let $R_i = R/Q_i$. From Proposition 7 we get immediately $Q_i + P_j = R$ for all $i \neq j$. This shows clearly that P_i/Q_i is the radical of R_i. By $R/P_i \simeq R_i/(P_i/Q_i)$ the rings R_i $(i \in I)$ are primary rings. Consider the direct product $\tilde{R} = \prod \tilde{R}_i$. By Proposition 10 we have $R \simeq \tilde{R}$. This completes the proof of Corollary 13.

PROPOSITION 15. *If in a topologically artinian ring R the products of any two maximal open left ideals commute, then so do those of any two maximal closed ideals, and R/P is a division ring for each maximal closed ideal P.*

PROOF. Let P be a maximal closed ideal. Since P contains the radical, R/P is a primitive ring, consequently it is the endomorphism ring of a vector space over a division ring. To show that R/P is a division ring, we assume indirectly that R/P is the endomorphism ring of a right vector space V over a division ring and the dimension of V is at least 2. Hence there is a basis $\{v_i : i \in I\}$ in V such that the cardinal number of I is greater than 1. Let i, j be any two distinct index in I and we define the endomorphisms e_i and e_j by setting

$$e_i(v_k) = \begin{cases} v_k, & k \neq i \\ 0, & k = i, \end{cases} \qquad e_j(v_k) = \begin{cases} v_k, & k \neq j \\ 0, & k = j. \end{cases}$$

It is obvious that $(R/P)e_i$ and $(R/P)e_j$ are two maximal open left ideals in R/P. For the endomorphism e_{ij} definde by

$$e_{ij}(v_k) = \begin{cases} v_k, & k \neq j, \quad k \neq i \\ v_i, & k = j \\ v_j, & k = i \end{cases},$$

we obtain $e_i e_{ij} = e_j$ from which it follows that $(R/P)e_i \cdot (R/P)e_j = (R/P)e_j$. Similarly we can see that $(R/P)e_j \cdot (R/P)e_i = (R/P)e_i$.

Since $(R/P)e_i$ and $(R/P)e_j$ are two distinct maximal open left ideals, we have that their products do not commute. This contradicts to the assumption. This implies that P is a maximal open left ideal in R and hence the validity of the proposition is verified.

As an immediate consequence of Proposition 15 and Corollary 13 we have

COROLLARY 16. *A topologically artinian ring satisfying $\bigcap \overline{J^n} = 0$ is a direct product of local rings if and only if the products of any two maximal open left ideals commute.*

In what follows, let R be a topologically artinian local ring satisfying $\bigcap \overline{P^n} = 0$ where P denotes its maximal open ideal. We assume that R satisfies the additional condition:

There exists no one-sided open ideal between P and P^2.

PROPOSITION 17. *If P^2 is not open in P, then $P = 0$.*

PROOF. For any open left ideal L in P the left ideal $P^2 + L$ is open and it holds $P \supset P^2 + L \supset P^2$. Hence $P^2 + L = P$. This shows that P^2 is everywhere dense in P, so $\overline{P^2} = P$ is true. By induction we get $\overline{P^n} = P$. Therefore $P = \bigcap \overline{P^n} = 0$.

PROPOSITION 18. *$\{P^n, n = 1, 2, \dots\}$ forms a fundamental system of neighbourhoods of zero.*

PROOF. Let L be any open left ideal in R. Since $L = L + (\bigcap \overline{P^n}) = \bigcap (L + P^n)$ and R/L is an artinian module, there is an integer n with $P^n \subseteq L$. This means by [4] Lemma 6 that L is a power of P. (Note that in this case P^k is a union of translates of P^t for every $t \geqslant k$). Suppose now $P^n = 0$ for some n, and let k be the least integer with this property, *i.e.* $P^k = 0$, $P^{k-1} \neq 0$. We prove that R is artinian in this case, or in other words, 0 is ipen. In fact, since all non-zero open left ideals are powers of P containing P^{k-1}, they cannot form a fundamental system of neighbourhoods of zero, hence 0 must also be open.

PROPOSITION 19. *Every non-zero one-sided ideal (closed or not) in R coincides with some $P^n (P^0 = R)$, $n = 0, 1, 2, \dots$.*

PROOF. Let $0 \neq L \neq R$ be any left ideal in R and $0 \neq c \in L$. The Rc is a closed left ideal in R. This implies $Rc = Rc + (\bigcap \overline{P^n}) =$

$= \bigcap (Rc + P^n)$. Since $Rc + P^n$ is open, $Rc + P^n = P^{k_n}$ for each n. By $\bigcap P^n = 0$, $Rc \neq 0$ there exists an integer t with $Rc = P^t$. Therefore L is open and hence by the proof of Proposition 18 L equals a power of P. By the assumption it is easy to see that R is topologically artinian from the right, too. By symmetry the assertion of Proposition 19 is true.

THEOREM 20. *Let R be a topologically artinian local ring satisfying $\bigcap \overline{P^n} = 0$ and suppose that there exists no one-sided ideal between P and P^2 where P denotes its maximal non-zero open ideal. Then R has no zero-divisors if and only if $P^n \neq 0$ for each n.*

PROOF. The necessity is obvious.

Conversely, by Proposition 19 there is an element a contained in P but not in P^2, and then $P = Ra = aR$ and by induction $P^n = a^nR = Ra^n$. If x and y are non-zero elements in R, then there are integers k, l such that $x \in P^k - P^{k+1}$, $y \in P^l - P^{l+1}$. Thus it follows that we can express x and y as $x = ua^k$, $y = va^l$ where u, v must be units in R. Therefore $xy = ua^k va^l = uv_1a^{k+l}$ where v_1 is a unit with the property $v_1a^k = a^kv$. This implies $xy \neq 0$.

If a and b are any two non-zero elements of R, from the proof of Theorem 20 and by Proposition 19 we have $Ra = P^k$, $Rb = P^l$ for some integers k, l. Hence a, b have a common left multiple $c = b_1a = a_1b = 0$. By Ore's well-known result one can now construct a quotient division ring Δ of R whose elements are the quotients $a^{-1}b$. We define the following valuations v on R: $\nu(0) = +\infty$; if $a \neq 0$ is any element of R, then there is an integer n with $a \in P^n - P^{n+1}$, and let $v(a) = n$. It is routine to check that v defines a discrete valuation on R in the sense of [1] Chap. VI. In the classical way we can extend v to Δ and it is easy to see that v is a discrete valuation on Δ. We show that R is the ring of the valuation v on Δ. In fact, let $x = b^{-1}a$ be any element of Δ which is not contained in R where $a, b \in R$. Then we can write $a = c^ku$, $b = c^lt$ for $c \in P - P^2$, and u, t units in R. Since $x \notin R$ and $x = b^{-1}a = (c^lt)^{-1}c^ku = t^{-1}c^{k-l}u$, v^{-1}, $u \in R$, we have $k < l$. Hence $x^{-1} = a^{-1}b = u^{-1}c^{l-k}t \in R$. Since R is complete, Δ is complete in the topology induced by v.

From the above we have

THEOREM 21. *Let R be a linearly compact local ring satisfying $\bigcap \overline{P^n} = 0$ where P denotes its maximal open ideal. The following conditions are equivalent.*

1) $P^n \neq 0$ *for each* n *and there exists no one-sided ideal between* P *and* P^2.

2) R *is the ring of a complete, discrete valuation on a division ring.*

THEOREM 22. *Let* R *be a linearly compact ring with* $\bigcap \overline{J^n} = 0$. R *is a direct product of rings of complete, discrete valuations on division rings, of local uni-serial rings, and of division rings if and only if the products of any two maximal open left ideals commute and there exists no one-sided open ideal between* P *and* P^2 *for each maximal open ideal* P *of* R.

In what follows we shall investigate topologically noetherian linearly compact rings. First we prove

THEOREM 23. *A linearly compact ring* R *is topologically noetherian in the equivalent Leptin-topology if and only if* ${}_*J = 0$, *i.e. its radical is l-transfinitely nilpotent.*

PROOF. Assume that R is topologically noetherian. For any open left ideal L of R we have $J[({}_*J + L)/L] = ({}_*J + L)/L$ and hence by Nakayama's Lemma we have $({}_*J + L)/L = 0$, consequently ${}_*J \subseteq L$ and therefore ${}_*J = 0$. Conversely if ${}_*J = 0$, then we prove by induction that $R/{}_\mu J$ is topologically noetherian for every ordinal μ from which the statement follows clearly. For $\mu = 1$ the assertion is obvious. If $R/{}_\mu J$ is topologically noetherian, then ${}_\mu J/{}_{\mu+1}J$ can be considered as R/J-module and hence it is obviously topologically noetherian in the Leptin-topology. Therefore $R/{}_{\mu+1}J$ is such, too. If λ is a limit ordinal and $R/{}_\mu J$ is topologically noetherian for all $\mu < \lambda$, there R/J is the inverse limit of $R/{}_\mu J$ is trivially topologically noetherian, too. This completes the proof.

A topological ring R is called of *cofinite type* if the set of all isomorphism classes of simple factors of all submodules in any finitely generated discrete R-module is finite. Similarly to Proposition 9 we can prove the next statement.

PROPOSITION 24. *Let* R *be a topologically noetherian linearly compact ring of cofinite type. If the set* $\{Pi : i \in I\}$ *of all maximal closed ideals in* R *satisfies* $\overline{P_i P_j} = \overline{P_j P_i}$, $\overline{P_i (P_j)_\lambda} = \overline{(P_i)_\lambda P_i}$ *and* $\overline{(P_j)_\lambda (P_i)_\xi} = \overline{(P_i)_\xi (P_j)_\lambda}$ *for any* $i \neq j$, $i, j \in I$ *and any limit ordinals* ξ, λ *then* $\bigcap_{i \in I} {}_*(P_i) = 0$.

PROOF. Similarly to Proposition 9 we obtain $\overline{(P_i)_\xi (P_j)_\lambda} = \overline{(P_j)_\lambda (P_i)_\xi}$ for any $i \neq j$, $i, j \in I$ and any ordinals ξ and η. Assume now indirectly that $\bigcap_{i \in I} {}_*(P_i) \neq 0$. Then there is an element $c \in R$ and an open left ideal L with $c \in \bigcap_i {}_*(P_i)$ and $c \notin L$. Since R is of cofinite type, there are finitely many maximal closed ideals P_i, say $P_1, \ldots, P_n$ such that every simple factor of each submodule in R/L is annihilated by some one of them. By Zorn's Lemma there exists a minimal submodule M of R/L such that $(R/L)/M$ is annihilated by ${}_*(P_1) \ldots {}_*(P_n)$. We claim $M = 0$. In fact, if $M \neq 0$, then M as a submodule of the noetherian module R/L is finitely generated. Therefore $JM \neq M$. Since M/JM can be considered as a module over R/J, it is a finite direct sum of simple R-modules by Proposition 6. Thus there is a submodule N of M such that M/N is annihilated by some P_i, say P_1. Henceforth we obtain now

$$[{}_*(P_1) \cdots {}_*(P_n)][(R/L/N)] =$$

$$= P_1\{[{}_*(P_1) \ldots {}_*(P_n)][(R/L)/N]\} \subseteq P_1 \cdot M/N = 0\,,$$

which contradicts to the minimality of M. Thus $M = 0$ holds. This implies by $c \in \bigcap_i {}_*(P_i) \subseteq {}_*(P_1) \ldots {}_*(P_n)$ that $c \cdot R/L = 0$, i.e. $c \in L$, which is impossible. Therefore $\bigcap_i {}_*(P_i) = 0$ and we are done.

Consider now a topologically noetherian linearly compact ring R satisfying the condition of Proposition 24. In the case $i \neq j$ by Proposition 4 the ideal $P_i + P_j$ is closed in R and hence we get $P_i + P_j = R$. Similarly to Proposition 7 we obtain ${}_\xi(P_i) + {}_\xi(P_j) = R$ for every ordinal ξ. Thus ${}_*(P_i) + {}_*(P_j) = R$ holds. Henceforth for any finite set $\{{}_*(P_1), \ldots, {}_*(P_n)\}$ we have by the Chinese Remainder Theorem

$${}_*(P_i) + \bigcap_{k \neq i} {}_*(P_k) = R\,, \qquad i = 1, \ldots, n\,.$$

As ${}_*(P_i)$ is a closed ideal of R, the factor ring $R_i = R/{}_*(P_i)$ is a topologically noetherian linearly compact ring, too. Similarly to Proposition 11, one can see that the rings R_i are primary rings. As it was done in Proposition 10, we can prove that R is isomorphic to the direct product $\prod R_i$ and this isomorphism is continuous, but in general, it is not a topological isomorphism. Furthermore, the topology in $\prod R_i$

induced by this continuous isomorphism is equivalent to the product topology. Therefore this isomorphism is topological if we endow R with the equivalent Leptin-topology. Thus the proof of the following theorem is similar to that of Theorem 12 and hence we omit it.

THEOREM 24. *Let R be a topologically noetherian linearly compact ring endowed with the equivalent Leptin-topology. The following conditions are equivalent.*

1) *R is a direct product of topologically noetherian linearly compact primary ring R_i.*

2) *R is of cofinite type and if $\{P_i: i \in I\}$ is the set of all maximal closed ideals in R, then the equalities $P_iP_j = P_jP_i$, $P_i(P_j)_\lambda = (P_j)_\lambda P_i$, $(P_i)_\mu(P_j)_\lambda = (P_j)_\lambda(P_i)_\mu$ hold for all $i \neq j$, $i, j \in I$ and limit ordinals μ, λ.*

3) *R is of cofinite type and the equalities $\overline{P_iP_j} = \overline{P_jP_i}$, $\overline{P_i(P_j)_\lambda} = \overline{P_i(P_j)_\lambda} = \overline{(P_j)_\lambda P_i}$, $\overline{(P_i)_\mu(P_j)_\lambda} = \overline{(P_j)_\lambda(P_i)_\mu}$ hold for all $i \neq j$, $i, j \in I$ and limit ordinals μ, λ.*

4) *Any ideal K of R such that $K = \overline{K^2}$ is a unital ring.*

ACKNOWLEDGEMENT. I express my sincere thanks to Professor A. Orsatti and the referee for many remarks and much helpful advice.

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Manoscritto pervenuto in redazione il 6 dicembre 1984.