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Integral Functionals Determined by Their Minima.

GIANNI DAL MASO - LUCIANO MODICA (*)

Introduction.

In this paper we study the following problem in Calculus of Variations: determine an integral functional

$$F(u, A) = \int_A f(x, Du(x))\, dx$$

by the knowledge of the minima of the Dirichlet's problems for F with linear boundary values, that is by knowing the numbers

$$m(p, A) = \min_u \{F(u, A): u(x) = p\cdot x\,, \quad \forall x \in \partial A\}$$

for every $p \in \mathbb{R}^n$ and for every bounded open subset A of $\mathbb{R}^n$.

Namely, we show that the integrand $f(x, p)$ can be calculated by a differentiation process of the set function $A \to m(p, A)$ along a family $(A_\varrho)_{\varrho>0}$ of open subsets of $\mathbb{R}^n$ which shrinks nicely to x as $\varrho \to 0^+$. According to W. Rudin ([13], ch. 8), a family (A_ϱ) is said to shrink to x nicely as $\varrho \to 0^+$ if for every $\varrho > 0$

$$A_\varrho \subseteq B(x, \varrho) = \{y \in \mathbb{R}^n : |y - x| < \varrho\}\,, \qquad |A_\varrho| \geqslant c|B(x, \varrho)|$$

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where $c > 0$ is a suitable real constant independent of ϱ and $|\cdot|$ denotes the Lebesgue measure in $\mathbf{R}^n$.

The main result we prove is the following.

THEOREM I. *Suppose that the function* $f: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ *satisfies the following hypotheses*:

(i) $f(x, p)$ *is measurable in* x *and convex in* p;

(ii) $\varphi_1(p) \leq f(x, p) \leq \varphi_2(p)$ $\forall (x, p) \in \mathbf{R}^n \times \mathbf{R}^n$,
where φ_1, φ_2: $\mathbf{R}^n \to \mathbf{R}$ *are convex functions and*

(iii) $\displaystyle\lim_{|p| \to +\infty} \frac{\varphi_1(p)}{|p|} = +\infty$.

Then, denoting

$$m(p, A) = \inf\left\{ \int_A f(y, Du(y))\, dy : u \in C^\infty(\mathbf{R}^n),\ u(y) = p\cdot y\ \forall y \in \partial A \right\},$$

there exists a measurable subset $N \subseteq \mathbf{R}^n$ *with* $|N| = 0$ *such that*

$$(*) \qquad f(x, p) = \lim_{\varrho \to 0^+} \frac{m(p, A_\varrho)}{|A_\varrho|}$$

for every $p \in \mathbf{R}^n$, $x \in \mathbf{R}^n \setminus N$ *and for every family* $(A_\varrho)_{\varrho > 0}$ *of open subsets of* $\mathbf{R}^n$ *which shrinks to* x *nicely as* $\varrho \to 0^+$.

Some comments. (*a*) The superlinearity hypothesis (iii) can be dropped if $f(x, p)$ depends only on p for large $|p|$ (see remark 1.3). (*b*) In the vector case (when $u(x)$ is a vector in $\mathbf{R}^m$ and f is defined on $\mathbf{R}^n \times \mathbf{R}^{nm}$) the same thesis (*) holds by assuming f quasi-convex but by strenghtening (ii) to

$$(\text{ii})' \qquad c_1 |p|^\alpha \leq f(x, p) \leq c_2(1 + |p|^\alpha)$$

with $0 < c_1 \leq c_2$ and $\alpha > 1$ (see theorem II). The proof in this case relies on a recent approximation result for quasi-convex functions due to P. Marcellini [10]. (*c*) The case of non-negative integrands f depending not only on x and Du but also on u is more delicate. As an example, we treat here the case of uniform continuity in u and $f(x, u, 0) = 0$ for every $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ (see theorem III).

An application of theorem I is a useful and meaningful characterization of the Γ-convergence of a sequence of equicoercive functionals: see theorem IV. This theorem is an important step for applying Ergodic Theory in nonlinear stochastic homogenization (see G. Dal Maso - L. Modica [3]).

A particular case of theorem I was obtained by E. De Giorgi and S. Spagnolo [5] when f is a quadratic form, i.e.

$$f(x, p) = \sum_{i,j=1}^{n} a_{ij}(x) p_i p_j .$$

Their proof relies on Meyers' estimate of the summability exponent for the gradients of the solutions to the Euler equation of the corresponding integral functional F. Recently, M. Giaquinta and E. Giusti [9] have found an analogous estimate for the gradients of the minima of integral functionals (even non-differentiable). Neverthless, we have preferred to employ an elementary and direct method for proving theorem I.

One may also consider the problem of determining an integral functional F by the knowledge of the values of other variational problems for F, for instance by knowing the numbers

$$(1) \qquad m(\lambda, w, A) = \inf \left\{ F(u, A) + \lambda \int_A |u - w|^2 dx : u \in C^\infty(A) \right\}$$

for every bounded open subset A of $\mathbb{R}^n$, $\lambda > 0$, $w \in L^2(A)$ or the numbers

$$(2) \qquad m(\varphi, A) = \inf \left\{ F(u, A) + \int_A \varphi u \, dx : u \in C_0^\infty(A) \right\}$$

for every bounded open subset A of $\mathbb{R}^n$ and $\varphi \in L^2(A)$.

In both cases suitable reformulations of theorem I continue to hold. The first case (1) has been studied in many papers about Γ-convergence (see, for example, E. De Giorgi - T. Franzoni [4], L. Carbone - C. Sbordone [1], G. Dal Maso - L. Modica [2]), the second case (2) is related to Fenchel's duality for convex functions (see, for example, I. Ekeland - R. Teman [6], R. T. Rockafellar [12]).

We thank the referee for some useful advice.

1. Proof of Theorem 1.

Let us begin by a particular case of theorem I.

1.1. *Proposition. Let* $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be a function such that* $f(x, p)$ *is measurable in* x*, convex in* p*, and bounded from below. If there exists a real constant* R *so that* $f(x, p)$ *does not depend on* x *for* $|p| \geqslant R$*, then the thesis* (*) *of theorem I holds.*

PROOF. Let us fix $x \in \mathbb{R}^n$. A straightforward application of Jensen's inequality gives that

$$\inf\left\{\int_{A_\varrho} f(x, Du(y))\,dy : u \in C^\infty(\mathbb{R}^n),\ u(y) = p\cdot y\ \forall y \in \partial A_\varrho\right\} =$$

$$= \int_{A_\varrho} f(x, p)\,dy = |A_\varrho| f(x, p) \qquad \forall p \in \mathbb{R}^n,\ \varrho > 0$$

so we easily obtain

$$\left| f(x, p) - \frac{m(p, A_\varrho)}{|A_\varrho|}\right| \leqslant \frac{1}{|A_\varrho|} \sup_{u \in C^\infty(\mathbb{R}^n)} \left| \int_{A_\varrho} [f(x, Du(y)) - f(y, Du(y))]\,dy\right| \leqslant$$

$$\leqslant \frac{1}{|A_\varrho|} \int_{A_\varrho} \sup_{q \in \mathbb{R}^n} |f(x, q) - f(y, q)|\,dy\,.$$

If we define

$$\omega(x, y, p) = |f(x, p) - f(y, p)| \qquad (x, y, p \in \mathbb{R}^n),$$

$$\varphi(x, y) = \sup_{p \in \mathbb{R}^n} \omega(x, y, p) \qquad (x, y \in \mathbb{R}^n),$$

it remains to prove that there exists a measurable subset $N \subseteq \mathbb{R}^n$ with $|N| = 0$ such that

$$\lim_{\varrho \to 0^+} \frac{1}{|A_\varrho|} \int_{A_\varrho} \varphi(x, y)\,dy = 0$$

for every $x \in \mathbb{R}^n \setminus N$ and (A_ϱ) which shrinks to x nicely as $\varrho \to 0^+$.

Since $f(x, p)$ depends only on p for $|p|\geqslant R$ and is convex in p, we have that

$$f(x, p)\leqslant \max_{|q|=R+1} f(x, q) = M\,, \qquad \forall x\in\mathbf{R}^n,\, p\in\mathbf{R}^n\colon |p|\leqslant R+1\,,$$

with M independent of x. On the other hand f is bounded from below, so it follows that all the functions $f(x, p)$ are Lipschitz continuous in p, uniformly with respect to $x\in\mathbf{R}^n$, on the ball $|p|\leqslant R$. If we observe that $\omega(x, y, p) = 0$ for every $p\in\mathbf{R}^n$ such that $|p|\geqslant R$, we may infer that

$$|\omega(x, y, p) - \omega(x, y, q)|\leqslant K|p-q| \qquad \forall x, y, p, q\in\mathbf{R}^n$$

for a suitable real constant K.

Now, let us choose a countable dense subset D of $\mathbf{R}^n$ and let us construct, by Lebesgue's differentiation theorem (see, for instance, [13], th. 8.8) a measurable subset N of $\mathbf{R}^n$ with $|N| = 0$ such that

$$\lim_{\varrho\to 0^+}\frac{1}{|A_\varrho|}\int_{A_\varrho}\omega(x, y, p)\,dy = 0$$

for every $x\in\mathbf{R}^n\setminus N$, $p\in D$ and (A_ϱ) which shrinks to x nicely as $\varrho\to 0^+$.

For every $\varepsilon>0$ there exists a finite number $p_1, \ldots, p_m$ of elements of D such that

$$\inf_{1\leqslant i\leqslant m}|p-p_i|<\varepsilon\,, \qquad \forall p\in\mathbf{R}^n\colon |p|\leqslant R,$$

so we have that

$$\varphi(x, y)\leqslant\sum_{i=1}^{m}\omega(x, y, p_i) + K\varepsilon\,, \qquad \forall x, y\in\mathbf{R}^n,$$

and we may conclude that

$$\limsup_{\varrho\to 0^+}\frac{1}{|A_\varrho|}\int_{A_\varrho}\varphi(x, y)\,dy\leqslant K\varepsilon\,.$$

for every $x\in\mathbf{R}^n\setminus N$ and (A_ϱ) which shrinks to x nicely as $\varrho\to 0^+$. By taking $\varepsilon\to 0^+$, proposition 1.1 is proved.

The general case of theorem I will be obtained by the following approximation lemma.

1.2 LEMMA. *If f satisfies the hypotheses of theorem I, then there exists an increasing sequence (f_h) of functions such that $f = \sup_h f_h$ and each function f_h fulfils the assumptions of proposition* 1.1.

PROOF. For every $h \in \mathbb{N}$ we define

$$\tilde{f}_h(x, p) = \inf_{z \in \mathbb{R}^n} [f(x, z) + h|z - p|] , \qquad \big((x, p) \in \mathbb{R}^n \times \mathbb{R}^n\big).$$

The sequence $(\tilde{f}_h)$ is the usual approximation from below of f by Lipschitz continuous functions. In fact $\tilde{f}_h(x, p)$ is Lipschitz continuous in p (with Lipschitz constant h), $\tilde{f}_h \leqslant \tilde{f}_{h+1} \leqslant f$ for every $h \in \mathbb{N}$ and it is easy to prove, by remarking that $f(x, p)$ is convex (hence continuous) in p, that $\sup_h \tilde{f}_h = f$. The same remark proves that

$$\inf_{z \in \mathbb{R}^n} [f(x, z) + h|z - p|] = \inf_{z \in \mathbb{Q}^n} [f(x, z) + h|z - p|] ,$$

so $\tilde{f}_h(x, p)$ is measurable in x. Finally, a direct calculation shows that $\tilde{f}_h(x, p)$ is convex in p.

Then, we define for $h \in \mathbb{N}$ and for $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$

$$f_h(x, p) = \max \{\varphi_1(p), \tilde{f}_h(x, p)\} .$$

If is obvious that $f_h(x, p)$ is measurable in x and convex in p. Since

$$\tilde{f}_h(x, p) \leqslant \varphi_2(0) + h|p| \qquad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n$$

the superlinearity hypothesis (iii) gives that there exist $c \in \mathbb{R}$ and $R_h > 0$ such that

$$c \leqslant \varphi_1(p) \leqslant f_h(x, p) \qquad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n$$

$$f_h(x, p) = \varphi_1(p) \qquad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n : |p| \geqslant R_h .$$

This concludes the proof of lemma 1.2.

Now, let us prove theorem I.

Proof of theorem I. First, we prove that there exists a measurable set $N' \subseteq \mathbb{R}^n$ with $|N'| = 0$ such that

$$\lim_{\varrho \to 0^+} \frac{1}{|A_\varrho|} \int_{A_\varrho} |f(y, p) - f(x, p)| \, dy = 0 \tag{3}$$

for every $x \in \mathbb{R}^n \setminus N'$, $p \in \mathbb{R}^n$ and (A_ϱ) which shrinks to x nicely as $\varrho \to 0^+$. Let D be a countable dense subset of $\mathbb{R}^n$. By Lebesgue's differentiation theorem (see, for instance, [13], th. 8.8), there exists a measurable set $N' \subseteq \mathbb{R}^n$ with $|N'| = 0$ such that (3) holds for every $x \in \mathbb{R}^n \setminus N'$, $p \in D$ and (A_ϱ) which shrinks to x nicely as $\varrho \to 0^+$. Since $f(x, p)$ is locally Lipschitz continuous in p uniformly with respect to x (by convexity and (ii)), it is easy to see that (3) holds for every $p \in \mathbb{R}^n$.

Now, we have at once

$$\frac{1}{|A_\varrho|} \int_{A_\varrho} f(y, p) \, dy \geqslant \frac{m(p, A_\varrho)}{|A_\varrho|}$$

so by (3)

$$f(x, p) \geqslant \limsup_{\varrho \to 0^+} \frac{m(p, A_\varrho)}{|A_\varrho|}$$

for every $x \in \mathbb{R}^n \setminus N'$, $p \in \mathbb{R}^n$ and (A_ϱ) which shrinks to x nicely as $\varrho \to 0^+$.

For the converse inequality, let (f_h) be the sequence given by lemma 1.2, $m_h(p, A_\varrho)$ be the corresponding minima and N_h be the measurable subsets of $\mathbb{R}^n$ with $|N_h| = 0$ given by proposition 1.1 for f_h. Define $N'' = \bigcup_{h=1}^{+\infty} N_h$. Since $f \geqslant f_h$ for every $h \in \mathbb{N}$, we have that

$$f_h(x, p) = \lim_{\varrho \to 0^+} \frac{m_h(p, A_\varrho)}{|A_\varrho|} \leqslant \liminf_{\varrho \to 0^+} \frac{m(p, A_\varrho)}{|A_\varrho|}$$

and, by taking the limit as $h \to +\infty$, we obtain

$$f(x, p) \leqslant \liminf_{\varrho \to 0^+} \frac{m(p, A_\varrho)}{|A_\varrho|}$$

for every $x \in \mathbb{R}^n \setminus N''$, $p \in \mathbb{R}^n$ and (A_ϱ) which shrinks to x nicely as $\varrho \to 0^+$. Then, theorem I is proved by choosing $N = N' \cup N''$.

1.3 REMARK. The coerciveness hypothesis (iii) in theorem I is crucial for the approximation lemma 1.2. A particular non-coercive case has been studied by N. Fusco and G. Moscariello [8], who consider non-negative quadratic forms

$$f(x, p) = \sum_{i,j=1}^{n} a_{ij}(x) p_i p_j$$

and obtain the formula (*) with limsup instead of lim. However, if $f(x, p)$ does not depend on x for large $|p|$, theorem 1 holds without any coerciveness hypothesis, as proposition 1.1 shows.

Theorem I can be generalized as follows.

THEOREM II. *Suppose that the function* $f: \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}$ *satisfies the following hypotheses*:

(i) $f(x, p)$ *is measurable in* x *and quasi-convex in* p (*in the Morrey's* [11] *sense*).

(ii) $c_1|p|^\alpha \leqslant f(x, p) \leqslant c_2(1 + |p|^\alpha)$ $\forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^{mn}$ *with* $0 < c_1 \leqslant c_2$, $\alpha > 1$ *real constants.*

Then the thesis (*) *of theorem I holds* (u *is a* m*-vector function,* p *is identified with a* $m \times n$ *matrix*).

PROOF. The proof of theorem I can be repeated only substituting lemma 1.2 by theorem 1.2 of P. Marcellini [10].

THEOREM III. *Suppose that* $f: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *satisfies the following hypotheses*:

(i) $f(x, s, p)$ *is measurable in* x, *continuous in* s, *convex in* p;

(ii) $0 \leqslant \varphi_1(p) \leqslant f(x, s, p) \leqslant \varphi_2(p)$ $\forall (x, s, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ *where* φ_1, $\varphi_2: \mathbb{R}^n \to \mathbb{R}$ *are convex functions and*

$$\lim_{|p| \to +\infty} \frac{\varphi_1(p)}{|p|} = +\infty;$$

(iii) $|f(x, s_1, p) - f(x, s_2, p)| \leqslant (1 + f(x, s_1, p))\, \omega(|s_1 - s_2|)$ $\forall x \in \mathbb{R}^n$, $s_1, s_2 \in \mathbb{R}$, $p \in \mathbb{R}^n$ *where* $\omega: \mathbb{R}_+ \to \mathbb{R}$ *is a function such that* $\lim_{t \to 0^+} \omega(t) = 0$;

(iv) $f(x, s, 0) = 0$ $\forall (x, s) \in \mathbb{R}^n \times \mathbb{R}$.

Then, letting $W^{1,\infty}(\mathbb{R}^n)$ be the space of the Lipschitz continuous functions on $\mathbb{R}^n$ and denoting

$$(4)\qquad m(x, s, p, A) =$$

$$= \inf_{u\in W^{1,\infty}(\mathbb{R}^n)}\left\{\int_A f(y, u(y), Du(y))\,dy : u(y) = s + p\cdot(y - x) \quad \forall y \in \partial A\right\},$$

there exists a measurable subset N of $\mathbb{R}^n$ with $|N| = 0$ such that

$$f(x, s, p) = \lim_{\varrho\to 0^+} \frac{m(x, s, p, A_\varrho)}{|A_\varrho|}$$

for every $x \in \mathbb{R}^n\setminus N$, $s \in \mathbb{R}$, $p \in \mathbb{R}^n$ and for every family $(A_\varrho)_{\varrho>0}$ of open subsets of $\mathbb{R}^n$ which shrinks to x nicely as $\varrho \to 0^+$.

Proof. Let us introduce the auxiliary function

$$m'(s, p, A) = \inf_{u\in W^{1,\infty}(\mathbb{R}^n)}\left\{\int_A f(y, s, Du(y))\,dy : u(y) = p\cdot y \quad \forall y \in \partial A\right\}$$

and note that

$$(5)\qquad m'(s, p, A) =$$

$$= \inf_{u\in W^{1,\infty}(\mathbb{R}^n)}\left\{\int_A f(y, s, Du(y))\,dy : u(y) = s + p\cdot(y - x) \quad \forall y \in \partial A\right\}$$

for every $x \in \mathbb{R}^n$. Hypothesis (iv) assures that the functionals

$$u \to \int_A f(y, u(y), Du(y))\,dy\,, \qquad u \to \int_A f(y, s, Du(y))\,dy$$

decrease by truncating the function u, hence the class of competing functions in the infima (4) and (5) can be restricted to the functions such that

$$u(y) = s + p\cdot(y - x) \qquad \forall y \in \partial A$$
$$|u(y) - s| \leqslant (\operatorname{diam} A')|p|$$

where $A' = A \cup \{x\}$ (« maximum principle »).

Let us fix $x \in \mathbb{R}^n$. Then, by (ii) and (iii), we have that

$$|m(x, s, p, A_\varrho) - m'(s, p, A_\varrho)| \leqslant \omega(2\varrho|p|)(1 + \varphi_2(p))|A_\varrho| \tag{6}$$

for every $s \in \mathbb{R}$, $p \in \mathbb{R}^n$, and for every open set $A_\varrho \subseteq B(x, \varrho)$.

Let D be a countable dense subset of $\mathbb{R}$. By theorem I there exists a measurable subset $N \subseteq \mathbb{R}^n$ with $|N| = 0$ such that

$$\lim_{\varrho \to 0^+} \frac{m'(s, p, A_\varrho)}{|A_\varrho|} = f(x, s, p) \tag{7}$$

for every $x \in \mathbb{R}^n - N$, $s \in D$, $p \in \mathbb{R}^n$, and for every family (A_ϱ) which shrinks to x nicely as $\varrho \to 0^+$. Since, by (ii) and (iii), we have

$$|m'(s_1, p, A_\varrho) - m'(s_2, p, A_\varrho)| \leqslant \omega(|s_1 - s_2|)(1 + \varphi_2(p))|A_\varrho|,$$

it is easy to prove that (7) holds for every $s \in \mathbb{R}$, and the thesis follows from (6).

1.4 REMARK. The same « freezing » technique of the previous proof could be extended also to the vector case. Indeed, the use of the maximum principle can be avoided by taking profit of a result by N. Fusco and J. Hutchinson ([7], lemma 4.1), but by assuming more regularity on f.

2. A characterization of Γ-convergence.

Let us fix $0 < c_1 \leqslant c_2$, $\alpha > 1$ and let $\mathcal{F} = \mathcal{F}(\alpha, c_1, c_2)$ be the set of all functionals $F: L^\alpha_{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{A}_0 \to \overline{\mathbb{R}}$ ($\mathcal{A}_0$ denotes the family of the bounded open subsets of $\mathbb{R}^n$) given by

$$F(u, A) = \begin{cases} \displaystyle\int_A f(x, Du(x))\, dx & \text{if } u_{|A} \in W^{1,\alpha}(A), \\ + \infty & \text{otherwise,} \end{cases}$$

where $f:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$ is any function such that $f(x,p)$ is measurable in x, convex in p and

$$c_1|p|^\alpha\leqslant f(x,p)\leqslant c_2(1+|p|^\alpha)\qquad\forall(x,p)\in\mathbf{R}^n\times\mathbf{R}^n\,.$$

Of course, $W^{1,\alpha}(A)$ denotes the usual first order Sobolev space with summability exponent α.

A notion of convergence for sequences of real-extended functions defined on a topological space, the Γ-convergence (see E. De Giorgi - T. Franzoni [4]), is particularly useful when applied to the sequences in $\mathcal{F}$. We refer to G. Dal Maso - L. Modica [2] for a systematic and self-contained study of the Γ-convergence on $\mathcal{F}$.

The crucial property of Γ-convergence is a general theorem on convergence of minima. In particular, we are interested here in the following proposition (see [2], prop. 1.18).

PROPOSITION 2.1. *Suppose that* (F_h) *is a sequence in* $\mathcal{F}$ *which* Γ-*converges as* $h\to+\infty$ *to* $F_\infty\in\mathcal{F}$. *Then, for every* $A\in\mathcal{A}_0$ *and* $u_0\in W^{1,\alpha}(A)$, *we have that*

$$\lim_{h\to+\infty}\min_u\{F_h(u,A):u-u_0\in W_0^{1,\alpha}(A)\}=\\=\min_u\{F_\infty(u,A):u-u_0\in W_0^{1,\alpha}(A)\}\,.$$

In this section, our aim is to prove a converse of the previous proposition and so to obtain a characterization of Γ-convergence in $\mathcal{F}$ by the convergence of the minima of Dirichlet problems.

THEOREM IV. *Let* (F_n) *be a sequence in* $\mathcal{F}$, *let* D *be a dense subset of* $\mathbf{R}^n$ *and let* $\mathcal{B}$ *be a family of bounded open subset of* $\mathbf{R}^n$ *which contains, for any* $x\in\mathbf{R}^n$, *a subfamily which shrinks to* x *nicely. Suppose that*

$$\lim_{h\to\infty}\min_u\{F_h(u,B):u-l_\xi\in W_0^{1,\alpha}(B)\}\,,$$

where $l_\xi(x)=\xi\cdot x$, *exists for every* $\xi\in D$ *and* $B\in\mathcal{B}$.

Then, there exists a functional $F_\infty\in\mathcal{F}$ *such that* (F_h) Γ-*converges to* F_∞ *and*

$$\lim_{h\to\infty}\min_u\{F_h(u,A):u-l_p\in W_0^{1,\alpha}(A)\}=\min_u\{F_\infty(u,A):u-l_p\in W_0^{1,\alpha}(A)\}$$

for every $p\in\mathbf{R}^n$ *and* $A\in\mathcal{A}_0$.

PROOF. By proposition 2.1 it is enough to prove that (F_h) Γ-converges to a functional $F_\infty \in \mathcal{F}$. It is possible (see [2], prop. 1.21 and cor. 1.22) to define a metric on $\mathcal{F}$ in such a way that $(\mathcal{F}, d)$ is a compact metric space and the convergence of a sequence in $(\mathcal{F}, d)$ is equivalent to Γ-convergence. By taking profit of this result, it will suffice to prove that, if $(F_{\sigma(h)})$ and $(F_{\tau(h)})$ are two subsequences of (F_h) which Γ-converge respectively to $F'_\infty \in \mathcal{F}$ and $F''_\infty \in \mathcal{F}$, then $F'_\infty = F''_\infty$. Indeed, by proposition 2.1 and by hypothesis

$$\min_u \{F'_\infty(u, B): u - l_\xi \in W_0^{1,\alpha}(B)\} = \min_u \{F''_\infty(u, B): u - l_\xi \in W_0^{1,\alpha}(B)\} .$$

for every $\xi \in D$ and $B \in \mathcal{B}$, hence theorem I yields that there exists $N \subseteq \mathbb{R}^n$ with $|N| = 0$ such that

$$f'_\infty(x, \xi) = f''_\infty(x, \xi) \qquad \forall x \in \mathbb{R}^n \setminus N,\ \forall \xi \in D$$

where f'_∞ and f''_∞ denote respectively the integrand of F'_∞ and F''_∞. Finally, $f'_\infty(x, p)$ and $f''_\infty(x, p)$ are convex, hence continuous, in p so $f'_\infty(x, p) = f''_\infty(x, p)$ for every $x \in \mathbb{R}^n \setminus N$ and $p \in \mathbb{R}^n$ and the thesis follows.

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