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Partial Hölder continuity of the spatial derivatives of the solutions to nonlinear parabolic systems with quadratic growth

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Partial Hölder Continuity of the Spatial Derivatives of the Solutions to Nonlinear Parabolic Systems with Quadratic Growth.

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SUNTO - Si considerano soluzioni $u \in L^2(-T, 0, H^2(\Omega, \boldsymbol{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \boldsymbol{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \boldsymbol{R}^N)$, $N \geq 1$, $0 < \gamma < 1$, del sistema non lineare, in forma di divergenza,

$$-\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du)$$

e si dimostra, nell'ipotesi che il sistema sia fortemente parabolico ad andamenti quadratici, la parziale hölderianità delle derivate spaziali $D_i u$ di u.

1. Introduction.

Let Ω be a bounded open subset of $\boldsymbol{R}^n$, with $n \geq 2$, whose boundary $\partial\Omega$ is sufficiently smooth, for instance of class C^3. Let $T > 0$ and $Q = \Omega \times (-T, 0)$. If $x = (x_1, x_2, \ldots, x_n)$ is a point of $\boldsymbol{R}^n$ and t is a real number, we set $X = (x, t)$.

By $Q(X^0, \sigma)$ we denote the subset of $\boldsymbol{R}^{n+1}$

$$B(x^0, \sigma) \times (t^0 - \sigma^2, t^0)$$

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where $X^0 = (x^0, t^0)$, $\sigma > 0$ and $B(x^0, \sigma)$ is the (open) ball of $\mathbf{R}^n$ centered at x^0 and with radius σ.

If u is a function from Q to $\mathbf{R}^N$ (N is an integer $\geqslant 1$), we set $Du = (D_1 u | \dots | D_n u)$ where $D_i = \partial/\partial x_i$. Clearly $Du \in \mathbf{R}^{nN}$. We shall denote by $p = (p^1 | \dots | p^n)$, $p^j \in \mathbf{R}^N$, the typical vector of $\mathbf{R}^{nN}$.

We consider in Q the second order non-linear system

$$(1.1) \qquad -\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du) ,$$

where $a^i(X, u, p)$, $i = 1, 2, \dots, n$, and $B^0(X, u, p)$ are vectors of $\mathbf{R}^N$ measurable in X and continuous in (u, p).

We suppose that system (1.1) is strongly parabolic, namely that vectors $a^i(X, u, p)$, $i = 1, 2, \dots, n$, are differentiable with respect to p and there exists a constant $\nu > 0$ such that

$$(1.2) \qquad \sum_{h,k=1}^{N} \sum_{i,j=1}^{n} \frac{\partial a_h^i(X, u, p)}{\partial p_k^j} \xi_h^i \xi_k^j \geqslant \nu \sum_{i=1}^{n} \|\xi^i\|^2 \text{ (1)}$$

for every system $\{\xi^i\}_{i=1,\dots,n}$ of vectors of $\mathbf{R}^N$ and every $(X, u, p) \in \in Q \times \mathbf{R}^N \times \mathbf{R}^{nN}$.

If vectors a^i and B^0 have « controlled » growth (see [7], n. 1), by a solution to system (1.1) we mean a vector $u \in L^2(-T, 0, H^1(\Omega, \mathbf{R}^N)) \cap \cap L^\infty(-T, 0, L^2(\Omega, \mathbf{R}^N))$ such that

$$(1.3) \qquad \int_Q \left\{ \sum_{i=1}^{n} (a^i(X, u, Du) | D_i \varphi) - \left(u \,\middle|\, \frac{\partial \varphi}{\partial t} \right) \right\} dX = \int_Q (B^0(X, u, Du) | \varphi) \, dX , \qquad \forall \varphi \in C_0^\infty(Q, \mathbf{R}^N) .$$

In the case of « natural » growth (see n. 4), we shall say that $u \colon Q \to \mathbf{R}^N$ is a solution to system (1.1) if $u \in L^2(-T, 0, H^2(\Omega, \mathbf{R}^N)) \cap \cap H^1(-T, 0, L^2(\Omega, \mathbf{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \mathbf{R}^N)$, $0 < \gamma < 1$ (2), and u satisfies (1.3), $\forall \varphi \in C_0^\infty(Q, \mathbf{R}^N)$.

(1) $(\cdot | \cdot)_k$ and $\|\cdot\|_k$ are the scalar product and the norm in $\mathbf{R}^k$, respectively. We shall drop the subscript k whenever there is no fear of confusion.

(2) Throughout this paper, the Hölder continuity is related to the parabolic metric

$$\delta(X, Y) = \max\{\|x - y\|, |t - \tau|^{\frac{1}{2}}\} , \qquad X = (x, t), \; Y = (y, \tau) .$$

In the work [7] S. Campanato has proved Hölder continuity and partial Hölder continuity results for the solutions of system (1.1), under the assumption that the vectors a^i and B^0 have controlled growths.

The aim of this work is to study the partial Hölder continuity in Q of the spatial derivatives of the solutions to systems of type (1.1) with non controlled growth.

In the case of non linear elliptic systems, some partial Hölder continuity results for the gradient of the solutions are given in the works [3], [8] and [4].

We start with the following remark. Let us suppose that the system (1.1) has natural growth and that the vectors $a^i(X, u, p)$, $i = 1, 2, \dots, n$, belong to $C^1(\bar{Q}\times \boldsymbol{R}^N \times \boldsymbol{R}^{nN})$. Let $u \in L^2(-T, 0, H^2(\Omega, \boldsymbol{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \boldsymbol{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \boldsymbol{R}^N)$, $0 < \gamma < 1$, be a solution of system (1.1).

Fixed an integer s, $1 \leqslant s \leqslant n$, in (1.3) let us assume

$$\varphi = D_s v^s \quad \text{with} \quad v^s \in C_0^\infty(Q, \boldsymbol{R}^N) . \tag{1.4}$$

We obtain that u is solution of the system

$$\int_Q \left\{ -\sum_{i=1}^n \left(D_s a^i(X, u, Du) | D_i v^s\right) + \left(D_s u \,\middle|\, \frac{\partial v^s}{\partial t}\right)\right\} dX = \int_Q \sum_{i=1}^n \left(\delta_{is} B^0(X, u, Du) | D_i v^s\right) dX , \qquad \forall v^s \in C_0^\infty(Q, \boldsymbol{R}^N) . \tag{1.5}$$

Because

$$D_s a^i(X, u, Du) = \frac{\partial a^i}{\partial x_s} + \sum_{k=1}^N \frac{\partial a^i}{\partial u_k} D_s u_k + \sum_{j=1}^n \sum_{k=1}^N \frac{\partial a^i}{\partial p_k^j} D_{js} u_k \text{ (3)} , \qquad i = 1, \dots, n ,$$

we can write (1.5) in the following way

$$\int_Q \left\{ \sum_{i,j=1}^n \left(A_{ij}(X, u, Du)\, D_{js} u | D_i v^s\right) - \left(D_s u \,\middle|\, \frac{\partial v^s}{\partial t}\right)\right\} dX = \int_Q \sum_{i=1}^n \left(F^{i,s}(X, u, Du) | D_i v^s\right) dX , \qquad \forall v^s \in C_0^\infty(Q, \boldsymbol{R}^N) , \qquad s = 1, 2, \dots, n , \tag{1.6}$$

(3) $D_{js} u_k = \dfrac{\partial^2 u_k}{\partial x_j \partial x_s}$, $\quad k = 1, \dots, N ; \quad D_{js} u = (D_{js} u_1, \dots, D_{js} u_N)$.

where

(1.7) $$A_{ij} = \{A_{ij}^{hk}\}\,, \qquad A_{ij}^{hk} = \frac{\partial a_h^i}{\partial p_k^j}\,,$$

(1.8) $$F^{i,s}(X, u, Du) = -\frac{\partial a^i}{\partial x_s} - \sum_{k=1}^{N} \frac{\partial a^i}{\partial u_k} D_s u_k - \delta_{is} B^0\,.$$

If now in (1.6) we add with respect to s and set $U = Du$, we obtain that U belongs to $L^2(-T, 0, H^1(\Omega, \boldsymbol{R}^{nN}))$ and verifies

(1.9) $$\int_Q \left\{ \sum_{i,j=1}^{n} (\mathcal{A}_{ij}(X, u, U)\, D_j U | D_i \Phi) - \left(U \Big| \frac{\partial \Phi}{\partial t}\right)\right\} dX =$$
$$= \int_Q \sum_{i=1}^{n} (F^i(X, u, U) | D_i \Phi)\, dX\,, \qquad \forall \Phi \in C_0^\infty(Q, \boldsymbol{R}^{nN})\,,$$

where $\mathcal{A}_{ij}$, $i, j = 1, 2, \dots, n$, is the following $nN \times nN$ matrix

(1.10) $$\mathcal{A}_{ij} = \left\{ \begin{array}{c|c|c} A_{ij} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & A_{ij} \end{array} \right\} \quad n^2 \text{ blocks}$$

whereas F^i, $i = 1, 2, \dots, n$, are the vectors of $\boldsymbol{R}^{nN}$ whose components are

(1.11) $$F^i = (F^{i,1} | F^{i,2} | \dots | F^{i,n})\,.$$

Let us remark that system (1.9) is strongly parabolic too. Infact for every system $\{\eta^i\}_{i=1,2,\dots,n}$, $\eta^i = (\eta^{i,1} | \eta^{i,2} | \dots | \eta^{i,n})$, $\eta^{i,s} \in \boldsymbol{R}^N$, of vectors of $\boldsymbol{R}^{nN}$ we have:

$$\sum_{i,j=1}^{n} (\mathcal{A}_{ij} \eta^j | \eta^i) = \sum_{i,j=1}^{n} \sum_{s=1}^{n} (A_{ij} \eta^{j,s} | \eta^{i,s}) =$$
$$= \sum_{i,j=1}^{n} \sum_{s=1}^{n} \sum_{h,k=1}^{N} \frac{\partial a_h^i}{\partial p_k^j} \eta_k^{j,s} \eta_h^{i,s} \geqslant \nu \sum_{i=1}^{n} \|\eta^i\|^2\,.$$

The matrices $\mathcal{A}_{ij}$ are bounded and uniformly continuous in $\bar{Q} \times$

$\times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$ if such are the derivatives

$$\frac{\partial a^i}{\partial p_k^j}, \quad i, j = 1, 2, \ldots, n, \; k = 1, 2, \ldots, N .$$

Then the study of the partial Hölder continuity in Q of the spatial derivatives of the solutions u to the system (1.1) is reduced to the study of the partial Hölder continuity of the solution $U = Du$ of systems of type (1.9) with coefficients that depend on X, u, U.

We shall be concerned with this type of systems in Section 3; in Section 4 we shall prove a partial Hölder continuity result for the spatial derivatives of the solutions to the system (1.1) in the case of natural growth. Finally in the Section 2 we mention a few results that will be used in the next sections.

2. Preliminary results.

LEMMA 2.1. *Let* $U \in L^2\big(t^0 - \sigma^2, t^0, H^1(B(x^0, \sigma), \boldsymbol{R}^{nN})\big) \cap H^{\frac{1}{2}}\big(t^0 - \sigma^2, t^0, L^2(B(x^0, \sigma), \boldsymbol{R}^{nN})\big)$. *Then*

$$\int_{Q(X^0,\sigma)} \| U - U_{Q(X^0,\sigma)}\|^2 \, dX \leqslant c\sigma^2 \left\{ \int_{Q(X^0,\sigma)} \sum_{i=1}^n \|D_i U\|^2 \, dX + \int_{t^0-\sigma^2}^{t^0} dt \int_{t^0-\sigma^2}^{t^0} d\xi \int_{B(x^0,\sigma)} \frac{\|U(x,t) - U(x,\xi)\|^2}{|t-\xi|^2} dx \right\} \text{ (4)} .$$

See [5], Lemma 2.I.

Now let $\mathcal{A}_{ij}^0$, $i, j = 1, 2, \ldots, n$, be constant $nN \times nN$ matrices, such that

$$\text{(2.1)} \qquad \sum_{i,j=1}^n (\mathcal{A}_{ij}^0 \eta^j | \eta^i) \geqslant \nu \sum_{i=1}^n \|\eta^i\|^2$$

for some $\nu > 0$ and any system $\{\eta^i\}_{i=1,2,\ldots,n}$ of vectors of $\boldsymbol{R}^{nN}$. Also let F^i, $i = 1, 2, \ldots, n$, be functions in $L^2(Q(X^0, \sigma), \boldsymbol{R}^{nN})$ and let $\mathcal{V} \in$

(4) $U_{Q(X^0,\sigma)} = -\!\!\!\!\!\!\int_{Q(X^0,\sigma)} U(X)\, dX = \dfrac{1}{\operatorname{meas} Q(X^0, \sigma)} \displaystyle\int_{Q(X^0,\sigma)} U(X)\, dX .$

$\in L^2\big(t^0 - \sigma^2, t^0, \boldsymbol{H}^1(B(x^0, \sigma), \boldsymbol{R}^{nN})\big)$ be a solution of the parabolic system

$$(2.2) \qquad \int\limits_{Q(X^0,\sigma)} \left\{ \sum_{i,j=1}^{n} (\mathcal{A}^0_{ij} D_j \mho | D_i \Phi) - \left(\mho \,\middle|\, \frac{\partial \Phi}{\partial t}\right) \right\} dX = \\ = \int\limits_{Q(X^0,\sigma)} \sum_{i=1}^{n} (F^i | D_i \Phi)\, dX\,, \qquad \forall \Phi \in C_0^\infty\big(Q(X^0, \sigma), \boldsymbol{R}^{nN}\big)\,.$$

Then we have

LEMMA 2.2. *Let* (2.1) *and* (2.2) *be satisfied. Then, for every* $\tau \in (0, 1)$,

$$\int\limits_{Q(X^0,\tau\sigma)} \sum_{i=1}^{n} \|D_i \mho\|^2\, dX \leqslant c \left\{ \tau^{n+2} \int\limits_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i \mho\|^2\, dX + \int\limits_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|F^i\|^2\, dX \right\}$$

and

$$\int\limits_{Q(X^0,\tau\sigma)} \|\mho - \mho_{Q(X^0,\tau\sigma)}\|^2\, dX \leqslant \\ \leqslant c \left\{ \tau^{n+4} \int\limits_{Q(X^0,\sigma)} \|\mho - \mho_{Q(X^0,\sigma)}\|^2\, dX + \tau^2 \sigma^2 \int\limits_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|F^i\|^2\, dX \right\}.$$

See [5], Lemma 2.II.

LEMMA 2.3. *Let* φ, ψ *be nonnegative functions defined in* $(0, \sigma]$. *Let* M, B *be nonnegative constants. Also let* $\alpha > 0$ *and* $A > 1$. *Assume that*

$$\varphi(\tau\varrho) \leqslant A \tau^\alpha \varphi(\varrho) + \psi(\varrho)$$

$$\psi(\tau\varrho) \leqslant B \tau^\alpha \psi(\varrho) + M\,,$$

for every $\tau \in (0, 1)$ *and* $\varrho \in (0, \sigma]$. *Then there exist constants* K *and* C, *depending only on* A, α, ε, *such that*

$$\varphi(\tau\sigma) \leqslant A \tau^{\alpha-\varepsilon} \{\varphi(\sigma) + KB\psi(\sigma)\} + CM$$

for every $\tau \in (0, 1)$ *and* $\varepsilon \in (0, \alpha)$.

See [4], Chap. I, Lemma 1.II.

LEMMA 2.4. *Let functions φ and ω_1 be nonnegative in $(0, d]$ and let ω_1 be nondecreasing; also, let function ω_2 be nonnegative and nondecreasing in $(0, +\infty)$. Let A, α be positive constants and let $0 \leqslant \beta < \alpha$. Suppose that*

$$\varphi(\tau\sigma) \leqslant \{A\tau^{\alpha} + \omega_1(\sigma) + \omega_2(\sigma^{-\beta}\varphi(\sigma))\}\varphi(\sigma)$$

for every $\tau \in (0, 1)$ and $\sigma \in (0, d]$. Also, suppose that, for some $\varepsilon \in (0, \alpha - \beta]$, there is a $\sigma_\varepsilon \in (0, d]$ such that

$$\omega_1(\sigma_\varepsilon) + \omega_2(\sigma_\varepsilon^{-\beta}\varphi(\sigma_\varepsilon)) < (1 + A)^{-\alpha/\varepsilon}.$$

Then, $\forall \tau \in (0, 1)$

$$\varphi(\tau\sigma_\varepsilon) \leqslant B\tau^{\alpha-\varepsilon}\varphi(\sigma_\varepsilon),$$

where $B = (1 + A)^{\alpha/\varepsilon}$.

See [4], Chap. I, Lemma 1.IV.

Finally, we need the following interpolation result.

THEOREM 2.1. *Let $u \in L^2(-T, 0, \boldsymbol{H}^2(\Omega, \boldsymbol{R}^N)) \cap \boldsymbol{H}^1(-T, 0, L^2(\Omega, \boldsymbol{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \boldsymbol{R}^N)$, $0 < \gamma < 1$. Then*

$$D_i u \in L^s(Q, \boldsymbol{R}^N), \qquad \forall 1 \leqslant s < \frac{4(n+2)}{n+2-2\gamma}, \qquad i = 1, 2, \ldots, n, \tag{2.3}$$

and

$$\int_Q \sum_{i=1}^n \|D_i u - (D_i u)_Q\|^s \, dX \leqslant \leqslant c(\operatorname{mis} Q)^{1-s(n+2-2\gamma)/4(n+2)} \left\{ \int_Q \left(\sum_{i,j=1}^n \|D_{ij} u\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dX \right\}^{s/4} \tag{2.4}$$

where c depends on n, s, γ, $[u]_{\gamma,\bar{Q}}$.

The proof of this theorem is achieved by a standard technique, taking into account Theorem 3.III of [2], Lemma 1 of the Appendix to [9] and the results of [12] (see also Theorem 2.III of [1]).

3. Local L^q regularity for systems of type (1.9).

We start by recalling the following L^q regularity result for linear systems.

THEOREM 3.1. *Let* $U \in L^2(-T, 0, H^1(\Omega, \mathbf{R}^{nN})) \cap L^\infty(-T, 0, L^2(\Omega, \mathbf{R}^{nN}))$ *be a solution in* Q *of the linear system*

$$-\sum_{i,j=1}^{n} D_i(\mathcal{A}_{ij}(X)\, D_j U) + \frac{\partial U}{\partial t} = -\sum_{i=1}^{n} D_i F^i(X)\,. \tag{3.1}$$

Here the $nN \times nN$ *matrices* $\mathcal{A}_{ij}(X)$, $i, j = 1, 2, \ldots, n$, *are defined and bounded in* Q *and satisfy the strong ellipticity condition. Then there exists* $\gamma_0 \in ((n-2)/n, 1)$ *such that, if*

$$F^i \in L^q(Q, \mathbf{R}^{nN})\,, \qquad 2 \leqslant q \leqslant \frac{2}{\gamma_0}\,, \qquad i = 1, 2, \ldots, n\,, \tag{3.2}$$

we have

$$D_i U \in L^q_{\mathrm{loc}}(Q, \mathbf{R}^{nN})\,, \qquad i = 1, 2, \ldots, n\,, \tag{3.3}$$

and

$$\Big(\int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i U\|^q\, dX\Big)^{1/q} \leqslant c\Big(\int_{Q(X^0,2\sigma)} \sum_{i=1}^{n} \|F^i\|^q\, dX\Big)^{1/q} + c\sigma^{(n+2)(1/q-\frac{1}{2})}\Big\{\int_{Q(X^0,2\sigma)} \Big(\sum_{i=1}^{n} \|D_i U\|^2 + \sigma^{-2}\|U-\lambda\|^2\Big)\, dX\Big\}^{\frac{1}{2}}$$

for every $\lambda \in \mathbf{R}^{nN}$ *and* $Q(X^0, 2\sigma) \Subset Q$ (5).

See [6], Section 4, Theorem 4.IV.

Now we are ready to prove, taking into account Theorem 3.1, the following L^q regularity result for non-linear systems:

THEOREM 3.2. *Assume that* u *is in* $L^2(-T, 0, H^2(\Omega, \mathbf{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \mathbf{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \mathbf{R}^N)$, $0 < \gamma < 1$, *and* $U = Du$ *is a solution in* Q

(5) $Q(X^0, \varrho) \Subset Q$ means that $B(x^0, \varrho) \Subset \Omega$ and $\varrho^2 < t^0 + T \leqslant T$.

of system

$$(3.4)\qquad -\sum_{i,j=1}^{n} D_i\big(\mathcal{A}_{ij}(X)\,D_j U\big) + \frac{\partial U}{\partial t} = -\sum_{i=1}^{n} D_i F^i(X, u, U)\,.$$

Here the $nN\times nN$ matrices $\mathcal{A}_{ij}(X)$, $i, j = 1, 2, \ldots, n$, and the nN-vectors $F^i(X, u, p)$, $i = 1, 2, \ldots, n$, are assumed to be as it follows: $\mathcal{A}_{ij}$ are defined and bounded in Q and satisfy the strong ellipticity condition; $F^i(X, u, p)$ are measurable in X and continuous in (u, p) and have the following growths

$$(3.5)\qquad \|F^i(X, u, p)\| \leqslant f^i(X) + M\sum_{j=1}^{n}\|p^j\|^2\,, \qquad i = 1, 2, \ldots, n\,,$$

$$(3.6)\qquad f^i(X) \in L^q(Q)\,, \qquad 2 \leqslant q < \frac{2}{\gamma_0} \wedge \frac{2(n+2)}{n+2-2\gamma}\,.$$

Then

$$(3.7)\qquad U \in L^4(Q, \mathbf{R}^{nN})\,, \qquad D_i U \in L^q_{\mathrm{loc}}(Q, \mathbf{R}^{nN})\,, \qquad i = 1, 2, \ldots, n\,,$$

and

$$(3.8)\qquad \Big(\int_{Q(X^0,\sigma)} \sum_{i=1}^{n}\|D_i U\|^q\,dX\Big)^{1/q} \leqslant c\Big(\int_{Q(X^0,2\sigma)} \sum_{i=1}^{n}|f^i|^q\,dX\Big)^{1/q} + c\sigma^{(n+2)(1/q-\frac12)}\cdot$$
$$\cdot\Big\{\int_{Q(X^0,2\sigma)}\Big(\sum_{i=1}^{n}\|D_i U\|^2 + \|U\|^4 + \sigma^{-2}\|U-\lambda\|^2 + \Big\|\frac{\partial u}{\partial t}\Big\|^2\Big)\,dX\Big\}^{\frac12}$$

for every $\lambda \in \mathbf{R}^{nN}$ and $Q(X^0, 2\sigma) \Subset Q$ with $\sigma \leqslant 1$.

Proof. Let $u \in L^2(-T, 0, H^2(\Omega, \mathbf{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \mathbf{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \mathbf{R}^N)$, $0 < \gamma < 1$, be such that $U = Du$ is solution in Q of the system (3.4), namely

$$(3.9)\qquad \int_Q \Big\{\sum_{i,j=1}^{n}\big(\mathcal{A}_{ij}(X)\,D_j U|D_i\Phi\big) - \Big(U\Big|\frac{\partial\Phi}{\partial t}\Big)\Big\}\,dX =$$
$$= \int_Q \sum_{i=1}^{n}\big(F^i(X, u, U)|D_i\Phi\big)\,dX\,, \qquad \forall \Phi \in C_0^\infty(Q, \mathbf{R}^{nN})\,.$$

Let $Q(X^0, 2\sigma) \Subset Q$ and $Q^* = \Omega^* \times (-\lambda^* T, 0)$, where $B(x^0, 2\sigma) \Subset \Omega^* \Subset \Omega$, $\lambda^* \in (0,1)$, $-\lambda^* T < t^0 - (2\sigma)^2$.

From Theorem 2.1 of Section 2 it follows

$$U \in L^{2s}(Q, \boldsymbol{R}^{nN}), \quad \forall 2 \leqslant s < \frac{2(n+2)}{n+2-2\gamma}, \tag{3.10}$$

therefore, for every integer j, $1 \leqslant j \leqslant n$,

$$\|U^j\|^2 \in L^s(Q), \quad \forall 2 \leqslant s < \frac{2(n+2)}{n+2-2\gamma}.$$

Then, if f^i verifies (3.6), taking into account (3.5), it results

$$F^i(X, u, U) \in L^q(Q, \boldsymbol{R}^{nN}), \quad i = 1, 2, \ldots, n. \tag{3.11}$$

On the other hand, by well-known results (see [13], [14]), we have

$$U \in L^\infty(-\lambda^* T, 0, L^2(\Omega^*, \boldsymbol{R}^{nN})) \cap H^{\frac{1}{2}}(-\lambda^* T, 0, L^2(\Omega^*, \boldsymbol{R}^{nN})). \tag{3.12}$$

From (3.11), (3.12) and from Theorem 3.1 (applied to Q^*) we obtain:

$$D_i U \in L^q\big(Q(X^0, \sigma), \boldsymbol{R}^{nN}\big), \quad i = 1, 2, \ldots, n, \tag{3.13}$$

and, $\forall \lambda \in \boldsymbol{R}^{nN}$

$$\begin{aligned}\Bigg(\int_{Q(X^0,\sigma)} \sum_{i=1}^n \|D_i U\|^q \, dX\Bigg)^{1/q} &\leqslant \\ \leqslant c\Bigg(\int_{Q(X^0,2\sigma)} \sum_{i=1}^n \|F^i(X, u, U)\|^q \, dX\Bigg)^{1/q} &+ c\sigma^{(n+2)(1/q-\frac{1}{2})} \cdot \\ \cdot \Bigg\{\int_{Q(X^0,2\sigma)} \Bigg(\sum_{i=1}^n \|D_i U\|^2 &+ \sigma^{-2}\|U-\lambda\|^2\Bigg) dX\Bigg\}^{\frac{1}{2}}.\end{aligned} \tag{3.14}$$

Now let us estimate the norms of the vectors $F^i(X, u, U)$, $i = 1, 2, \ldots, n$, in the right side of inequality (3.14). In particular, taking into account (3.5), we need to estimate the integral

$$\int_{Q(X^0,2\sigma)} \sum_{j=1}^{n} \|U^j\|^{2q}\, dX .$$

If j is an integer, $1 \leqslant j \leqslant n$, we have:

$$(3.15)\qquad \left(\int_{Q(X^0,2\sigma)} \|U^j\|^{2q}\, dX\right)^{1/q} \leqslant c\left(\int_{Q(X^0,2\sigma)} \|U^j - (U^j)_{Q(X^0,2\sigma)}\|^{2q}\, dX\right)^{1/q} + \\ + c\sigma^{(n+2)/q}\|(U^j)_{Q(X^0,2\sigma)}\|^2 .$$

On the other hand, (2.4) enables us to write

$$(3.16)\qquad \left(\int_{Q(X^0,2\sigma)} \|U^j - (U^j)_{Q(X^0,2\sigma)}\|^{2q}\, dX\right)^{1/q} \leqslant \\ \leqslant c\sigma^{(n+2)(1/q-\frac{1}{2})+\gamma}\left\{\int_{Q(X^0,2\sigma)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX\right\}^{\frac{1}{2}}$$

where c does not depend on σ but only on $n, q, \gamma, [u]_{\gamma,\bar{Q}}$.

Moreover we easily have

$$(3.17)\qquad \sigma^{(n+2)/q}\|(U^j)_{Q(X^0,2\sigma)}\|^2 \leqslant c\sigma^{(n+2)(1/q-\frac{1}{2})}\left(\int_{Q(X^0,2\sigma)} \|U\|^4\, dX\right)^{\frac{1}{2}} .$$

Then from (3.15), (3.16) and (3.17) we obtain

$$\left(\int_{Q(X^0,2\sigma)} \|U^j\|^{2q}\, dX\right)^{1/q} \leqslant c\sigma^{(n+2)(1/q-\frac{1}{2})+\gamma}\left\{\int_{Q(X^0,2\sigma)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX\right\}^{\frac{1}{2}} + \\ + c\sigma^{(n+2)(1/q-\frac{1}{2})}\left(\int_{Q(X^0,2\sigma)} \|U\|^4\, dX\right)^{\frac{1}{2}}$$

from this, for $\sigma \leqslant 1$, it follows that

$$(3.18)\qquad \left(\int_{Q(X^0,2\sigma)} \|U^j\|^{2q}\, dX\right)^{1/q} \leqslant \\ \leqslant c\sigma^{(n+2)(1/q-\frac{1}{2})}\left\{\int_{Q(X^0,2\sigma)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|U\|^4\right) dX\right\}^{\frac{1}{2}} .$$

From (3.5) and (3.18) we obtain the estimates of vectors F^i in $L^q(Q(X^0, 2\sigma), \mathbf{R}^{nN})$:

$$(3.19)\qquad \left(\int_{Q(X^0,2\sigma)} \sum_{i=1}^{n} \|F^i(X, u, U)\|^q \, dX\right)^{1/q} \leqslant c\left(\int_{Q(X^0,2\sigma)} \sum_{i=1}^{n} |f^i(X)|^q \, dX\right)^{1/q} + c\sigma^{(n+2)(1/q-\frac{1}{2})}\left\{\int_{Q(X^0,2\sigma)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + \|U\|^4\right) dX\right\}^{\frac{1}{2}}.$$

Estimate (3.8) easily follows from (3.14) and (3.19).

We need a different version of Theorem 3.2.

We set

$$(3.20)\qquad \xi = (n+2)\left(1 - \frac{2}{p}\right) \quad \text{with } p > n+2$$

$$(3.21)\qquad \Phi(X^0, \sigma) = \sigma^{\xi} + \int_{Q(X^0,\sigma)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \|U\|^4 + \sigma^{-2}\|U - U_{Q(X^0,\sigma)}\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX .$$

If $p > n+2$, we have

$$p > q\,, \qquad \xi > n$$

for every number q in the interval $\left[2, \dfrac{2}{\gamma_0} \wedge \dfrac{2(n+2)}{n+2-2\gamma}\right)$, $\gamma \in (0, 1)$.

THEOREM 3.3. *Under the assumptions of Theorem* 3.2 *with*

$$f^i \in L^p(Q)\,, \qquad p > n+2\,,$$

we have

$$(3.22)\qquad \left(\int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i U\|^q \, dX\right)^{1/q} \leqslant c\sigma^{(n+2)(1/q-\frac{1}{2})}[\Phi(X^0, 2\sigma)]^{\frac{1}{2}}\,,$$

for every $q \in \left[2, \dfrac{2}{\gamma_0} \wedge \dfrac{2(n+2)}{n+2-2\gamma}\right)$ *and* $Q(X^0, 2\sigma) \Subset Q$ *with* $\sigma \leqslant 1$.

PROOF. Estimate (3.22) follows from (3.8) and from the inequality

$$\Bigg(\int\limits_{Q(X^0,2\sigma)} \sum_{i=1}^{n} |f^i|^q\, dX\Bigg)^{1/q} \leqslant c\Bigg(\int\limits_{Q(X^0,2\sigma)} \sum_{i=1}^{n} |f^i|^p\, dX\Bigg)^{1/p} \sigma^{(n+2)(1/q-\frac{1}{2})+\xi/2} .$$

It is self evident that Theorems 3.2 and 3.3 also hold for the $u \in L^2(-T, 0, H^2(\Omega, \boldsymbol{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \boldsymbol{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \boldsymbol{R}^N)$, $0 < \gamma < 1$, such that $U = Du$ is a solution in Q of a system of the type (1.9) with $\mathcal{A}_{ij}(X, u, p)$, $i, j = 1, 2, \ldots, n$, bounded $nN \times nN$ matrices defined in $Q \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$ and such that

$$\sum_{i,j=1}^{n} (\mathcal{A}_{ij}(X, u, p)\eta^j|\eta^i) \geqslant \nu \sum_{i=1}^{n} \|\eta^i\|^2 , \qquad \nu > 0 , \tag{3.23}$$

for every system $\{\eta^i\}_{i=1,2,\ldots,n}$ of vectors of $\boldsymbol{R}^{nN}$ and every $(X, u, p) \in Q \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$.

4. Partial Hölder continuity of the spatial derivatives of the solutions to system (1.1).

Let $u \in L^2(-T, 0, H^2(\Omega, \boldsymbol{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \boldsymbol{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \boldsymbol{R}^N)$, $0 < \gamma < 1$, be a solution of the non-linear system

$$-\sum_{i=1}^{n} D_i a^i(X, u, Du) + \frac{\partial u}{\partial t} = B^0(X, u, Du) , \tag{4.1}$$

where $a^i(X, u, p)$, $i = 1, 2, \ldots, n$, and $B^0(X, u, p)$ are vectors of $\boldsymbol{R}^N$ measurable in X, continuous in (u, p) and verifying the following conditions:

(4.2) $\qquad a^i(X, u, p) \in C^1(\bar{Q} \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN})$, $\quad i = 1, 2, \ldots, n$;

(4.3) *there exists a constant* $\nu > 0$ *such that*

$$\sum_{h,k=1}^{N} \sum_{i,j=1}^{n} \frac{\partial a_h^i(X, u, p)}{\partial p_k^j} \xi_h^i \xi_k^j \geqslant \nu \sum_{i=1}^{n} \|\xi^i\|^2$$

for every system $\{\xi^i\}_{i=1,2,\ldots,n}$ *of vectors of* $\boldsymbol{R}^N$ *and every* $(X, u, p) \in Q \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$;

(4.4) *the vectors* $\partial a^i/\partial p^j_k$, $i, j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, N$, *are uniformly continuous in* $\bar{Q} \times \mathbf{R}^N \times \mathbf{R}^{nN}$;

(4.5) $\forall (X, u, p) \in \bar{Q} \times \mathbf{R}^N \times \mathbf{R}^{nN}$ *it results*

$$\sum_{k=1}^{N} \sum_{j=1}^{n} \left\| \frac{\partial a^i}{\partial p^j_k} \right\| \leqslant M, \qquad i = 1, 2, \ldots, n$$

and

$$\|a^i\| + \sum_{s=1}^{n} \left\| \frac{\partial a^i}{\partial x_s} \right\| + \sum_{k=1}^{N} \left\| \frac{\partial a^i}{\partial u_k} \right\| \leqslant M\left(1 + \sum_{j=1}^{n} \|p^j\|\right), \qquad i = 1, 2, \ldots, n;$$

(4.6) $\forall (X, u, p) \in \bar{Q} \times \mathbf{R}^N \times \mathbf{R}^{nN}$ *it results*

$$\|B^0(X, u, p)\| \leqslant g^0(X) + M \sum_{j=1}^{n} \|p^j\|^2$$

with $g^0(X) \in L^p(Q)$, $p > n + 2$.

Set $U = Du$, then we have

$$U \in L^2(-T, 0, H^1(\Omega, \mathbf{R}^{nN})) . \tag{4.7}$$

Also, taking into account what we pointed out in Section 1, U is solution of the system

$$-\sum_{i,j=1}^{n} D_i(\mathcal{A}_{ij}(X, u, U)\, D_j U) + \frac{\partial U}{\partial t} = -\sum_{i=1}^{n} D_i F^i(X, u, U) , \tag{4.8}$$

i.e.:

$$\int_Q \left\{ \sum_{i,j=1}^{n} (\mathcal{A}_{ij}(X, u, U)\, D_j U | D_i \Phi) - \left(U \left| \frac{\partial \Phi}{\partial t} \right.\right) \right\} dX =$$

$$= \int_Q \sum_{i=1}^{n} (F^i(X, u, U) | D_i \Phi)\, dX , \qquad \forall \Phi \in C_0^\infty(Q, \mathbf{R}^{nN}) ,$$

where $\mathcal{A}_{ij}$, $i, j = 1, 2, \ldots, n$, is the $nN \times nN$ matrix defined by (1.10) and (1.7), and F^i, $i = 1, 2, \dots, n$, is the vector of $\mathbf{R}^{nN}$ given by

$$F^i = \left(F^{i,1}|F^{i,2}| \dots |F^{i,n}\right) ,$$

$$F^{i,s}(X, u, U) = -\frac{\partial a^i}{\partial x_s} - \sum_{k=1}^{N} \frac{\partial a^i}{\partial u_k} D_s u_k - \delta_{is} B^0 .$$

The assumptions on the vectors a^i imply that the matrices $\mathcal{A}_{ij}(X, u, p)$, $i, j = 1, 2, \dots, n$, are uniformly continuous and bounded in $\bar{Q} \times \times \mathbf{R}^N \times \mathbf{R}^{nN}$, and satisfy the strong ellipticity condition (3.23). Consequently there exists a bounded continuous function $\omega(\eta)$, defined for $\eta \geqslant 0$, which is increasing, concave, such that $\omega(0) = 0$ and $\forall X, Y \in \bar{Q}$, $\forall u, v \in \mathbf{R}^N$ and $\forall p, \bar{p} \in \mathbf{R}^{nN}$

$$\text{(4.9)} \qquad \left\{\sum_{i,j=1}^{n} \|\mathcal{A}_{ij}(X, u, p) - \mathcal{A}_{ij}(Y, v, \bar{p})\|^2\right\}^{\frac{1}{2}} \leqslant \leqslant \omega\left(\delta^2(X, Y) + \|u - v\|^2 + \|p - \bar{p}\|^2\right)$$

where $\delta(X, Y)$ is the parabolic distance:

$$\text{(4.10)} \qquad \delta(X, Y) = \max\left\{\|x - y\|, |t - \tau|^{\frac{1}{2}}\right\} , \qquad X = (x, t), \; Y = (y, \tau) .$$

The vectors $F^i(X, u, p)$, $i = 1, 2, \dots, n$, are measurable in X, continuous in (u, p) and have the following growth

$$\text{(4.11)} \qquad \|F^i(X, u, p)\| \leqslant f^0(X) + c\sum_{j=1}^{n} \|p^j\|^2 , \qquad i = 1, 2, \dots, n ,$$

where $f^0(X) = c\{1 + g^0(X)\} \in L^p(Q)$, $p > n + 2$.

By Theorem 2.1 we have:

$$U \in L^s(Q, \mathbf{R}^{nN}) , \qquad \forall 1 \leqslant s < \frac{4(n + 2)}{n + 2 - 2\gamma} ,$$

and $\forall Q(X^0, r) \Subset Q$, $j = 1, 2, \dots, n$,

$$\text{(4.12)} \qquad \int_{Q(X^0,r)} \| U^j - (U^j)_{Q(X^0,r)}\|^s \, dX \leqslant \leqslant c r^{(n+2)(1-s(n+2-2\gamma)/4(n+2))} \left\{ \int_{Q(X^0,r)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX\right\}^{s/4} .$$

From (4.12), written for $s=4$, it follows $\forall\tau\in(0,1)$:

$$\int_{Q(X^0,\tau r)}\|U^j\|^4\,dX\leqslant c\int_{Q(X^0,r)}\|U^j-(U^j)_{Q(X^0,r)}\|^4\,dX+c(\tau r)^{n+2}\|(U^j)_{Q(X^0,r)}\|^4\leqslant$$

$$\leqslant cr^{2\gamma}\int_{Q(X^0,r)}\left(\sum_{i=1}^n\|D_iU\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2\right)dX+c\tau^{n+2}\int_{Q(X^0,r)}\|U\|^4\,dX$$

and hence

$$\int_{Q(X^0,\tau r)}\|U\|^4\,dX\leqslant cr^{2\gamma}\int_{Q(X^0,r)}\left(\sum_{i=1}^n\|D_iU\|^2+\left\|\frac{\partial u}{\partial t}\right\|^2\right)dX+c\tau^{n+2}\int_{Q(X^0,r)}\|U\|^4\,dX\,. \tag{4.13}$$

Let us recall that if $Q(X^0,\sigma)\Subset Q$ then $U\in L^\infty\big(t^0-\sigma^2,t^0,L^2(B(x^0,\sigma),\boldsymbol{R}^{nN})\big)\cap H^{\frac{1}{2}}\big(t^0-\sigma^2,t^0,L^2(B(x^0,\sigma),\boldsymbol{R}^{nN})\big)$ and the following inequality holds

$$\int_{Q(X^0,\sigma)}\sum_{i=1}^n\|D_iU\|^2\,dX+\int_{t^0-\sigma^2}^{t^0}dt\int_{t^0-\sigma^2}^{t^0}d\xi\int_{B(x^0,\sigma)}\frac{\|U(x,t)-U(x,\xi)\|^2}{|t-\xi|^2}\,dx\leqslant$$

$$\leqslant c\left\{1+\int_Q\left(\sum_{i=1}^n\|D_iU\|^2+\|U\|^4\right)dX\right\} \tag{4.14}$$

where c depends on the distance of $Q(X^0,\sigma)$ from the parabolic boundary of Q [6] and on the $L^2(Q)$-norm of f^0.

From (4.14) and Lemma 2.1, we deduce

$$\sigma^{-2}\int_{Q(X^0,\sigma)}\|U-U_{Q(X^0,\sigma)}\|^2\,dX\leqslant c\left\{1+\int_Q\left(\sum_{i=1}^n\|D_iU\|^2+\|U\|^4\right)dX\right\} \tag{4.15}$$

where the constant c depends on the usual arguments.

Now we are ready to prove the following

LEMMA 4.1. *Let $u\in L^2(-T,0,H^2(\Omega,\boldsymbol{R}^N))\cap H^1(-T,0,L^2(\Omega,\boldsymbol{R}^N))\cap C^{0,\gamma}(\bar{Q},\boldsymbol{R}^N)$, $0<\gamma<1$, be a solution of the system* (4.1). *Let conditions* (4.2)-(4.6) *be fulfilled. Then, $\forall Q(X^0,\sigma)\Subset Q$ with $\sigma\leqslant 2$, $\forall\tau\in(0,1)$ and*

[6] $[\Omega\times\{-T\}]\cup[\partial\Omega\times(-T,0)]$.

$\forall \lambda \in (n, \xi)$,

$$(4.16) \qquad \Phi(X^0, \tau\sigma) \leqslant K\Phi(X^0, \sigma)\{\tau^\lambda + \sigma^{2\gamma} + [\omega(c\sigma^{-n}\Phi(X^0, \sigma))]^{1-2/q}\}$$

where q is an arbitrary number in the interval $\left(2, \dfrac{2}{\gamma_0} \wedge \dfrac{2(n+2)}{n+2-2\gamma}\right)$, ξ *and* $\Phi(X^0, \sigma)$ *are defined as in* (3.20) *and* (3.21).

PROOF. Consider $Q(X^0, 2\sigma) \Subset Q$ with $\sigma \leqslant 1$. We set $U = Du$. Taking into account the remark made at the beginning of this Section, we have that U is solution of system (4.8) and belongs to the space $L^2(-T, 0, H^1(\Omega, \mathbf{R}^{nN})) \cap [L^\infty \cap H^{\frac{1}{2}}](t^0 - \sigma^2, t^0, L^2(B(x^0, \sigma), \mathbf{R}^{nN}))$.

If we set

$$\mathcal{A}^0_{ij} = \mathcal{A}_{ij}(X^0, u_{Q(X^0,\sigma)}, U_{Q(X^0,\sigma)}) ,$$

then, in $Q(X^0, \sigma)$, U can be written in the form

$$U = \mathcal{V} + W$$

where

$$W \in L^2\big(t^0 - \sigma^2, t^0, H^1_0(B(x^0, \sigma), \mathbf{R}^{nN})\big) \cap L^\infty\big(t^0 - \sigma^2, t^0, L^2(B(x^0, \sigma), \mathbf{R}^{nN})\big)$$

is the solution of the problem

$$(4.17) \qquad \int\limits_{Q(X^0,\sigma)} \left\{\sum_{i,j=1}^n (\mathcal{A}^0_{ij} D_j W | D_i \Phi) - \left(W \left| \frac{\partial \Phi}{\partial t}\right.\right)\right\} dX =$$

$$= \int\limits_{Q(X^0,\sigma)} \sum_{i=1}^n \left(\sum_{j=1}^n [\mathcal{A}^0_{ij} - \mathcal{A}_{ij}(X, u, U)]\, D_j U | D_i \Phi\right) dX ,$$

$$\forall \Phi \in L^2\big(t^0 - \sigma^2, t^0, H^1_0(B(x^0, \sigma), \mathbf{R}^{nN})\big) \cap H^1\big(t^0 - \sigma^2, t^0, L^2(B(x^0, \sigma), \mathbf{R}^{nN})\big):$$

$$: \Phi(x, t^0) = 0 \qquad \text{in } B(x^0, \sigma) ,$$

whereas

$$\mathcal{V} \in L^2\big(t^0 - \sigma^2, t^0, H^1(B(x^0, \sigma), \mathbf{R}^{nN})\big) \cap L^\infty\big(t^0 - \sigma^2, t^0, L^2(B(x^0, \sigma), \mathbf{R}^{nN})\big)$$

is solution of the system

$$(4.18)\qquad \int_{Q(X^0,\sigma)} \left\{ \sum_{i,j=1}^{n} (\mathcal{A}^0_{ij} D_j \mathcal{V} | D_i \Phi) - \left(\mathcal{V} \,\middle|\, \frac{\partial \Phi}{\partial t} \right) \right\} dX =$$

$$= \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} (F^i(X, u, U) | D_i \Phi)\, dX\,, \qquad \forall \Phi \in C_0^\infty(Q(X^0, \sigma), \boldsymbol{R}^{nN})\,.$$

By well-known results (see [13]), the problem (4.17) admits an unique solution; this solution belongs to the space $H^{\frac{1}{2}}\big(t^0 - \sigma^2, t^0, L^2(B(x^0, \sigma), \boldsymbol{R}^{nN})\big)$ and the following estimate holds:

$$(4.19)\qquad \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i W\|^2\, dX + \int_{t^0-\sigma^2}^{t^0} dt \int_{t^0-\sigma^2}^{t^0} d\xi \int_{B(x^0,\sigma)} \frac{\|W(x,t) - W(x,\xi)\|^2}{|t-\xi|^2}\, dx \leqslant$$

$$\leqslant c \int_{Q(X^0,\sigma)} \sum_{i,j=1}^{n} \|\mathcal{A}^0_{ij} - \mathcal{A}_{ij}(X, u, U)\|^2 \cdot \sum_{i=1}^{n} \|D_i U\|^2\, dX\,.$$

Moreover from (4.9), (3.22) $\left(\text{with } q \in \left(2, \dfrac{2}{\gamma_0} \wedge \dfrac{2(n+2)}{n+2-2\gamma}\right)\right)$, taking into account the fact that ω is bounded and concave ([7]), it follows

$$(4.20)\qquad \int_{Q(X^0,\sigma)} \sum_{i,j=1}^{n} \|\mathcal{A}^0_{ij} - \mathcal{A}_{ij}(X, u, U)\|^2 \cdot \sum_{i=1}^{n} \|D_i U\|^2\, dX \leqslant$$

$$\leqslant c\Phi(X^0, 2\sigma) \left[\fint_{Q(X^0,\sigma)} \omega(\sigma^2 + \|u - u_{Q(X^0,\sigma)}\|^2 + \|U - U_{Q(X^0,\sigma)}\|^2)\, dX \right]^{1-2/q} \leqslant$$

$$\leqslant c\Phi(X^0, 2\sigma) \left[\omega\left(\sigma^2 + \fint_{Q(X^0,\sigma)} \|u - u_{Q(X^0,\sigma)}\|^2\, dX + \fint_{Q(X^0,\sigma)} \|U - U_{Q(X^0,\sigma)}\|^2\, dX \right) \right]^{1-2/q}.$$

([7]) Therefore $\displaystyle\fint_{Q(X^0,\sigma)} \omega(\varphi)\, dX \leqslant \omega\left(\fint_{Q(X^0,\sigma)} \varphi\, dX \right)$.

Now the following estimate holds for the term $\displaystyle\fint_{Q(X^0,\sigma)} \|u - u_{Q(X^0,\sigma)}\|^2\, dX$:

$$\int_{Q(X^0,\sigma)} \|u - u_{Q(X^0,\sigma)}\|^2\, dX \leqslant \fint_{Q(X^0,\sigma)} dx\, dt \int_{Q(X^0,\sigma)} \|u(x,t) - u(y,\xi)\|^2\, dy\, d\xi \leqslant$$

$$\leqslant c\Bigg\{ \int_{t^0-\sigma^2}^{t^0} dt \fint_{B(x^0,\sigma)} dx \int_{B(x^0,\sigma)} \|u(x,t) - u(y,t)\|^2\, dy +$$

$$+ \sigma^{-2} \int_{t^0-\sigma^2}^{t^0} dt \int_{t^0-\sigma^2}^{t^0} d\xi \int_{B(x^0,\sigma)} \|u(y,t) - u(y,\xi)\|^2\, dy \Bigg\} \leqslant$$

$$\leqslant c\sigma^2 \int_{Q(X^0,\sigma)} \|U\|^2\, dX + c\sigma^4 \int_{Q(X^0,\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^2 dX;$$

on the other hand

$$\int_{Q(X^0,\sigma)} \|U\|^2\, dX \leqslant c\sigma^{(n+2)/2} \Bigg(\int_{Q(X^0,\sigma)} \|U\|^4\, dX \Bigg)^{\frac{1}{2}} \leqslant c\sigma^{n+2} + \int_{Q(X^0,\sigma)} \|U\|^4\, dX\,,$$

hence we obtain

$$\fint_{Q(X^0,\sigma)} \|u - u_{Q(X^0,\sigma)}\|^2\, dX \leqslant \tag{4.21}$$

$$\leqslant c\sigma^2 + c\sigma^{-n} \int_{Q(X^0,\sigma)} \|U\|^4\, dX + c\sigma^{-n+2} \int_{Q(X^0,\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^2 dX\,.$$

From (4.19), (4.20) and (4.21), taking into account the fact that $\sigma \leqslant 1$ and $\xi < n + 2$, we deduce

$$\int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i W\|^2\, dX + \tag{4.22}$$

$$+ \int_{t^0-\sigma^2}^{t^0} dt \int_{t^0-\sigma^2}^{t^0} d\xi \int_{B(x^0,\sigma)} \frac{\|W(x,t) - W(x,\xi)\|^2}{|t - \xi|^2}\, dx \leqslant$$

$$\leqslant c\Phi(X^0, 2\sigma)\Bigg[\omega\Bigg(c\sigma^2 + c\sigma^{-n}\int\limits_{Q(X^0,\sigma)}(\|U\|^4 + \sigma^{-2}\|U - U_{Q(X^0,\sigma)}\|^2)\,dX +$$

$$+ c\sigma^{-n+2}\int\limits_{Q(X^0,\sigma)}\left\|\frac{\partial u}{\partial t}\right\|^2 dX\Bigg)\Bigg]^{1-2/q} \leqslant c\Phi(X^0, 2\sigma)[\omega(c\sigma^{-n}\Phi(X^0, \sigma))]^{1-2/q}.$$

From (4.22) and Lemma 2.1 it follows, $\forall\tau \in (0, 1]$:

$$\int\limits_{Q(X^0,\tau\sigma)}\left(\sum_{i=1}^n \|D_i W\|^2 + (\tau\sigma)^{-2}\|W - W_{Q(X^0,\tau\sigma)}\|^2\right) dX \leqslant \tag{4.23}$$

$$\leqslant c\Phi(X^0, 2\sigma)[\omega(c\sigma^{-n}\Phi(X^0, \sigma))]^{1-2/q}.$$

If we use Lemma 2.2, then we get the following estimate on $\mathcal{V}$: $\forall\tau \in (0, 1)$ and $\varrho \leqslant \sigma$

$$\int\limits_{Q(X^0,\tau\varrho)}\left(\sum_{i=1}^n \|D_i \mathcal{V}\|^2 + (\tau\varrho)^{-2}\|\mathcal{V} - \mathcal{V}_{Q(X^0,\tau\varrho)}\|^2\right) dX \leqslant$$

$$\leqslant c\tau^{n+2}\int\limits_{Q(X^0,\varrho)}\left(\sum_{i=1}^n \|D_i \mathcal{V}\|^2 + \varrho^{-2}\|\mathcal{V} - \mathcal{V}_{Q(X^0,\varrho)}\|^2\right) dX +$$

$$+ c\int\limits_{Q(X^0,\varrho)}\sum_{i=1}^n \|F^i(X, u, U)\|^2\,dX$$

and, denoting

$$\psi(X^0, \varrho) = \varrho^\xi + \int\limits_{Q(X^0,\varrho)}\|U\|^4\,dX\,,$$

from (4.11), it follows

$$\int\limits_{Q(X^0,\varrho)}\sum_{i=1}^n \|F^i(X, u, U)\|^2\,dX \leqslant c\int\limits_{Q(X^0,\varrho)}|f^0(X)|^2\,dX + c\int\limits_{Q(X^0,\varrho)}\|U\|^4\,dX \leqslant \tag{4.24}$$

$$\leqslant c\varrho^{(n+2)(1-2/p)}\Bigg(\int\limits_{Q(X^0,\varrho)}|f^0(X)|^p\,dX\Bigg)^{2/p} + c\int\limits_{Q(X^0,\varrho)}\|U\|^4\,dX \leqslant c\psi(X^0, \varrho)\,.$$

Hence, $\forall 0 < \varrho \leqslant \sigma$ and $\forall \tau \in (0,1)$, we have

$$\int\limits_{Q(X^0,\tau\varrho)} \left(\sum_{i=1}^{n} \|D_i \mathcal{V}\|^2 + (\tau\varrho)^{-2}\|\mathcal{V} - \mathcal{V}_{Q(X^0,\tau\varrho)}\|^2\right) dX \leqslant \tag{4.25}$$

$$\leqslant c\tau^{n+2}\int\limits_{Q(X^0,\varrho)} \left(\sum_{i=1}^{n} \|D_i \mathcal{V}\|^2 + \varrho^{-2}\|\mathcal{V} - \mathcal{V}_{Q(X^0,\varrho)}\|^2\right) dX + c\psi(X^0, \varrho) .$$

Now, $\forall \tau \in (0,1)$ and $0 < \varrho \leqslant \sigma$, the following estimate is true:

$$\psi(X^0, \tau\varrho) \leqslant c\tau^{\xi}\psi(X^0, \varrho) + c\sigma^{2\gamma}\Phi(X^0, \sigma) ; \tag{4.26}$$

in fact, from (4.13) (written for $r = \varrho$), we deduce

$$\int\limits_{Q(X^0,\tau\varrho)} \|U\|^4 \, dX \leqslant c\varrho^{2\gamma}\int\limits_{Q(X^0,\varrho)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX + c\tau^{n+2}\int\limits_{Q(X^0,\varrho)} \|U\|^4 \, dX \leqslant$$

$$\leqslant c\sigma^{2\gamma}\Phi(X^0, \sigma) + c\tau^{\xi}\int\limits_{Q(X^0,\varrho)} \|U\|^4 \, dX ;$$

hence, adding the term $(\tau\varrho)^{\xi}$ to the first and the last side, the inequality (4.26) follows.

From (4.25), (4.26), by virtue of Lemma 2.3, it follows that $\forall \tau \in (0,1)$ and $\forall \lambda \in (n, \xi)$

$$\int\limits_{Q(X^0,\tau\sigma)} \left(\sum_{i=1}^{n} \|D_i \mathcal{V}\|^2 + (\tau\sigma)^{-2}\|\mathcal{V} - \mathcal{V}_{Q(X^0,\tau\sigma)}\|^2\right) dX \leqslant \tag{4.27}$$

$$\leqslant c\tau^{\lambda}\int\limits_{Q(X^0,\sigma)} \left(\sum_{i=1}^{n} \|D_i \mathcal{V}\|^2 + \sigma^{-2}\|\mathcal{V} - \mathcal{V}_{Q(X^0,\sigma)}\|^2\right) dX + c\Phi(X^0, \sigma)\{\tau^{\lambda} + \sigma^{2\gamma}\} .$$

Since $U = \mathcal{V} + W$ in $Q(X^0, \sigma)$, from (4.23) and (4.27) we get in a standard way that $\forall \tau \in (0,1)$ and $\forall \lambda \in (n, \xi)$

$$\int\limits_{Q(X^0,\tau\sigma)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + (\tau\sigma)^{-2}\|U - U_{Q(X^0,\tau\sigma)}\|^2\right) dX \leqslant \tag{4.28}$$

$$\leqslant c\Phi(X^0, 2\sigma)\{\tau^{\lambda} + \sigma^{2\gamma} + [\omega(c\sigma^{-n}\Phi(X^0, 2\sigma))]^{1-2/q}\} ,$$

whereas from (4.26) we have $\forall\tau \in (0, 1)$

$$(4.29)\qquad (\tau\sigma)^{\xi} + \int_{Q(X^0,\tau\sigma)} \|U\|^4\, dX \leqslant c\Phi(X^0, 2\sigma)\{\tau^{\xi} + \sigma^{2\gamma}\}\ .$$

Now let us show that also the term $\int_{Q(X^0,\tau\sigma)} \|\partial u/\partial t\|^2\, dX$ is less or equal to the right side of (4.28).

Since $u \in L^2(-T, 0, H^2(\Omega, \mathbf{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \mathbf{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \mathbf{R}^N)$, $0 < \gamma < 1$, is a solution of the system (4.1), we have $\forall\tau \in (0, 1)$:

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} D_i a^i(X, u, U) + B^0(X, u, U) \qquad \text{a.e. in } Q(X^0, \tau\sigma)\ ,$$

and hence, by virtue of the growths (4.5) and (4.6),

$$(4.30)\qquad \left\|\frac{\partial u}{\partial t}\right\|^2 \leqslant c\left\{|f^0(X)|^2 + \|U\|^4 + \sum_{i=1}^{n}\|D_i U\|^2\right\}$$

and hence

$$(4.31)\qquad \int_{Q(X^0,\tau\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^2 dX \leqslant c\int_{Q(X^0,\tau\sigma)} |f^0(X)|^2\, dX + c\int_{Q(X^0,\tau\sigma)} \|U\|^4\, dX + \\ + c\int_{Q(X^0,\tau\sigma)} \sum_{i=1}^{n} \|D_i U\|^2\, dX \leqslant c\Big(\int_{Q(X^0,\tau\sigma)} |f^0(X)|^p\, dX\Big)^{2/p} (\tau\sigma)^{\xi} + \\ + c\int_{Q(X^0,\tau\sigma)} \|U\|^4\, dX + c\int_{Q(X^0,\tau\sigma)} \sum_{i=1}^{n} \|D_i U\|^2\, dX\ .$$

From (4.13) (written for $r = \sigma$), we get

$$(4.32)\qquad \int_{Q(X^0,\tau\sigma)} \|U\|^4\, dX \leqslant c\sigma^{2\gamma}\int_{Q(X^0,\sigma)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dX + \\ + c\tau^{n+2}\int_{Q(X^0,\sigma)} \|U\|^4\, dX;$$

from (4.28), it follows afterwards

$$(4.33)\qquad \int\limits_{Q(X^0,\tau\sigma)} \sum_{i=1}^{n} \|D_i U\|^2\, dX \leqslant$$
$$\leqslant c\Phi(X^0, 2\sigma)\{\tau^\lambda + \sigma^{2\gamma} + [\omega(c\sigma^{-n}\Phi(X^0, 2\sigma))]^{1-2/q}\}\,.$$

Then, from (4.31), (4.32) and (4.33), we obtain $\forall\tau \in (0, 1)$ and $\forall\lambda \in \in (n, \xi)$:

$$(4.34)\qquad \int\limits_{Q(X^0,\tau\sigma)} \left\|\frac{\partial u}{\partial t}\right\|^2 dX \leqslant c\Phi(X^0, 2\sigma)\{\tau^\lambda + \sigma^{2\gamma} + [\omega(c\sigma^{-n}\Phi(X^0, 2\sigma))]^{1-2/q}\}\,.$$

The estimates (4.28), (4.29), (4.34) hold trivially also for $\tau \in [1, 2)$; then adding such inequalities we deduce $\forall\tau \in (0, 2)$, $\forall\lambda \in (n, \xi)$, $\forall Q(X^0, 2\sigma) \Subset Q$, $\sigma \leqslant 1$

$$\Phi(X^0, \tau\sigma) \leqslant c\Phi(X^0, 2\sigma)\{\tau^\lambda + \sigma^{2\gamma} + [\omega(c\sigma^{-n}\Phi(X^0, 2\sigma))]^{1-2/q}\}$$

and the lemma follows.

Now we can easily achieve the partial Hölder continuity in Q of the derivatives $D_i u$ of the solutions to the system (4.1) making use of the some technique used in Section 3 of [5].

Let

$$Q_0 = \left\{X \in Q:\ \min\lim_{\sigma\to 0} \sigma^{-n}\Phi(X, \sigma) > 0\right\}.$$

It is well-known that

$$\lim_{\sigma\to 0} \sigma^{-n} \int\limits_{Q(X,\sigma)} \left(\sum_{i=1}^{n} \|D_i U\|^2 + \|U\|^4 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dY = 0 \qquad \text{a.e. in } Q$$

and

$$\lim_{\sigma\to 0} \fint\limits_{Q(X,\sigma)} \|U - U_{Q(X,\sigma)}\|^2\, dY = 0 \qquad \text{a.e. in } Q\,,$$

hence

$$\operatorname{meas} Q_0 = 0\,.$$

We can give a more accurate estimate of the measure of Q_0, considering a suitable Hausdorff measure. From (3.10) and (3.13), via Hölder inequality, we get $\forall Q(X, \sigma) \Subset Q$:

$$\sigma^{-n} \int_{Q(X,\sigma)} \| U \|^4 \, dY \leqslant c \left\{ \sigma^{q-n-2} \int_{Q(X,\sigma)} \| U \|^{2q} \, dY \right\}^{2/q}, \tag{4.35}$$

$$\sigma^{-n} \int_{Q(X,\sigma)} \sum_{i=1}^{n} \| D_i U \|^2 \, dY \leqslant c \left\{ \sigma^{q-n-2} \int_{Q(X,\sigma)} \sum_{i=1}^{n} \| D_i U \|^q \, dY \right\}^{2/q} \tag{4.36}$$

for every $q \in \left(2, \frac{2}{\gamma_0} \wedge \frac{2(n+2)}{n+2-2\gamma}\right)$.

Moreover from (4.30), that holds for a.e. $Y \in Q(X, \sigma)$ with $Q(X, \sigma) \Subset Q$, we deduce

$$\sigma^{-n} \int_{Q(X,\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dY \leqslant c\sigma^{-n} \int_{Q(X,\sigma)} |f^0|^2 \, dY + c\sigma^{-n} \int_{Q(X,\sigma)} \| U \|^4 \, dY + c\sigma^{-n} \int_{Q(X,\sigma)} \sum_{i=1}^{n} \| D_i U \|^2 \, dY \leqslant$$

$$\leqslant c\sigma^{\xi - n} + c\sigma^{-n} \int_{Q(X,\sigma)} \| U \|^4 \, dY + c\sigma^{-n} \int_{Q(X,\sigma)} \sum_{i=1}^{n} \| D_i U \|^2 \, dY,$$

after which, taking into account (4.35) and (4.36), it follows

$$\sigma^{-n} \int_{Q(X,\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 dY \leqslant c\sigma^{\xi - n} + \tag{4.37}$$

$$+ c \left\{ \sigma^{q-n-2} \int_{Q(X,\sigma)} \left(\sum_{i=1}^{n} \| D_i U \|^q + \| U \|^{2q} \right) dY \right\}^{2/q}.$$

Using Lemma 2.III of [7] (see also [9], Lemma 2.II) we get

$$\sigma^{-n-2} \int_{Q(X,\sigma)} \| U - U_{Q(X,\sigma)} \|^2 \, dY \leqslant c\sigma^{-n} \int_{Q(X,\sigma)} \left(\sum_{i=1}^{n} \| D_i U \|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right) dY$$

and hence, because of (4.36) and (4.37),

$$\sigma^{-n-2}\int_{Q(X,\sigma)}\|U - U_{Q(X,\sigma)}\|^2\,dY \leqslant c\sigma^{\xi-n} + \\ + c\left\{\sigma^{q-n-2}\int_{Q(X,\sigma)}\left(\sum_{i=1}^{n}\|D_i U\|^q + \|U\|^{2q}\right)dY\right\}^{2/q}. \tag{4.38}$$

From (4.35), (4.36), (4.37) and (4.38) it follows

$$\sigma^{-n}\Phi(X,\sigma) \leqslant c\sigma^{\xi-n} + c\left\{\sigma^{q-n-2}\int_{Q(X,\sigma)}\left(\sum_{i=1}^{n}\|D_i U\|^q + \|U\|^{2q}\right)dY\right\}^{2/q};$$

consequently, by a well-known result (see [11])

$$\mathcal{H}_{n+2-q}(Q_0) = 0$$

where $\mathcal{H}_{n+2-q}$ is the $(n+2-q)$-dimensional Hausdorff measure with respect to the parabolic metric (4.10).

Now, taking into account Lemmas 4.1 and 2.4 and using the some technique of [5] (Lemma 3.II), we can prove that $\forall X^0 \in Q\setminus Q_0$ and $\forall \lambda \in (n,\xi)$ it is possible to find $\sigma_\lambda < 1$ and $r > 0$ with $Q(X^0, r+\sigma_\lambda) \subset\subset Q$ such that $\forall Y \in Q(X^0, r)$ and $\forall \tau \in (0,1)$

$$\Phi(Y, \tau\sigma_\lambda) \leqslant c\tau^\lambda \Phi(Y, \sigma_\lambda)\,. \tag{4.39}$$

In particular, Q_0 is closed in Q.

Then from (4.39) and taking into account the definition of Φ, it follows that if $X^0 \in Q\setminus Q_0$, $Y \in Q(X^0, r)$, $\tau \in (0,1)$

$$\int_{Q(Y,\tau\sigma_\lambda)}\|U - U_{Q(Y,\tau\sigma_\lambda)}\|^2\,dX \leqslant c\tau^{\lambda+2}\Phi(Y,\sigma_\lambda)\,, \qquad \forall \lambda \in (n,\xi)\,.$$

Finally from (4.15) we deduce

$$\int_{Q(Y,\tau\sigma_\lambda)}\|U - U_{Q(Y,\tau\sigma_\lambda)}\|^2\,dX \leqslant \\ \leqslant c\tau^{\lambda+2}\left\{1 + \int_Q \left(\sum_{i=1}^{n}\|D_i U\|^2 + \|U\|^4 + \left\|\frac{\partial u}{\partial t}\right\|^2\right)dX\right\}.$$

The previous inequality ensures that (see [10]):

$$U \in C^{0,\mu}\left(\overline{Q(X^0, r)}, \boldsymbol{R}^{nN}\right), \qquad \forall \mu < 1 - \frac{n+2}{p}.$$

Therefore we have proved the following

THEOREM 4.1. *Let* $u \in L^2(-T, 0, H^2(\Omega, \boldsymbol{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \boldsymbol{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \boldsymbol{R}^N)$, $0 < \gamma < 1$, *be a solution of the system* (4.1). *Let conditions* (4.2)-(4.6) *be fulfilled. Then there exists a set* $Q_0 \subset Q$, *closed in* Q, *such that*

$$D_i u \in C^{0,\mu}(Q \setminus Q_0, \boldsymbol{R}^N), \qquad \forall \mu < 1 - \frac{n+2}{p}, \; i = 1, 2, \ldots, n,$$

and

$$\mathcal{H}_{n+2-q}(Q_0) = 0$$

for every $q \in \left(2, \frac{2}{\gamma_0} \wedge \frac{2(n+2)}{n+2-\gamma}\right)$.

REMARK 4.1. In the case of natural growth, the problem of the local differentiability of the solutions $u \in L^2(-T, 0, H^1(\Omega, \boldsymbol{R}^N)) \cap C^{0,\gamma}(\bar{Q}, \boldsymbol{R}^N)$, $0 < \gamma < 1$, to the system (4.1) is open. In the case of controlled growth, this problem is to consider solved.

REMARK 4.2. The assumptions of Theorem 4.1 on the matrices A_{ij} do not imply those required in Theorem 5.II of [9]; hence our result is not a particular case of the results obtained in [9].

BIBLIOGRAPHY

[1] S. CAMPANATO, *Equazioni paraboliche del secondo ordine e spazi* $\mathcal{L}^{2,\theta}(\Omega, \delta)$, Ann. Mat. Pura Appl., **73** (1966), pp. 55-102.

[2] S. CAMPANATO, *Maggiorazioni interpolatorie negli spazi* $H^{m,p}_{\lambda}(\Omega)$, Ann. Mat. Pura Appl., **75** (1967), pp. 261-276.

[3] S. CAMPANATO, *Partial Hölder continuity of the gradient of solutions of some nonlinear elliptic systems*, Rend. Sem. Mat. Padova, **59** (1978), pp. 147-165.

[4] S. CAMPANATO, *Sistemi ellittici in forma divergenza. Regolarità all'interno*, « Quaderni » Ann. Scuola Norm. Sup. Pisa, 1980.

[5] S. CAMPANATO, *Partial Hölder continuity of solutions of quasilinear parabolic systems of second order with linear growth*, Rend. Sem. Mat. Padova, **64** (1981), pp. 59-75.

[6] S. CAMPANATO, *L^p regularity and partial Hölder continuity for solutions of second order parabolic systems with strictly controlled growth*, Ann. Mat. Pura Appl., **128** (1981), pp. 287-316.

[7] S. CAMPANATO, *On the nonlinear parabolic systems in divergence form. Hölder continuity and partial Hölder continuity of the solutions*, Ann. Mat. Pura Appl., **137** (1984), pp. 83-122.

[8] S. CAMPANATO - P. CANNARSA, *Differentiability and partial Hölder continuity of the solutions of nonlinear elliptic systems of order 2m with quadratic growth*, Ann. Scuola Norm. Sup. Pisa, (4), **8** (1981), pp. 285-309.

[9] P. CANNARSA, *Second order non variational parabolic systems*, Boll. Un. Mat. Ital., (5), **18**-C (1981), pp. 291-315.

[10] G. DA PRATO, *Spazi $\mathfrak{L}^{(p,\theta)}(\Omega, \delta)$ e loro proprietà*, Ann. Mat. Pura Appl., **69** (1965), pp. 383-392.

[11] E. GIUSTI, *Precisazione delle funzioni $H^{1,p}$ e singolarità delle soluzioni deboli di sistemi ellittici non lineari*, Boll. Un. Mat. Ital., **2** (1969), pp. 71-76.

[12] F. JOHN - L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), pp. 415-426.

[13] O. A. LADYZENSKAJA - V. A. SOLONNIKOV - N. N. URAL'CEVA, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, Vol. **23**, Amer. Mathem. Society, Providence, 1968.

[14] J. L. LIONS, *Equations différentielles opérationnelles et problèmes aux limites*, Grundlehren B. 111, Springer-Verlag, Berlin, 1961.

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