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The curve $\tilde{C}_4 = (\lambda^4, \lambda^3\mu, \lambda\mu^3, \mu^4) \subset \mathbb{P}^3_k$, is not set-theoretic complete intersection of two quartic surfaces

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The Curve $\tilde{\mathfrak{C}}_4 = (\lambda^4, \lambda^3\mu, \lambda\mu^3, \mu^4) \subset \mathbf{P}_k^3$, is Nnot Set-Theoretic Complete Intersection of Two Quartic Surfaces.

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RIASSUNTO - In questa nota si dimostra che la quartica di Cremona $\tilde{\mathfrak{C}}_4 = (\lambda^4, \lambda^3\mu, \lambda\mu^3, \mu^4) \subset \mathbf{P}_k^3$, ove k è un campo algebricamente chiuso di caratteristica $p \neq 2, 3$, non è sottoinsieme intersezione completa di due superficie di ordine quattro.

Introduction.

In a previous paper [1] it has been proved that the Cremona quartic curve $\tilde{\mathfrak{C}}_4 = (\lambda^4, \lambda^3\mu, \lambda\mu^3, \mu^4) \subset \mathbf{P}_k^3$, where k is an algebraically closed field of characteristic $p \neq 2, 3$, is not set-theoretic complete intersection of any pair of surfaces of degrees 3 and 4. This result has been generalized in [2], where it is proved that the same holds for any rational non singular quartic curve of $\mathbf{P}_k^3$.

The method used in [2] is not however applicable in order to prove that $\tilde{\mathfrak{C}}_4$ is not set-theoretic complete intersection of two quartic surfaces, because it is based essentially on the fact that a cubic surface $\tilde{\mathcal{F}}$ on which $\tilde{\mathfrak{C}}_4$ were s.t.c.i., should have at least one singular point on $\tilde{\mathfrak{C}}_4$, so that one can use the theory of monoid surfaces.

In order to prove our statement in this paper we have chosen a different approach: first, stating a result on flexes of plane quartics (see § 2) and exploiting some formulas (see § 1) by which one can step down deeply, but gradually, into the analysis of the contact

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of hyperosculating surfaces, we prove that if there exist $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ such that $\tilde{\mathcal{F}} \cdot \tilde{\mathcal{G}} = 4\tilde{\mathcal{C}}_4$, $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ must have a double point on each of the two flexes of $\tilde{\mathcal{C}}_4$; then with the help of the above formulas and the properties of the resultant we can conclude our proof.

§ 1. In what follows the field k will be algebraically closed and of characteristic $p \neq 2, 3$. By surface or curve we shall always mean an algebraic reduced surface or curve. If V is an algebraic variety of $\mathbf{P}^3_k$, we put $V^a = V \cap \mathbf{A}^3_k$, where $\mathbf{A}^3_k$ is the affine space canonically embedded in $\mathbf{P}^3_k$ by the map

$$(x, y, z) \to (1, x, y, z) .$$

With $\hat{V}$ we denote the projective closure in $\mathbf{P}^3_k$ of the affine variety V of $\mathbf{A}^3_k$. $\pi_{\mathcal{F}}(P)$ will denote the tangent plane to a surface $\mathcal{F}$ in its simple point P.

REMARK 1. Let $\mathcal{F}$ and $\mathcal{G}$ surfaces in $\mathbf{A}^3_k$ and $\mathcal{F} \cap \mathcal{G} = \mathcal{C}$, where $\mathcal{C}$ is an irreducible curve and let $P \in \mathcal{C}$ a point which is non singular for $\mathcal{F}$ and $\mathcal{G}$. If $\mathcal{F} \cdot \mathcal{G} = \mu\mathcal{C}$, with $\mu > 1$, $\mathcal{F}$ and $\mathcal{G}$ have the same tangent plane in P.

REMARK 2. Let $\mathcal{F}$ and $\mathcal{G}$ surfaces in $\mathbf{A}^3_k$ and $\mathcal{F} \cap \mathcal{G} = \mathcal{C}$, where $\mathcal{C}$ is an irreducible curve which is non singular for $\mathcal{F}$; then for any P belonging a suitable open Zariski-subset of $\mathcal{C}$, if α is an arbitrary plane through P transversal to $\mathcal{C}$, $\alpha \cap \mathcal{F}$ has a single reduced and irreducible component through P, non singular in P, and we have:

$$i(\mathcal{C}, \mathcal{F} \cap \mathcal{G}) = i(P, (\alpha \cap \mathcal{F}) \cap (\alpha \cap \mathcal{G})) .$$

(the second multiplicity beeing calculated in the D.V.R. $k[\alpha \cap \mathcal{F}]_P$).

SKETCH OF THE PROOF. If $H \in k[X, Y, Z]$ is such that its projection $\bar{H}$ in $k[\mathcal{F}]_P$ is a uniformizing parameter, we have $\bar{G} = \bar{H}^\mu(\bar{C}/\bar{D})$, from which we get

$$\bar{D}\bar{G} = \bar{H}^\mu \bar{C} \tag{\&}$$

with $\mathcal{G} = \{G = 0\}$, and $\{C = 0\}$, $\{D = 0\}$ surfaces not containing $\mathcal{C}$.

Lifting (&) in $k[X, Y, Z]$, we get, with a suitable $A \in k[X, Y, Z]$,

$$DG = H^\mu C + AF \tag{\&'}$$

It is then clear that one can find on a open Zariski-subset of $\mathcal{C}$ for every point P of which we can assume what follows:

1) P is a simple point of $\mathcal{C}$;
2) no other component of $\mathcal{F} \cap \mathcal{H}$ passes through P (where $\mathcal{H} = \{H = 0\}$);
3) $D(P) \neq 0$ and $C(P) \neq 0$.

As a consequence of 1), 2) P is a simple point for $\mathcal{F}$ and $\mathcal{H}$ and $\pi_{\mathcal{F}}(P) \neq \pi_{\mathcal{H}}(P)$, moreover both $\alpha \cap \mathcal{F}$ and $\alpha \cap \mathcal{H}$ have a single irreducible and reduced component through P non singular in P and they are transversal in P to each other. Indicating with q the image of the polynomial Q of $k[X, Y, Z]$ in $k[\alpha]$ (which can be identified with a ring of polynomials in two variables) and with $\bar{q}$ the projection of q into $k[\alpha \cap \mathcal{F}]_P$ one gets from (&′):

$$\bar{d}\bar{g} = \bar{h}^{\mu}\bar{c} + \bar{a}\bar{f}$$

where $= 0$, $\bar{d}$, $\bar{c}$ are units (by 3)) and $\bar{h}$ is a uniformizing parameter in $k[\alpha \cap \mathcal{F}]_P$ by the consequences of 1) and 2) mentioned above. This means that the curve $\alpha \cap \mathcal{G}$ (defined on α by g) has with $\alpha \cap \mathcal{F}$ multiplicity of intersection μ in P.

PROPOSITION 1. *Let $\mathcal{F} = \{F = 0\}$ and $\mathcal{G} = \{G = 0\}$ be surfaces of $\mathbf{A}^3_k$, $\mathcal{C}$ a curve non singular for $\mathcal{F}$ and let be $\mathcal{F} \cdot \mathcal{G} = \mu\mathcal{C}$; for every point $P \in \mathcal{C}$ non singular for $\mathcal{F}$ let be r_P the multiplicity of $\mathcal{G}$ in P and $\Gamma(P)$ the tangent cone in P to $\mathcal{G}$. Then the following holds in a suitable open Zariski-subset U of $\mathcal{C}$:*

$$\mu > r_P \Rightarrow \pi_{\mathcal{F}}(P) \subseteq \Gamma(P)\,.$$

PROOF. Let us take for U the open Zariski-subset of $\mathcal{C}$ in which Remark 2 holds. Let α be a plane through P transversal to $\mathcal{C}$ which is not one of the (possibly 0 and at most r_P) planes components of $\Gamma(P)$.

Then $\mathcal{F}_\alpha = \mathcal{F} \cap \alpha$ has a simple point in P and $\mathcal{G}_\alpha = \mathcal{G} \cap \alpha$ has an r_P-fold point in P. By Remark 2 $i(P, \mathcal{F}_\alpha \cap \mathcal{G}_\alpha) = \mu > r_P$; then the tangent $\pi_{\mathcal{F}}(P) \cap \alpha$ in P to $\mathcal{F}_\alpha$ must be a component of the «tangent cone» in P to $\mathcal{G}_\alpha = \Gamma(P) \cap \alpha$. This happens for the generic plane through P; hence $\pi_{\mathcal{F}}(P)$ contains infinitely many straight lines of $\Gamma(P)$, so $\pi_{\mathcal{F}}(P) \subset \Gamma(P)$.

COROLLARY 1. *Let be* $\mathcal{F} = \{F = 0\}$ *and* $\mathcal{G} = \{G = 0\}$ *surfaces of* $\mathbf{A}_k^3$ *and* $\mathcal{F} \cap \mathcal{G} = \mathcal{C}$ *with* $\mathcal{C}$ *simple curve for* $\mathcal{F}$.

$$F = F_1^{(P)} + F_2^{(P)} + F_3^{(P)} + \dots, \qquad G = G_1^{(P)} + G_2^{(P)} + G_3^{(P)} + \dots$$

the Taylor expansions of F *and* G *at the point* $P = (x, y, z)$ ($F_i^{(P)}$ *and* $G_i^{(P)}$ *are forms in* X-x, Y-y, Z-z). *Then in a suitable open Zariski-subset of* $\mathcal{C}$ *the following implications hold*:

a) $\{\mathcal{F}\cdot\mathcal{G} = \mu\mathcal{C},\ \mu \geqslant 2\} \Rightarrow F_1^{(P)} | G_1^{(P)}$;

b) $\{\mathcal{F}\cdot\mathcal{G} = \mu\mathcal{C}, \mu \geqslant 3\} \Rightarrow F_2^{(P)} | (G_1^{(P)} - \varrho_P F_2^{(P)})$;

c) $\{\mathcal{F}\cdot\mathcal{G} = \mu\mathcal{C},\ \mu \geqslant 4\} \Rightarrow F_1^{(P)} | (G_3^{(P)} - \varrho_P F_3^{(P)} - \Phi_1^{(P)} F_2^{(P)})$

where in b) *and in c*) $\varrho_P \in k$ *is such that* $G_1^{(P)} = \varrho_P F_1^{(P)}$ (*by a*)) *and in c*) $\Phi_1^{(P)}$ *is the linear form such that* $G_2^{(P)} - \varrho_P F_2^{(P)} = \Phi_1^{(P)} F_1^{(P)}$ (*by b*)).

PROOF. If $G_1^{(P)} \neq 0$, $G_1^{(P)}$ defines the tangent plane at P to $\mathcal{G}$ and *a*) is an immediate consequence of Remark 1 (applied in the o.Z.-sub. U of $\mathcal{C}$ where $F_1^{(P)} \neq 0$). As for *b*), for every point P in the o.Z.-sub. $U'(\supset U)$ of $\mathcal{C}$ in which Prop. 1 holds for $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{C}$ let us put $\mathcal{H}_P = \{H_P = G - \varrho_P F = 0\}$ with ϱ_P s.t. $G_1^{(P)} = \varrho_P F_1^{(P)}$ (by *a*)). Since in the D.V.R. $k[\mathcal{F}]_\mathcal{C}$ we have $\bar{G} = \bar{H}_P$ Remark 2 and Prop. 1 hold for $\mathcal{F}$, $\mathcal{H}_P$ and $\mathcal{C}$ in the same o.Z.-sub. U'. The initial form of H_P is $G_2^{(P)} - \varrho_P F_2^{(P)}$ (by *a*)) which, if it is not zero, defines the tangent cone Γ_2 in P to $\mathcal{H}_P$, whence *b*) applying Prop. 1.

In a similar way *c*) is deduced applying Prop. 1 to $\mathcal{F}$ and $\mathcal{H}'_P = \{G - \varrho_P F - \Phi_1^{(P)} F = 0\}$ where ϱ_P is s.t. $G_1^{(P)} = \varrho_P F_1^{(P)}$ (by *a*)) and $\Phi_1^{(P)}$ is s.t. $G_2^{(P)} - \varrho_P F_2^{(P)} = \Phi_1^{(P)} F_1^{(P)}$ (by *b*)).

DEFINITION 1. Given a polynomial $F \in k[X, Y, Z]$ we define:

$$D_{XX}(F) = F_{XX}F_Z^2 - 2F_{XZ}F_XF_Z + F_{ZZ}F_X^2$$

$$D_{XY}(F) = F_{XY}F_Z^2 - F_{XZ}F_YF_Z - F_{YZ}F_XF_Z + F_{ZZ}F_XF_Y$$

$$D_{YY}(F) = F_{YY}F_Z^2 - 2F_{YZ}F_YF_Z + F_{ZZ}F_Y^2$$

$$\begin{aligned} D_{YYY}(F) = {} & F_{YYY}F_Z^4 - 3F_{YYZ}F_YF_Z^3 + 3F_{YZZ}F_Z^2F_Y^2 - \\ & - 9F_{YZ}F_{ZZ}F_Y^2F_Z + 6F_{YZ}^2F_YF_Z^2 + 3F_{YY}F_{ZZ}F_YF_Z^2 - \\ & - 3F_{YY}F_{YZ}F_Z^3 + 3F_{ZZ}^2F_Y^3 - F_{ZZZ}F_Y^3F_Z \end{aligned}$$

In the sequel, if $\mathcal{C} \subset \mathbf{A}_k^3$ is a curve, $I(\mathcal{C})$ will be the ideal of $\mathcal{C}$ in $k[X, Y, Z]$.

PROPOSITION 2. *Let us consider in* $\mathbb{A}^3_k$ *the surfaces* $\mathcal{F} = \{F = 0\}$ *and* $\mathcal{G} = \{G = 0\}$, *and let be* $\mathcal{F}\cdot\mathcal{G} = \mu\mathcal{C}$, *with* $\mathcal{C}$ *irreducible curve. Moreover we assume that* $F_Z \neq 0$ *and* $G_Z \neq 0$ *mod.* $I(\mathcal{C})$. *If* $\mu \geqslant 3$, *we have the following equalitites of rational functions on* $\mathcal{C}$:

$$(1)\qquad D_{XX}(F)/F_Z^3 = D_{XX}(G)/G_Z^3, \qquad D_{XY}(F)/F_Z^3 = D_{XY}(G)/G_Z^3,$$
$$D_{YY}(F)/F_Z^3 = D_{YY}(G)/G_Z^3 .$$

If $\mu \geqslant 4$, *besides the above equalities* (1), *we have also the following*:

$$(2)\qquad D_{YYY}(F)/F_Z^5 = D_{YYY}(G)/G_Z^5 .$$

PROOF. Let us consider a point $P = (x, y, z)$ in the o.Z.-sub. where Cor. 1 holds and furthermore $F_Z \neq 0$ and $G_Z \neq 0$. For the sake of semplicity let us suppose $P = O = (0, 0, 0)$. Let be:

$$F = aX + bY + cZ + dX^2 + eY^2 + fZ^2 + gXY + hXZ + \\ + lYZ + AX^3 + BY^3 + CZ^3 + DX^2Y + EX^2Z + \\ + FXY^2 + GXZ^2 + HXYZ + MY^2Z + NYZ^2 + \dots$$

$$G = G_1 + G_2 + G_3 + \dots$$

with G_i homogeneous polynomials of $k[X, Y, Z]$ of degree i. In the first case, from Cor. 1, *a*) and *b*), we have

$$G_1 = \varrho(aX + bY + cZ) \qquad (\varrho c \neq 0, \text{ by } G_Z \neq 0 \text{ mod. } I(\mathcal{C})),$$

$$G_2 = \varrho(dX^2 + eY^2 + fZ^2 + gXY + hXZ + lYZ) + \\ + (a'X + b'Y + c'Z)(aX + bY + cZ) = (\varrho d + a'a)X^2 + \\ + (\varrho e + b'b)Y^2 + (\varrho f + c'c)Z^2 + (\varrho g + a'b + b'a)XY + \\ + (\varrho h + c'a + a'c)XZ + (\varrho l + b'c + c'b)YZ .$$

The partial derivatives beeing calculated in O, we find:

$$F_X = a, \quad F_Y = b, \quad F_Z = c, \quad F_{XX} = 2d, \quad F_{YY} = 2e,$$
$$F_{ZZ} = 2f, \quad F_{XY} = g, \quad F_{XZ} = h, \quad F_{YZ} = l,$$

and

$$G_X = \varrho a, \quad G_Y = \varrho b, \quad G_Z = \varrho c, \quad G_{XX} = 2(\varrho d + a'a), \dots$$

Then we have the following:

$$D_{XX}(G)/G_Z^3 = 2(\varrho d + a'a)(\varrho c)^2 - 2(\varrho h + c'a + a'c)\cdot$$
$$\cdot(\varrho a)(\varrho c) + 2(\varrho f + c'c)(\varrho a)^2/(\varrho c)^3 =$$
$$= (2dc^2 - 2hac + 2fa^2)/c^3 = D_{XX}(F)/F_Z^3 .$$

Since the above equality holds in an o.Z.-sub. of $\mathcal{C}$ we can assert that the rational functions on $\mathcal{C}$:

$$D_{XX}(G)/G_Z^3 = D_{XX}(F)/F_Z^3$$

are equal.

Same proof for the remaining two formulas (1).

As for (2), it can be deduced in a completly similar way applying, besides *a*), *b*), also *c*) of Cor. 1.

COROLLARY 2. *In the hypotheses of Prop.* 2, *if* $\mu \geqslant 3$, *equalities* (1) *become the following equalities mod* $I(\mathcal{C})$:

$$(1_1) \quad F_{XX}F_ZG_Z - 2F_{XZ}F_XG_Z + F_{ZZ}F_XG_X =$$
$$= G_{XX}F_Z^2 - 2G_{XZ}F_XF_Z + G_{ZZ}F_X^2$$

$$(1_2) \quad F_{XY}F_ZG_Z - F_{XZ}F_YG_Z - F_{YZ}F_XG_Z + F_{ZZ}F_XG_Y =$$
$$= G_{XY}F_Z^2 - G_{XZ}F_YF_Z - G_{YZ}F_XF_Z + G_{ZZ}F_XF_Y$$

$$(1_3) \quad F_{YY}F_ZG_Z - 2F_{YZ}F_YG_Z + F_{ZZ}F_YG_Y =$$
$$= G_{YY}F_Z^2 - 2G_{YZ}F_YF_Z + G_{ZZ}F_Y^2 .$$

In order to deduce (1_1), for example, we have, from the first of (1), of Prop. 1:

$$F_{XX}F_Z^2G_Z^3 - 2F_{XZ}F_XF_ZG_Z^3 + F_{ZZ}F_X^2G_Z^3 =$$
$$= G_{XX}G_Z^2F_Z^3 - 2G_{XZ}G_XG_ZF_Z^3 + G_{ZZ}G_X^2F_Z^3 .$$

Taking into account that $F_XG_Z = F_ZG_X$ mod $I(\mathcal{C})$, by the condition of tangence of $\mathcal{F}$ and $\mathcal{G}$ along $\mathcal{C}$, one gets

$$(F_Z G_Z^2)(F_{XX}F_Z G_Z - 2F_{XZ}F_X G_Z + F_{ZZ}F_X G_X) = \\ = (F_Z G_Z^2)(G_{XX}F_Z^2 - 2G_{XZ}F_X F_Z + G_{ZZ}F_X^2)$$

whence (1) is deduced, simplifying by $F_Z G_Z^2$ ($\neq 0 \mod I(\mathcal{C})$).

REMARK 3. In the hypotheses $F_X \neq 0 \mod I(\mathcal{C})$ one can prove that (1_1), assuming the tangence condition of $\mathcal{F}$ and $\mathcal{G}$ along $\mathcal{C}$, is equivalent to the fact that the generic plane $Y = y$ cuts $\mathcal{F}$ and $\mathcal{G}$ in two curves having in $P = (x, y, z) \in \mathcal{C}$ multiplicity of intersection $\mu \geqslant 3$.

Analogous meaning for (1_2) and (1_3) relatively to the pianes $Z = z$ and $X = x$. If follows that, if $\mathcal{F}\cdot\mathcal{G} = \mu\mathcal{C}$, with $\mu \geqslant 3$, (1_1), (1_2) and (1_3) are generally indipendent.

REMARK 4. In the case $F_X G_X \neq 0 \mod I(\mathcal{C})$, or $F_Y G_Y \neq 0 \mod I(\mathcal{C})$ one can deduce in a completely similar way formulas analogous to (1_i) ($i = 1, 2, 3$) above. For example, if $F_Y G_Y \neq 0 \mod I(\mathcal{C})$, one gets the

$$(1_4) \quad F_{XX}F_Y G_Y - 2F_{XY}F_X G_Y + F_{YY}F_X G_X = \\ = G_{XX}F_Y^2 - 2G_{XY}F_X F_Y + G_{YY}F_X^2 .$$

COROLLARY 3. *In the hypoteses of Prop. 2, if $\mu \geqslant 4$, the equality 2) becomes the following equality mod $I(\mathcal{C})$:*

$$(2) \quad (F_{YYY}F_Z^2 - 3F_{YYZ}F_Y F_Z + 3F_{YZZ}F_Y^2 + 6F_{YZ}^2 F_Y - \\ - 3F_{YY}F_{YZ}F_Z)G_Z - (3F_{YZ}F_{ZZ}F_Y + F_{ZZZ}F_Y^2)G_Y + \\ + (-3F_{YY}F_Y F_Z + 6F_{YZ}F_Y^2)G_{ZZ} + (3F_{YY}F_Z^2 - \\ - 6F_{YZ}F_Y F_Z - 3F_{ZZ}F_Y^2)G_{YZ} + 3F_{ZZ}F_Y F_Z G_{YY} + \\ + F_Y^3 G_{ZZZ} - 3F_Y^2 F_Z G_{YZZ} + 3F_Y F_Y^2 G_{YYZ} - F_Z^3 G_{YYY} = 0 .$$

SKETCH OF THE PROOF. Multiply 2) by $F_Z^5 G_Z^5$ ($\neq 0 \mod I(\mathcal{C})$) and notice that $F_Z G_Y = F_Y G_Z \mod I(\mathcal{C})$ makes it possible to divide both members by $F_Z^2 G_Z^3$. After suitable subtitution, following equation (1_3) in the second member, one can simplify again by dividing by G_Z: then one gets the relation (2).

REMARK 5. Three more formulas can be obtained in the case $\mu \geqslant 4$ in a similar manner. We omit them for the sake of brevity and because only the above formula will be used in the sequel.

§ 2. PROPOSITION 3. *No flex point of a quartic curve* $\tilde{\mathfrak{Q}}$ *of* $\mathbf{P}_k^2$ *can be set-theoretic complete intersection with another quartic curve, possibly reduced, of* $\mathbf{P}_k^2$.

REMARK 6. Let be $\mathfrak{C}$ a curve, F a flex point of $\mathfrak{C}$, t the tangent in F to $\mathfrak{C}$. It is easy to show that for every curve $\mathfrak{D}$ s.t. $i(F, \mathfrak{C} \cap \mathfrak{D}) \geqslant 3$ one has $i(F, t \cap \mathfrak{D}) \geqslant 3$.

PROOF OF PROP. 3. We can choose an affine open set of P_k^2, identified with $\mathbf{A}_k^2$, s.t., if $\mathfrak{Q}$ is the affine part of our quartic $\tilde{\mathfrak{Q}}$, we have that the flex F is $O = (0, 0)$, the tangent to $\mathfrak{Q}$ in the flex O is $\{X = 0\}$ and that $\tilde{\mathfrak{Q}} \cdot \{\widetilde{X=0}\} = 3O + Y_\infty$. The equation of $\mathfrak{Q}$ is then

$$X + c_1X^2 + c_2XY + c_3X^3 + c_4X^2Y + c_5XY^2 - \\ - Y^3 + c_6X^4 + c_7X^3Y + c_8X^2Y^2 + c_9XY^3 = 0 \,.$$

Let us notice that, if there exists a quartic $\mathfrak{Q}' \subset \mathbf{A}_k^2$, such that $i(O, \mathfrak{Q} \cap \mathfrak{Q}') = 16$ then every quartic ($\neq \mathfrak{Q}$) of the (affine) pencil generated by $\mathfrak{Q}$ and $\mathfrak{Q}'$ satisfies the same condition. Let be $\mathfrak{Q}''$ the quartic of the pencil which is singular in O:

$$\mathfrak{Q}'' = d_1X^2 + d_2XY + d_3X^3 + d_4X^2Y + d_5XY^2 + d_6Y^3 + \\ + d_7X^4 + d_8X^3Y + d_9X^2Y^2 + d_{10}XY^3 + d_{11}Y^4 = 0 \,.$$

In order to find the conditions for which $i(O, \mathfrak{Q} \cap \mathfrak{Q}'') \geqslant 16$ we shall proceed as follows. Let us write the equation of $\mathfrak{Q}$ in the form

$$Y^3 = XM, \quad \text{with } M = 1 + c_1X + c_2Y + c_3X^2 + c_4XY + c_5Y^2 + \\ + c_6X^3 + c_7X^2Y + c_8XY^2 + c_9Y^3;$$

we have:

$$\mathfrak{Q} \cap \mathfrak{Q}'' = \begin{cases} Y^3 = XM \\ d_1X^2 + d_2XY + d_3X^3 + d_4X^2Y + d_5XY^2 + d_7X^4 + \\ \quad + d_8X^3Y + d_9X^2Y^2 + d_{10}XY^3 + (d_6 + d_{11}Y)XM = 0 \end{cases}$$

$$\mathfrak{Q} \cap \mathfrak{Q}'' = \begin{cases} Y^3 = XM \\ X = 0 \end{cases} + \begin{cases} Y^3 = XM \\ Q_1 = 0 \end{cases}$$

where

$$Q_1 = d_1 X + d_2 Y + d_3 X^2 + d_4 XY + d_5 Y^2 + d_7 X^3 + d_8 X^2 Y + \\ + d_9 XY^2 + d_{10} Y^3 + (d_6 + d_{11} Y) M = 0 .$$

Having put $\tilde{\mathfrak{Q}}_1 = \{\widetilde{Q_1 = 0}\}$, it must be $i(O, \tilde{\mathfrak{Q}} \cap \tilde{\mathfrak{Q}}_1) \geqslant 13$: by Remark 6 then we have $d_6 = d_2 + d_6 c_2 + d_{11} = d_5 + d_6 c_5 + d_{11} c_2 = 0$.
As $d_6 = 0$, it must be $d_{11} \neq 0$, or $\tilde{\mathfrak{Q}}''$ would have $\{\widetilde{X = 0}\}$ as its component, but then $Y_\infty \in \tilde{\mathfrak{Q}}$: absurd because $\tilde{\mathfrak{Q}} \cap \tilde{\mathfrak{Q}}'' = \{O\}$. We can suppose then $d_{11} = 1$ and the conditions above become:

1) $d_6 = 0$;

2) $d_2 + 1 = 0$;

3) $d_5 + c_2 = 0$.

Iterating this process one finds the following conditions:

4) $d_1 + d_{10} + c_5 = 0$;

5) $d_4 + c_1 + c_9 + (d_{10} + c_5)c_2 = 0$;

6) $d_9 + c_4 + (d_{10} + c_5) c_5 + c_2 c_9 = 0$;

7) $d_3 + c_8 + (d_{10} + c_5)(c_1 + c_9) + c_5 c_9 = 0$;

8) $d_8 + c_3 + (d_{10} + c_5)(c_4 + c_2 c_9) + c_1 c_9 + c_2 c_8 + c_2 c_5 c_9 + c_9^2 = 0$;

9) $c_7 + (d_{10} + c_5)(c_8 + c_5 c_9) + c_4 c_9 + c_5 c_8 + c_5^2 c_9 + c_2 c_9^2 = 0$;

10) $d_7 + (d_{10} + c_5)(c_3 + c_1 c_9 + c_9^2) + c_1 c_8 + 2c_8 c_9 + c_1 c_5 c_9 + 2c_5 c_9^2 = 0$;

11) $(d_{10} + c_5)(c_7 + c_4 c_9 + c_2 c_9^2) + c_6 + c_3 c_9 + c_4 c_8 + c_4 c_5 c_9 +$
$+ c_1 c_9^2 + 2c_2 c_8 c_9 + 2c_2 c_5 c_9^2 + c_9^3 = 0$;

12) $(d_{10} + c_5)(c_8 c_9 + c_5 c_9^2) + c_7 c_9 + c_8^2 + c_4 c_9^2 +$
$+ 3c_5 c_8 c_9 + 2c_5^2 c_9^2 + c_2 c_9^3 = 0$;

13) $(d_{10} + c_5)(c_6 + c_3 c_9 + c_1 c_9^2 + c_9^3) + c_3 c_8 + c_3 c_5 c_9 +$
$+ 2c_1 c_8 c_9 + 2c_1 c_5 c_9^2 + 3c_8 c_9^2 + 3c_5 c_9^3 = 0.$

Let us rewrite 12) in the following form

$$c_9[c_7 + (d_{10} + c_5)(c_8 + c_5c_9) + c_4c_9 + c_5c_8 + \\ + c_5^2c_9 + c_2c_9^2] + (c_8 + c_5c_9)^2 = 0 \ .$$

Taking into account 9), it follows

$$(\&) \qquad c_8 + c_5c_9 = 0 \ .$$

Then from 9) again we get

$$(\&\&) \qquad c_7 + c_4c_9 + c_2c_9^2 = 0$$

and from (&), (&&) and 11), we get finally

$$(\&\&\&) \qquad c_6 + c_3c_9 + c_1c_9^2 + c_9^3 = 0 \ .$$

By (&), (&&), (&&&) we can write the equation of $\mathfrak{Q}$ in the following form:

$$\begin{aligned} &X + c_1X^2 + c_2XY + c_3X^3 + c_4X^2Y + c_5XY^2 - Y^3 - \\ &\quad - (c_3c_9 + c_1c_9^2 + c_9^3)X^4 - (c_4c_9 + c_2c_9^2)X^3Y - \\ &\quad - c_5c_9X^2Y^2 + c_9XY^3 = (1 - c_9X)\cdot \\ &\quad \cdot[c_5XY^2 + (c_2c_9 + c_4)X^2Y + (c_3 + c_1c_9 + c_9^2)X^3 - \\ &\quad - Y^3 + (c_1 + c_9)X^2 + c_2XY + X] = 0 \ . \end{aligned}$$

It follows that $\tilde{\mathfrak{Q}}$ is reducible in a cubic $\tilde{\mathfrak{C}}$ and a straight line $\tilde{r}$ not passing through O. Every projective quartic $\tilde{\mathfrak{Q}}'(\neq \tilde{\mathfrak{Q}})$ must intersect $\tilde{r}$, and also $\tilde{\mathfrak{Q}}$, in a point different from O: absurd. This completes the proof.

Remark 7. In particular cases a flex on a plane quartic $\tilde{\mathfrak{Q}} \subset \mathbf{P}_k^2$ can be s.t.c.i. of $\tilde{\mathfrak{Q}}$ with a cubic $\tilde{\mathfrak{C}}$. For example let $\tilde{\mathfrak{C}}$ be a cubic, P a flex of $\tilde{\mathfrak{C}}$, $\tilde{t}$ the tangent to $\tilde{\mathfrak{C}}$ at P and $\tilde{r}$ an arbitrary straight line not passing through P. Then every quartic $\tilde{\mathfrak{Q}}$ of the form $\lambda\mathfrak{C}\ \tilde{r} + t^4$, $\lambda \in k$, has a flex in P which obviously is s.t.c.i. of $\tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{Q}}$. One can also easily verify that the quartics of this kind are the only quartics having a flex s.t.c.i. with a cubic.

§ 3. Let us consider the affine open set $\mathbf{A}^3_k$ in which the Cremona quartic curve $\tilde{\mathcal{C}}_4$ has the parametric representation (t, t^3, t^4). In $\mathbf{A}^2_k$ the generic quartic surface $\mathcal{F} = \{F = 0\}$ containing $\mathcal{C}_4$ is

$$\mathcal{F} = \{F = A_3 Z^3 + A_2 Z^2 + A_1 Z + A_0 = 0\}$$

where

$$\begin{aligned}
A_0 &= a_1 Y + a_3 XY + a_6 Y^2 - a_4 X^2 Y + a_{10} XY^2 - a_1 X^3 + \\
&\quad + a_{14} Y^3 - (a_7 + a_8) X^2 Y^2 - (a_6 + a_9) X^3 Y - \\
&\quad - (a_{11} + a_{17}) XY^3 - (a_2 + a_3) X^4 - (a_{15} + a_{16}) Y^4 \\
A_1 &= a_2 + a_4 X + a_5 Y + a_8 XY + a_9 X^2 + a_{11} Y^2 - a_{13} XY^2 - \\
&\quad - (a_{12} + a_{14}) X^2 Y - (a_5 + a_{10}) X^3 + a_{18} Y^3 \\
A_2 &= a_7 + a_{12} X + a_{13} Y + a_{16} XY + a_{17} X^2 \\
A_3 &= a_{15} - a_{18} X \,.
\end{aligned}$$

We shall denote the coefficients of the equation of another quartic surface $\mathcal{G} = \{G = 0\}$ containing $\mathcal{C}_4$ with b's in place of a's.

Let us notice that, since $\tilde{\mathcal{C}}_4$ has infinitely many trisecants that constitute an array of the only quadric $\{X_0 X_3 - X_1 X_2 = 0\}$ containing $\tilde{\mathcal{C}}_4$, no irreducible quartic surface containing $\tilde{\mathcal{C}}_4$ can have it as double curve.

Our goal is to prove the following

PROPOSITION 4. *There does not exist any pair of quartic surfaces $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ such that*

$$(*) \qquad \tilde{\mathcal{F}} \cdot \tilde{\mathcal{G}} = 4\tilde{\mathcal{C}}_4$$

The result is reached in three steps, A), B), C):

A) if $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ satisfy (*) and are non singular in the two flexes of $\tilde{\mathcal{C}}_4$, O and Z_∞, they must be there tangent to the hyperoscurating planes of $\tilde{\mathcal{C}}_4$, $\{Z = 0\}$ and the plane at infinity;

B) if $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ satisfy (*) they must have double points in the flexes of $\tilde{\mathcal{C}}_4$, O and Z_∞;

C) if $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ contain $\tilde{\mathcal{C}}_4$ and have double points in O and Z_∞, the molteplicity of intersection of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ along $\tilde{\mathcal{C}}_4$ cannot be 4.

PROOF of A). Given the symmetric roles of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ in (*), it is enough to prove the statement for $\tilde{\mathcal{F}}$. One can easily see that if $a_{18} \neq 0$, $\mathcal{F}_\infty = \tilde{\mathcal{F}} \cap \{X_0 = 0\}$ is a plane quartic with a flex in $Z_\infty = \tilde{\mathcal{C}}_4 \cap \cap \{X_0 = 0\}$. (*) implies that $\tilde{\mathcal{F}} \cap \tilde{\mathcal{G}} = \tilde{\mathcal{C}}_4$, hence $\mathcal{F}_\infty \cap \mathcal{G}_\infty = Z_\infty$: this contradicts Prop. 3, § 2), applied to $\mathcal{F}_\infty$ and its flex Z_∞. It follows $a_{18} = 0$, which means that, if $\tilde{\mathcal{F}}$ is non singular in Z_∞, it is there tangent to the plane at infinity.

Since $\tilde{\mathcal{C}}_4$ is a fixed curve in the linear isomorphism of $\mathbf{P}^3_k$:

$$\tau\colon (X_0, X_1, X_2, X_3) \to (X_3, X_2, X_1, X_0)$$

which interchanges the two flexes of $\tilde{\mathcal{C}}_4$ O and Z_∞, one has:

$$\tilde{\mathcal{F}} \cdot \tilde{\mathcal{G}} = 4\tilde{\mathcal{C}}_4 \Rightarrow \tau(\tilde{\mathcal{F}}) \cdot \tau(\tilde{\mathcal{G}}) = 4\tilde{\mathcal{C}}_4 .$$

Let be $\tau(\tilde{\mathcal{F}})^a = \{F' = 0\}$ and denote with a'_i the coefficients of F': one has $a'_{18} = a_1$; beeing, by the above argument, $a'_{18} = 0$, we have $a_1 = 0$ too; this completes the proof of A).

The condition (*) implies, by Remark 1, that $\mathcal{F}$ and $\mathcal{G}$ have the same tangent plane in their simple points along $\mathcal{C}_4$, so it must be:

$$\operatorname{rank} \begin{pmatrix} F_X & F_Y & F_Z \\ G_X & G_Y & G_Z \end{pmatrix} = 1 \quad \mod I(\mathcal{C}_4) .$$

For every point $P = (t, t^3, t^4) \in \mathcal{C}_4$ we have the equalities in the polynomial ring $k[t]$:

$$F_X(P) + 3t^2 F_Y(P) + 4t^3 F_Z(P) = 0 ,$$

$$G_X(P) + 3t^2 G_Y(P) + 4t^3 G_Z(P) = 0 .$$

By this the condition of tangence of $\mathcal{F}$ and $\mathcal{G}$ along $\mathcal{C}_4$ is, more simply:

$$(o) \qquad F_Z(P) G_Y(P) - F_Y(P) G_Z(P) = 0 , \qquad (P \in \mathcal{C}_4) .$$

REMARK 8. We notice that (*) implies the existence of a pencil Φ of quartic surfaces any pair of which satifies (*) again. In this pencil surely we find a quartic singular in Z_∞ and so we can suppose that in (*) $\tilde{\mathcal{G}}$ is such a surface, that is $b_{15} = 0$.

(o) gives the following system of bilinear equations in the a_i's and b_i's (beeing actually $a_{18} = a_1 = b_{18} = b_{15} = b_1 = 0$):

List 1

a) $-b_{16}a_{15} = 0$

b) $-b_{16}a_{13} + b_{13}(a_{15} + a_{16}) = 0$

c) $-b_{17}a_{15} - b_{16}a_{11} + b_{11}(a_{15} + a_{16}) = 0$

d) $-b_{17}a_{13} - b_{16}(a_{12} + a_{14}) + b_{14}(2a_{15} + a_{16}) + b_{13}a_{17} + b_{12}(a_{15} + a_{16}) = 0$

e) $-b_{17}a_{11} - b_{16}(2a_7 + a_8) + b_{14}a_{13} - b_{13}a_{14} + b_{11}a_{17} + b_8(a_{15} + a_{16}) + 2b_7(a_{15} + a_{16}) = 0$

f) $-b_{17}(a_{12} + a_{14}) - b_{16}(2a_5 + a_{10}) + b_{14}(a_{11} + a_{17}) + b_{12}a_{17} - b_{11}a_{14} + b_{10}(2a_{15} + a_{16}) + b_5(3a_{15} + 2a_{16}) = 0$

g) $-b_{17}(2a_7 + a_8) - b_{16}(2a_6 + a_9) + b_{14}a_{12} - b_{13}(a_5 + a_{10}) - b_{12}a_{14} + b_{10}a_{13} + b_9(a_{15} + a_{16}) + b_8a_{17} + 2b_7a_{17} +$
$+ b_6(3a_{15} + 2a_{16}) + b_5a_{13} = 0$

h) $-b_{17}(2a_5 + a_{10}) - b_{16}a_4 + b_{14}(2a_7 + a_8) - b_{13}a_6 - b_{11}(a_5 + a_{10}) + b_{10}(a_{11} + a_{17}) - b_8a_{14} - 2b_7a_{14} + b_6a_{13} +$
$+ b_5(a_{11} + 2a_{17}) + b_4(a_{15} + a_{16}) = 0$

i) $-b_{17}(2a_6 + a_9) - b_{16}(3a_2 + 2a_3) + b_{14}a_5 - b_{12}(a_5 + a_{10}) - b_{11}a_6 + b_{10}a_{12} + b_9a_{17} + b_6(a_{11} + 2a_{17}) + b_5(a_{12} - a_{14})$
$+ b_3(3a_{15} + 2a_{16}) + b_2(4a_{15} + 3a_{16}) = 0$

l) $-b_{17}a_4 + b_{14}(a_6 + a_9) - b_{13}(a_2 + a_3) - b_{12}a_6 + b_{10}(2a_7 + a_8) - b_9a_{14} - b_8(a_5 + a_{10}) - 2b_7(a_5 + a_{10}) +$
$+ b_6(a_{12} - a_{14}) + b_5(2a_7 + a_8) + b_4a_{17} + b_3a_{13} + b_2a_{13} = 0$

m) $-b_{17}(3a_2 + 2a_3) + b_{14}a_4 - b_{11}(a_2 + a_3) + b_{10}a_5 - b_8a_6 - 2b_7a_6 + b_6(2a_7 + a_8) - b_5a_{10} - b_4a_{14} + b_3(a_{11} + 2a_{17}) +$
$+ b_2(a_{11} + 3a_{17}) = 0$

n) $b_{14}(2a_2 + a_3) - b_{12}(a_2 + a_3) + b_{10}(a_6 + a_9) - b_9(a_5 + a_{10}) - b_6a_{10} + b_5a_9 + b_3(a_{12} - a_{14}) + b_2(a_{12} - 2a_{14}) = 0$

o) $b_{10}a_4 - b_9a_6 - b_8(a_2 + a_3) - 2b_7(a_2 + a_3) + b_6a_9 + b_5a_4 - b_4(a_5 + a_{10}) + b_3(2a_7 + a_8) + b_2(2a_7 + a_8) = 0$

p) $b_{10}(2a_2 + a_3) + b_6a_4 + b_5a_2 - b_4a_6 - b_3a_{10} - b_2(a_5 + 2a_{10}) = 0$

q) $-b_9(a_2 + a_3) + b_6a_2 + b_3a_9 - b_2(a_6 - a_9) = 0$

r) $-b_4(a_2 + a_3) + b_3a_4 + b_2a_4 = 0$

s) $b_3a_2 - b_2a_3 = 0$

PROOF of B). We want to prove first that (*) implies that also $\tilde{\mathfrak{F}}$ is singular in Z_∞, *i.e.* that $a_{15} = 0$. Let us suppose then that $a_{15} \neq 0$ and remember that the characteristic of k is $\neq 2, 3$. From a) of list 1) we have $\boxed{b_{16} = 0}$, which means $Y_\infty \in \tilde{\mathfrak{G}}$. As $Y_\infty \notin \tilde{\mathcal{C}}_4$ it follows $Y_\infty \notin \tilde{\mathfrak{F}}$, that is $a_{15} + a_{16} \neq 0$. From b) of list 1) it follows $\boxed{b_{13} = 0}$. Now let us notice that the common tangent plane to $\mathcal{F}$ and $\mathcal{G}$ in the generic point $P = (t, t^3, t^4)$ of $\mathcal{C}_4$ cannot pass through O. Indeed this situation would imply the following identity in $k[t]$:

$$F_X(P) + t^2 F_Y(P) + t^3 F_Z(P) = 0;$$

on the other hand we already have (see above):

$$F_X(P) + 3t^2 F_Y(P) + 4t^3 F_Z(P) = 0 \ ,$$

by subtracting the previous equations, and simplifying by t^2, we get:

$$2F_Y(P) + 3tF_Z(P) = 0$$

the vanishing of the coefficient of degree 9 in t yields $a_{15} = 0$: contradiction.

This situation has the following consequence. Let τ be the linear isomorphism considered above (see proof of A)) and let $\tilde{\mathfrak{F}}' = \tau(\tilde{\mathfrak{F}})$, $\tilde{\mathfrak{G}}' = \tau(\tilde{\mathfrak{G}})$; τ transforms the tangent plane $\pi(P)$ to $\tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{G}}$ in P in the tangent plane $\pi(\tau(P))$ to $\tilde{\mathfrak{F}}'$ and $\tilde{\mathfrak{G}}'$ in $\tau(P)$; as $\tau(O) = Z_\infty$ and $\tau(Z_\infty) = O$, the tangent plane $\pi(\tau(P))$ cannot pass (for generic P) through Z_∞, because this would imply for $\pi(P)$ to pass through O for generic P, so we can say that, if $\mathcal{F}' = \{F' = 0\}$, $F'_Z \neq 0 \bmod I(\mathcal{C}_4)$; but then also $G'_Z \neq 0 \bmod I(\mathcal{C}_4)$, because, for generic $P \in \mathcal{C}_4$, $\mathcal{F}'$ and $\mathcal{G}'$ have common tangent plane in $\pi(P)$. So we can apply to $\mathcal{F}'$ and $\mathcal{G}'$ formula (1_3), § 1. As $\tilde{\mathfrak{G}}$ is singular in Z_∞ and $\tilde{\mathfrak{F}}$ is tangent to the plane $\{X_0 = 0\}$ in Z_∞, we have that $\mathcal{G}'$ is singular in O and $\mathcal{F}'$ is tangent to $\tau(\{X_0 = 0\})^a = \{Z = 0\}$ in O, so we have $G'_X(O) = G'_Y(O) = G'_Z(O) = F'_X(O) = F'_Y(O) = 0$. Moreover the coefficient of Y^2 in G' is b_{17} and from (1_3) we get immediately $2a_{15}^2 b_{17} = 0$ from which we have $\boxed{b_{17} = 0}$. Now c) of list 1) gives $\boxed{b_{11} = 0}$. From d) we get then

$$(B_1) \qquad (a_{15} + a_{16}) b_{12} + (2a_{15} + a_{16}) b_{14} = 0 \ .$$

Calculating both members of formulas (1_i), $i = 1, 2, 3, 4$ (see Cor. 2) in the generic point (t, t^3, t^4) of $\mathcal{C}_4$ yields identities of polynomials in $k[t]$. For the sake of brevity we shall denote by $H_{1.i}(n)$ the difference of the coefficients of t^n in the two members of formulas (1_i). Similarly $H_2(n)$ will mean the coefficient of t^n in the first member of formula (2) (see Cor. 3). Of course for every i and n we have $H_{1.i}(n) = 0$ and $H_2(n) = 0$.

At this point we distinguish two subcases according as $F_Z G_Z \neq 0$ or $F_Z G_Z = 0 \mod I(\mathcal{C}_4)$.

First case: We can use formulas (1_i) $i = 1, 2, 3$, and (2), Cor. 2, § 1. We find

$$(B_2) \qquad \frac{1}{18} H_{1,1}(23) = a_{15}(a_{15} + a_{16})b_{12} + a_{15}^2 b_{14} = 0 \,.$$

Subtracting B_1), multiplied by a_{15} ($\neq 0$), from B_2), we get

$$a_{15}(a_{15} + a_{16})b_{14} = 0$$

$$\Rightarrow \boxed{b_{14} = 0} \Rightarrow \boxed{b_{12} = 0}\,.$$

From e) of list 1) we get

$$(B_3) \qquad 2b_7 + b_8 = 0 \,.$$

Now we have:

$$(B_4) \qquad \frac{1}{6} H_{1,1}(22) = (3a_{15}^2 + 2a_{15}a_{16} - a_{16}^2)b_7 + (3a_{15}^2 + 4a_{15}a_{16} + a_{16})b_8 = 0.$$

Comparing (B_3) and (B_4) gives

$$-3(a_{15} + a_{16})^2 b_7 = 0 \Rightarrow \boxed{b_7 = 0} \Rightarrow \boxed{b_8 = 0}\,.$$

Actually $\tilde{\mathcal{G}}$ is a monoid with a triple point in Z_∞. Using as above the linear isomorphism τ, we apply formula (2) to $\mathcal{F}' = \tau(\tilde{\mathcal{F}})^a = \{F' = 0\}$ and $\mathcal{G}' = \tau(\tilde{\mathcal{G}})^a = \{G' = 0\}$ calculated in $O = (0, 0, 0)$. As before $G'_Y(O) = G'_Z(O) = F'_Y(O) = 0$; moreover $G'_{YZ}(O) = b_{12} = 0$ (as we have just found). Formula (2), § 1, calculated in O, gives then

$$-F_Z'^3(O)G'_{YYY}(O) = -a_{15}^3[-6(b_5 + b_{10})] = 0 \,,$$

so we have

$$(B_5) \qquad b_5 + b_{10} = 0 .$$

From f) of list 1) we obtain:

$$(B_6) \qquad (3a_{15} + 2a_{16})b_5 + (2a_{15} + a_{16})b_{10} = 0 .$$

(B_5) and (B_6) give $(a_{15} + a_{16})b_5 = 0 \Rightarrow \boxed{b_5 = 0} \Rightarrow \boxed{b_{10} = 0}$.

From g) of list 1) we get:

$$(B_7) \qquad (a_{15} + a_{16})b_9 + (3a_{15} + 2a_{16})b_6 = 0 ;$$

on the other hand we have also:

$$(B_8) \qquad \frac{1}{18} H_{1,1}(20) = (2a_{15}^2 + 3a_{15}a_{16} + a_{16}^2)b_9 + \\ + (3a_{15}^2 + 3a_{15}a_{16} + a_{16}^2)b_6 = 0 .$$

and

$$(B_9) \qquad -\frac{1}{6} H_2(21) = (20a_{15}^3 + 32a_{15}^2 a_{16} + 13a_{15}a_{16}^2 + a_{16}^3)b_9 + \\ + (88a_{15}^3 + 126a_{15}^2 a_{16} + 53a_{15}a_{16}^2 + 6a_{16}^3)b_6 = 0 .$$

It is easy to see that (B_7), (B_8), (B_9), thought of as a linear system in (b_6, b_9), has only the trivial solution. So it must be $\boxed{b_6 = b_9 = 0}$. From h) of list 1) we get $\boxed{b_4 = 0}$; whereas from i) of the same list it follows:

$$(B_{10}) \qquad (4a_{15} + 3a_{16})b_2 + (3a_{15} + 2a_{16})b_3 = 0 .$$

Now we have also:

$$(B_{11}) \qquad \frac{1}{6} H_{1,1}(18) = (18a_{15}^2 + 20a_{15}a_{16} + 6a_{16}^2)b_2 + \\ + (18a_{15}^2 + 21a_{15}a_{16} + 7a_{16}^2)b_3 = 0$$

and

$$(B_{12}) \qquad \frac{1}{6} H_2(19) = (-36a_{15}^3 - 32a_{15}^2 a_{16} + 4a_{15}a_{16}^2 + 5a_{16}^3)\, b_2 + \\ + (-16a_{15}^3 + 17a_{15}a_{16}^2 + 6a_{16}^3)\, b_3 = 0\,.$$

As before, from (B_{10}), (B_{11}), (B_{12}), we conclude that $\boxed{b_2 = b_3 = 0}$. Hence G is the null polynomial: absurd.

Second case: Now we must have $F_Z = G_Z = 0 \mod I(\mathrm{C}_4)$ or, if it were *e.g.* $G_Z \neq 0 \mod I(\mathrm{C}_4)$, it would be $F_Y = 0 \mod I(\mathrm{C}_4)$ and from the identity

$$F_X(P) + t^2 F_Y(P) + t^3 F_Z(P) = 0\,, \qquad \forall P = (t, t^3, t^4) \in \mathrm{C}_4$$

it follows $F_X = 0 \mod I(\mathrm{C}_4)$ too, so $\mathcal{F}$ would be singular along C_4. From the same identity and from the analogous one relative to G, $F_Z = G_Z = 0$ implies that $F_Y G_Y \neq 0 \mod I(\mathrm{C}_4)$ or $\mathcal{F}$ or $\mathcal{G}$ would be singular along C_4. Taking into account $F_Z = G_Z = 0 \mod I(\mathrm{C}_4)$, we have:

$$b_{10} = b_9 = b_4 = b_2 = 0\,, \qquad b_{12} = b_{14}\,, \qquad b_8 = -2b_7\,,$$
$$2a_{16} = -3a_{15}\,, \qquad a_{13} = a_{10} = a_9 = a_4 = a_2 = 0\,,$$
$$a_{12} = a_{14}\,, \qquad a_8 = -2a_7\,, \qquad a_{11} = -2a_{17}\,.$$

Considering formula (1_4), now applicable, we have by the above equalities:

$$H_{1,4}(25) = 16a_{15}^2 b_{12} \Rightarrow \boxed{b_{12} = 0} \Rightarrow \boxed{b_{14} = 0}$$
$$H_{1,4}(24) = -8a_{15}^2 b_7 \Rightarrow \boxed{b_7 = 0} \Rightarrow \boxed{b_8 = 0}.$$
$$H_{1,4}(23) = 24a_{15}^2 b_5 \Rightarrow \boxed{b_5 = 0}\,;$$

with these results, we have furthermore

$$H_{1,4}(22) = 24a_{15}^2 b_6 \Rightarrow \boxed{b_6 = 0}$$
$$H_{1,4}(20) = 24a_{15} b_3 \Rightarrow \boxed{b_3 = 0}$$

again G would be the null polynomial: absurd.

So we can say that (*) implies $a_{15} = 0$, that is for $\tilde{\mathcal{F}}$ to have a singular point in Z_∞. Given this, the conclusion of the proof of (B), that is the fact that $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ have double points in O and Z_∞, is obtained from the following

REMARK 9. R_1: From what has been proved up to this point we can say that:

$$(*) \qquad \tilde{\mathcal{F}} \cdot \tilde{\mathcal{G}} = 4\tilde{\mathrm{C}}_4$$

implies that every quartic of the pencil Φ generated by $\tilde{\mathcal{F}}$ abd $\tilde{\mathcal{G}}$ has a singular point in Z_∞.

R_2: By R_1 we can suppose $\tilde{\mathcal{G}}$ to be the uniquely determined quartic of Φ which passes through $Y_\infty (\notin \tilde{\mathrm{C}}_4)$.

R_3: $\mathcal{G}$ cannot be a cone with vertex Z_∞, or else the cone $\{\widetilde{Y - X^3 = 0}\}$ would be its component.

R_4: Every quartic surface which passes through $\tilde{\mathrm{C}}_4$ and with a triple point in Z_∞, contains the line $Y_\infty Z_\infty$. Hence from (*) it follows that at most one of the surfaces $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ can have a triple point in Z_∞ and, should this happen, it must be $\tilde{\mathcal{G}}$, which already contains Y_∞ (see R_2).

R_5: $Y_\infty \notin \tilde{\mathcal{F}} \Rightarrow a_{16} \neq 0 \Rightarrow F_Z \neq 0 \mod \mathrm{I}(\mathrm{C}_4)$.

R_6: From R_5 it follows that the canonical projection of $Y - X^3$ in the local ring $k[\mathcal{F}]_{I(\mathrm{C}_4)}$ is a uniformizing parameter.

R_7: Let be $\mathrm{Res}_Z(F, G) \in k[X, Y]$ the resultant, relative to Z, of the polynomials F, G: one sees directly that $\deg(\mathrm{Res}_Z(F, G)) \leqslant 12$ if $\tilde{\mathcal{F}}$ and $\mathcal{G}$ both have a singular point on Z_∞, whereas it is of degree < 12 if $\tilde{\mathcal{G}}$ has a triple point in Z_∞(see R_2).

R_8: $\tilde{\mathcal{F}} \cdot \tilde{\mathcal{G}} = 4\tilde{\mathrm{C}}_4 \Rightarrow (Y - X^3)^4 | \mathrm{Res}_Z(F, G)$.

R_9: From R_7 and R_8 $\tilde{\mathcal{G}}$ cannot have a triple point in Z_∞.

R_{10}: From R_i, $i = 1, \ldots, 7$, we have that (*) implies for both $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ to have a double point in Z_∞.

In order to conclude that (*) implies for both $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ to have a double point also in O we use again in an obvious way the linear isomorphism τ that has been considered in the previous proof of A) and also here above during the proof of B).

Remark 10. In the sequel we can suppose $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ to be as in Remark 9, R_2. After what has been proved up to this moment [A) and B)] we have the following situation for the coefficients of F and G:

$$a_{18} = a_{15} = a_2 = a_1 = 0 \neq a_{16}$$

($a_{16} \neq 0$ because $Y_\infty \notin \tilde{\mathcal{F}}$), and

$$b_{18} = b_{16} = b_{15} = b_{13} = b_{11} = b_2 = b_1 = 0 \,.$$

($b_{13} = b_{11} = 0$ beeing consequence of b) and c) of list 1) and $b_{16} = 0$ because $Y_\infty \in \tilde{\mathcal{G}}$).

Remark 11. The tangent cones to $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ in Z_∞ cannot have a common component $\tilde{\mathcal{H}}$: indeed $a_{16} \neq 0 \Rightarrow \mathcal{H} = \tilde{\mathcal{H}} \cap \mathbf{A}^3_k \neq \emptyset$; if $\mathcal{H} = \{H = 0\}$ we have $\deg H < 3$ and $H|\mathrm{Res}_Z(F, G)$, hence it follows $H(Y - X^3)^4|\mathrm{Res}_Z(F, G)$ which is absurd, beeing $\deg(\mathrm{Res}_Z(F, G)) \leqslant 12$ (see above Remark 9, R_7).

Proof of C). We distinguish two subcases according as it is $b_{17} \neq 0$ or $b_{17} = 0$.

First case: $b_{17} \neq 0$.

$\mathcal{F}$ can be taken as the quartic of the pencil Φ^a (see Remark 9, R_1 and R_2) for which $a_{17} = 0$. Let us take $b_{17} = a_{16} = 1$. In this situation $\mathcal{G}_\infty$ splits into the line $t = \{X = 0\}_\infty$ and in a cubic $\mathcal{C}_3$ with a flex in Z_∞ and with tangent in it the line t. Moreover in the plane at infinity we have $t \cdot \mathcal{F}_\infty = 4Z_\infty$. From (*) and $\tilde{\mathcal{C}}_4 \cap \{X_0 = 0\} = Z_\infty$ it follows:

$$\begin{aligned} 16Z_\infty = \mathcal{F}_\infty \cdot \mathcal{G}_\infty = \mathcal{F}_\infty \cdot (\mathcal{C}_3 + t) = \mathcal{F}_\infty \cdot \mathcal{C}_3 + \mathcal{F}_\infty \cdot t = \\ = \mathcal{F}_\infty \cdot \mathcal{C}_3 + 4Z_\infty \Rightarrow \mathcal{F}_\infty \cdot \mathcal{C}_3 = 12Z_\infty \,. \end{aligned}$$

Let us consider the pencil φ of plane quartics $\mathcal{F}_\infty + \lambda t^4$: since also $t^4 \cdot \mathcal{C}_3 = 12Z_\infty$, every quartic $\mathcal{Q}$ of φ satisfies to

$$(') \qquad \mathcal{Q} \cdot \mathcal{C}_3 = 12Z_\infty \,.$$

Let be $P_0 \in \mathcal{C}_3$, $P_0 \neq Z_\infty$, and let be $\mathcal{Q}_0 = \mathcal{F}_\infty - \lambda t^4$ the quartic of φ passing through P_0. By this and ('), $\mathcal{C}_3$ must be component of $\mathcal{Q}_0$, whence if follows:

$$(″) \qquad \mathcal{F}_\infty = r\mathcal{C}_3 + \lambda_0 t^4 ,$$

where r is a line through Z_∞ because $\mathcal{F}_\infty$ is singular in Z_∞. In the affine open subset $(Z \neq 0)$ of the plane at infinity (where X, Y, Z are projective coordinates) we have:

$$\mathcal{F}_\infty^{(a)} = XY - (a_5 + a_{10})X^3 - (a_{12} + a_{14})X^2 Y - a_{13}XY^2 - Y^4 -$$
$$- a_3 X^4 - a_{11}XY^3 - (a_6 + a_9)X^3 Y - (a_7 + a_8)X^2 Y^2 = 0 ,$$

$$\mathcal{C}_3^{(a)} = X - (b_5 + b_{10})X^2 - (b_{12} + b_{14})XY - b_3 X^3 - Y^3 -$$
$$- (b_6 + b_9)X^2 Y - (b_7 + b_8)XY^2 = 0 .$$

In $(″)$ then we have $r = \{Y = 0\}$ and $\lambda_0 = - a_3$, whence the following equalities:

(C_1) $a_5 + a_{10} = 0$, $\quad (C_2)$ $b_5 + b_{10} = a_{12} + a_{14}$, $\quad (C_3)$ $b_{12} + b_{14} = a_{13}$,

(C_4) $b_3 = a_6 + a_9$, $\quad (C_5)$ $b_6 + b_9 = a_7 + a_8$, $\quad (C_6)$ $b_7 + b_8 = a_{11}$.

Taking into account the (C_i) here above and $e)$ $f)$, $g)$, $h)$ of list 1) we find first:

$$(\&) \qquad \begin{cases} b_{12} = a_{13} - b_{14} , \quad b_{10} = a_{12} + a_{14} + a_{11}b_{14} , \\ b_9 = a_8 + a_{12}a_{13} + (a_{12} + a_{14})b_{14} , \quad b_8 = a_{11} + a_{13}b_{14} , \\ b_7 = - a_{13}b_{14} , \quad b_6 = a_7 - a_{12}a_{13} - (a_{12} + a_{14})b_{14} , \\ b_5 = - a_{11}b_{14} , \\ b_4 = a_5 - a_7 a_{13} - a_{11}a_{12} + a_{12}a_{13} - (2a_7 + a_8 - a_{12}a_{13})b_{14} . \end{cases}$$

Now $a_{16} \neq 0 \Rightarrow F_Z G_Z \neq 0 \mod I(\mathcal{C}_4)$, so we can use formulas (1_3) and (2) with the notations $H_{1,3}(n)$ and $H_2(n)$ introduced in the proof of B). Taking into account equalities $(\&)$ listed above, we find:

$$H_{1,3}(15) = -2(b_{14} + a_{13})(a_7 - a_{12}a_{13}) = 0 .$$

It can be seen that $a_7 - a_{12}a_{13} = 0$ would imply for the tangent cones to $\mathcal{F}$ and $\tilde{\mathcal{G}}$ in Z_∞ to have the common component $\widetilde{\{X + a_{13} = 0\}}$

against what has been stated in Remark 11. So it must be $b_{14} = -a_{13}$ and the (&) become:

$$(\&\&) \quad \begin{cases} b_{14} = -a_{13}\,, \quad b_{12} = 2a_{13}\,, \quad b_{10} = a_{12} + a_{14} - a_{11}a_{13}\,, \\ b_9 = a_8 - a_{13}a_{14}\,, \quad b_8 = a_{11} - a_{13}^2\,, \quad b_7 = a_{13}^2\,, \\ b_6 = a_7 + a_{13}a_{14}, \quad b_5 = a_{11}a_{13}, \quad b_4 = a_5 + a_7a_{13} + a_8a_{13} - a_{11}a_{12}\,. \end{cases}$$

From i) of list 1) we get:

$$(C_7) \qquad 2b_3 = a_5a_{13} - a_7a_{11} - a_{12}^2 - a_{12}a_{14} + 2a_6 + a_9$$

and by comparing (C_7) with (C_4) we get:

$$(C_8) \qquad a_9 = a_5a_{13} - a_7a_{11} - a_{12}(a_{12} + a_{14})\,.$$

Given this, we find: $H_{1,3}(14) = 2(a_5a_{13} - a_7a_{11})$, whence

$$(C_9) \qquad a_5a_{13} - a_7a_{11} = 0\,.$$

Hence (C_8) becomes:

$$(C_{10}) \qquad a_9 = -a_{12}(a_{12} + a_{14})\,.$$

Now from l) and m) of list 1), in view of (C_9) and (C_{10}), we obtain

$$(C_{11}) \qquad a_4 = -2a_6a_{13} + 3a_7a_{12} + a_7a_{14} + a_8a_{12} + a_{12}a_{13}a_{14}\,,$$

$$(C_{12}) \qquad 2a_3 = -3a_7a_{12}a_{13} - a_8a_{12}a_{13} - a_{12}a_{13}^2a_{14} + \\ + a_6a_{13}^2 + a_5a_{12} + 2a_7^2 + a_7a_8 - a_{11}a_{12}^2\,.$$

Given this, we find

$$H_{1,3}(13) = 2(a_6a_{13} + a_{12}^2a_{13} - a_7a_{14} - 2a_7a_{12})$$

and

$$H_{1,3}(12) = -2(a_3 - a_{12}a_{13}^2a_{14} + a_7a_{13}a_{14} - a_{11}a_{12}^2 - \\ - a_8a_{12}a_{13} - 3a_7a_{12}a_{13} + a_7a_8 + 3a_7^2 + a_5a_{12})\,.$$

From these two last equations it follows

$$(C_{13}) \qquad a_6 a_{13} = -a_{12}^2 a_{13} + a a_{14} + 2a_7 a_{12},$$

$$(C_{14}) \qquad a_3 = a_{12} a_{13}^2 a_{14} - a_7 a_{13} a_{14} + a_{11} a_{12}^2 + a_8 a_{12} a_{13} + \\ + 3a_7 a_{12} a_{13} - a_7 a_8 - 3a_7^2 - a_5 a_{12}.$$

Summing both members of (C_{12}) and (C_{14}) and taking into account (C_{13}), we obtain:

$$(C_{15}) \qquad 3a_3 = -(a_7 - a_{12} a_{13})^2.$$

Now we turn to formula (2) of Cor. 3: we find

$$H_2(17) = 12(-5a_3 + a_{12}^2 a_{13}^2 - 2a_{12} a_{13}^2 a_{14} - 2a_{11} a_{12}^2 - \\ - 2a_8 a_{12} a_{13} - 2a_7 a_{13} a_{14} - 8a_7 a_{12} a_{13} + \\ + 4a_6 a_{13}^2 + 3a_7^2 + 2a_7 a_8 + 2a_5 a_{12})$$

whence

$$(C_{16}) \qquad 5a_3 = a_{12}^2 a_{13}^2 - 2a_{12} a_{13}^2 a_{14} - 2a_{11} a_{12}^2 - 2a_8 a_{12} a_{13} - \\ - 2a_7 a_{13} a_{14} - 8a_7 a_{12} a_{13} + 4a_6 a_{13}^2 + 3a_7^2 + 2a_7 a_8 + 2a_5 a_{12}.$$

Summing both members of (C_{12}), multiplied by -2, and (C_{16}), we get in view of (C_9), (C_{13}):

$$(C_{17}) \qquad a_3 = -(a_7 - a_{12} a_{13})^2.$$

Comparing (C_{15}) with (C_{17}), we find $(a_3 = 0)$ and

$$a_7 = a_{12} a_{13}:$$

again by Remark 11 this is inconsistent with (*) because the affine tangent cones to $\tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{G}}$ in Z_∞ would have the common component $\{\widetilde{X + a_{13} = 0}\}$.

This concludes the proof of (C) in the case $b_{17} \neq 0$.

Second case: $b_{17} = 0$.

We assume again $a_{16} = 1$. From *b*), *c*), *d*), *e*) and *f*) of list 1) we have

$$b_{13} = b_{11} = 0 , \quad b_{14} = -b_{12} ,$$

$$b_8 = -2b_7 + a_{13}b_{12} , \quad b_{10} = -2b_5 + a_{11}b_{12} .$$

Given these equalities, we find moreover

$$H_{1,3}(18) = -2(b_7 - a_{13}b_{12}) , \quad H_{1,3}(17) = -2(b_5 - a_{11}b_{12}) .$$

Hence

$$(C_{18}) \qquad b_7 = a_{13}b_{12} ,$$

$$(C_{19}) \qquad b_5 = a_{11}b_{12} .$$

Now from $b_8 = -2b_7 + a_{13}b_{12}$ and $b_{10} = -2b_5 + a_{11}b_{12}$ we get

$$(C_{20}) \qquad b_8 = -a_{13}b_{12} ,$$

$$(C_{21}) \qquad b_{10} = -a_{11}b_{12} .$$

From g) of list 1) we find on one hand

$$b_9 = (a_{12} + a_{14} - a_{13}a_{17})b_{12} - 2b_6 ;$$

on the other hand we get, in view of (C_{18}) and (C_{19})

$$H_{1,3}(16) = (-8a_{13}a_{17} + 8a_{14} + 8a_{12})b_{12} - 6b_9 - 16b_6 .$$

From the last equalities one derives

$$(C_{22}) \qquad b_6 = (a_{12} + a_{14} - a_{13}a_{17})b_{12} ,$$

$$(C_{23}) \qquad b_9 = (-a_{12} - a_{14} + a_{13}a_{17})b_{12} .$$

Now, from h) and i) of list 1), we find

$$(C_{24}) \qquad b_4 = (2a_7 + a_8 - a_{11}a_{17} - a_{12}a_{13} + a_1^2\, a_{17})b_{12} ,$$

$$(C_{25}) \qquad 2b_3 = (2a_5 + a_{10} - a_{11}a_{12} - a_{12}a_{17} - a_{14}a_{17} + a_{11}a_{13}a_{17} + a_{13}a_{17}^2)b_{12} .$$

Finally, taking into acconut all the previous results, we find

$$H_{1,3}(15) = 2(a_7 - a_{12}a_{13} + a_{13}^2 a_{17}) b_{12} = 0:$$

beeing $b_{12} \neq 0$, or else G would be the null polynomial, from the last equality it follows

$$a_7 = a_{12} a_{13} - a_{13}^2 a_{17} .$$

Now the tangent cones to $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ in Z_∞ have the affine equations:

$$(X + a_{13})(a_{17}X + Y + a_{12} - a_{13}a_{17}) = 0 \qquad \text{and } b_{12}(X + a_{13}) = 0$$

and again applying Remark 11, we reach the conclusion of the proof of (C).

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