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The Weyl Fractional Operator of a System of Polynomials.

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SUMMARY - In this paper the Weyl fractional operator of a general system of polynomials is obtained and the main result (Eq. (2.3) below) provides an extension to a recent result of Srivastava [7]. Some possible applications are also indicated briefly.

1. Introduction.

In his recent paper, Srivastava [7] derived the Weyl fractional integral of a general class of polynomials. This result, Eq. (2.7) on p. 221 of [7], is given by

$$(1.1)\qquad \{\Gamma(\mu)\}^{-1}\int_z^\infty (t-z)^{\mu-1}\{e^{-pt}S_n^q(t)\}\,dt = \\ = s^{-\mu}e^{-pz}\sum_{r=0}^{[n/q]}(-n)_{qr}(-1/p)^r C_{n,r}L_r^{(-\mu-r)}(pz)\,,$$

provided that $\min\{\mathrm{Re}\,(p), \mathrm{Re}\,(\mu)\} > 0$, where the polynomial system $S_n^q(z)$ is defined by ([6], p. 1, eq. (1))

$$(1.2)\qquad S_n^q(z) = \sum_{r=0}^{[n/q]}(-n)_{qr}C_{n,r}z^r/r!\,,$$

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q being any arbitrary positive integer, and the coefficients $C_{n,r}$ (n, $r \geqslant 0$) is any arbitrary sequence of real or complex numbers. It being understood that $(a)_n$ stands for the usual Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$.

The purpose of this paper is to extend (1.1) to hold true for all real values of μ, by invoking the Weyl fractional calculus. The main result (2.3) below provides a generalization to Srivastava's result (1.1). {Incidentally, a multidimensional generalization of (1.1) has recently been considered by Raina [5]}. The usefulness of the result is exhibited by giving briefly some applications, mentioned in the concluding section.

2. Weyl operator and the main result.

Denote by A the class of all functions f which are differentiable any number of times, and if $f(x)$ and all of its derivatives are $O(x^{-v})$, as x increases without limit.

The Weyl fractional operator of a function $g(z)$ is defined as follows:

$$W_z^\alpha\{g(z)\} = \frac{(-1)^\alpha}{\Gamma(-\alpha)}\int_z^\infty (t-z)^{-\alpha-1} g(t)\,dt\,, \qquad \text{for } \alpha < 0\,. \tag{2.1}$$

For $\alpha \geqslant 0$,

$$W_z^\alpha\{g(z)\} = \frac{d^m}{dz^m}\{W_z^{m-\alpha} g(z)\}\,, \tag{2.2}$$

m being a positive integer such that $m > \alpha$. The representations (2.1) and (2.2) exist, whenever $g \in A$, see Miller [3].

The main result to be established is now given as follows:
For all (real) values of α,

$$W_z^\alpha\{e^{-pz} S_n^q(z)\} = (-p)^\alpha e^{-pz} \sum_{r=0}^{[n/q]} (-n)_{qr}(-1/p)^r \cdot C_{n,r} L_r^{(\alpha-r)}(pz)\ (*)\,, \tag{2.3}$$

(*) A new variation of (2.3) has recently been considered by R. K. Raina in his paper [Indian J. pure appl. Math., **16** (7) (1985), pp. 770-774].

where $\operatorname{Re}(p) > 0$, $S_n^q(z)$ is the class of polynomials defined by (1.2), and $L_r^{(v)}(z)$ denotes the Laguerre polynomials

$$L_r^{(v)}(z) = (r!)^{-1}(1+v)_r \; {}_1F_1(-r; 1+v; z), \quad r \geqslant 0. \tag{2.4}$$

3. Derivation of (2.3).

For $\alpha < 0$, invoking the definitions (1.2) and (2.1), we have

$$W_z^\alpha \{e^{-pz} S_n^q(z)\} = \frac{(-1)^\alpha}{\Gamma(-\alpha)} \int_z^\infty (t-z)^{-\alpha-1} \{e^{-pt} S_n^q(t)\}\, dt = \tag{3.1}$$

$$= \frac{(-1)^\alpha}{\Gamma(-\alpha)} \sum_{r=0}^{[n/q]} \frac{(-n)_{qr}}{r!} C_{n,r} \left\{ \int_z^\infty (t-z)^{-\alpha-1} e^{-pt} t^r\, dt \right\}.$$

Evaluating the integral by appealing appropriately to a *special case* of the known formula due to Raina and Koul [4, p. 191, eq. (14a)] (or else introducing the transformation $t - z = x$, and then evaluating the resulting Laplace transform with the aid of the formula [7, p. 220, eq. (2.2)]), we arrive at (2.3) for $\alpha < 0$.

On the other hand, when $\alpha \geqslant 0$, we apply (2.2), then by virtue of (1.2) and (3.1), we get

$$W_z^\alpha \{e^{-pz} S_n^q(z)\} = (-p)^{\alpha-m} \sum_{r=0}^{[n/q]} (-n)_{qr} (-1/p)^r C_{n,r} \cdot \tag{3.2}$$

$$\cdot \frac{d^m}{dz^m} \{e^{-pz} L_r^{(\alpha-m-r)}(pz)\}.$$

From (2.4) and the Kummer's transformation

$${}_1F_1(a; b; x) = e^x \, {}_1F_1(b-a; b; -x), \tag{3.3}$$

we find that

$$e^{-pz} L_r^{(\alpha-m-r)}(pz) = (r!)^{-1}(1+\alpha-m-r)_r \cdot \tag{3.4}$$

$$\cdot {}_1F_1(1+\alpha-m; 1+\alpha-m-r; -pz).$$

Using the derivative formula (see [2], p. 285, eq. (1))

$$\frac{d^n}{dz^n}\,{}_1F_1(a;b;\beta z)=\frac{(a)_n}{(b)_n}\beta^n\,{}_1F_1(a+n;b+n;\beta z)\,, \tag{3.5}$$

then (3.4) gives

$$\frac{d^m}{dz^m}\{e^{-pz}L_r^{(\alpha-m-r)}(pz)\}=$$
$$=\frac{\Gamma(1+\alpha)}{r!\,\Gamma(1+\alpha-r)}(-p)^m\,{}_1F_1(1+\alpha;1+\alpha-r;-pz)\,. \tag{3.6}$$

Now (3.2) in conjunction with (2.4), (3.3) and (3.6) lead once again to (2.3) for $\alpha\geqslant 0$. This establishes (2.3) for all real values of α.

Alternatively, (2.3) can be established directly by expressing

$$W_z^\alpha\{e^{-pz}S_n^q(z)\}=\sum_{r=0}^{[n/q]}\frac{(-n)_{qr}}{r!}C_{n,r}W_z^\alpha\{z^r e^{-pz}\}\,. \tag{3.7}$$

Applying as before the formula [4, p. 191, eq. (14a)], (3.7) readily yields the desired result (2.3).

4. Applications.

The generalized formula (2.3) can find many applications giving the Weyl fractional operators of all such classical orthogonal polynomials which are special cases of the polynomial system (1.2). The details of these various polynomials to which (1.2) reduces to are fairly well described in [7]. As a simple illustration, we apply (2.3) to obtain the Weyl operator of one of the Konhauser biorthogonal polynomials $Z_n^\mu(z;k)$ defined by ([1], p. 304, eq. (5))

$$Z_n^\mu(z;k)=\frac{\Gamma(kn+\mu+1)}{n!}\sum_{j=0}^{n}(-n)_j z^{kj}/(j!)\Gamma(kj+\mu+1)\,, \tag{4.1}$$

where $\mu>-1$, k is a positive integer.

Setting

$$q = 1, \; C_{n,r} = \frac{\Gamma(1+\mu+kn)}{\Gamma(1+\mu+kr)}, \qquad k \geqslant 1, \tag{4.2}$$

in (1.2), (2.3) in view of (4.1) yields then the result

$$W_z^\alpha \{e^{-pz} Z_n^\mu(z^{1/k}; k)\} = (-p)^\alpha e^{-pz} (n!)^{-1} \Gamma(1+\mu+kn) \cdot \sum_{r=0}^{n} \frac{(-n)_r}{\Gamma(1+\mu+kr)} (-1/p)^r L_r^{(\alpha-r)}(pz), \tag{4.3}$$

valid for all real α, where $\operatorname{Re}(p) > 0$.

Also, from (3.7), it follows that

$$W_z^\alpha \{e^{-pz} S_n^q(z^k)\} = (-p)^\alpha e^{-pz} \sum_{r=0}^{[n/q]} \frac{(-n)_{qr}}{r!} C_{n,r} \cdot (-1/p)^{kr} (kr!) L_{kr}^{(\alpha-kr)}(pz), \tag{4.4}$$

α real and k is any positive integer.

Setting the polynomial $S_n^q(z)$ to the form (4.1) in accordance with the choice of parameters given in (4.2), we get the following result from (4.4):

$$W_z^\alpha \{e^{-pz} Z_n^\mu(z; k)\} = (-p)^\alpha e^{-pz} (n!)^{-1} \Gamma(1+\mu+kn) \cdot \sum_{r=0}^{n} \frac{(-n)_r (kr!)}{r! \, \Gamma(1+\mu+kr)} (-1/p)^{kr} L_{kr}^{(\alpha-kr)}(pz), \tag{4.5}$$

which holds for all real values of α, provided that $\operatorname{Re}(p) > 0$.

For $\alpha = -v$, (4.3) and (4.5) by virtue of (2.1) correspond to the known results given by Srivastava [7, p. 225, Eqns. (3.24) and (3.25)]. It may also be noted from (2.2) that if $\alpha = m$ (m being a positive integer), then one can write down at once the derivative formulas for

$$\frac{d^m}{dz^m} \{e^{-pz} Z_n^\mu(z^{1/k}; k)\} \quad \text{and} \quad \frac{d^m}{dz^m} \{e^{-pz} Z_n^\mu(z; k)\},$$

by merely putting $\alpha = m$, in the relations (4.3) and (4.5).

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