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Permutation models and topological groups

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Permutation Models and Topological Groups.

NORBERT BRUNNER - JEAN E. RUBIN (*)

SUMMARY - We investigate the symmetry structure of permutation models with topological methods. The main result is: The automorphism group of a permutation model is locally compact, if and only if the model satisfies a class form of the multiple choice axiom. The model is then a finite support model

1. Introduction.

Permutation models are the standard device for proving independence results for the axiom of choice (AC) in set theory without the axiom of foundation. In this note we investigate topological properties of automorphism groups. We use the following notation:

In the sequel, V denotes the real world, satisfying $NBG^0 + AC$ (v. Neumann-Bernays-Gödel set theory without the axiom of foundation), $U \in V$ is a set of urelements $\big(u = \{u\}$ for $u \in U\big)$ and $SM \in V$ is a transitive model of NBG^0. In the examples we will use only $SM_1 = \mathcal{P}^{<\varkappa}(U) = \bigcup\limits_{\alpha<\varkappa} \mathcal{P}^{\alpha}(U)$, where $\varkappa > |U|$ $\big(|\cdot| =$ cardinality in $V\big)$ is inaccessible and $SM_2 = \{x \in \mathcal{P}^{<\varkappa}(U)\colon |TC(x) \cap U| < \varkappa\}$ $\big(TC(x)$ is the transitive closure of $x\big)$, where $\varkappa = |U|$ is inaccessible. $\big(\mathcal{P}^0(U) = U$ and $\mathcal{P}^{\alpha+1}(U) = \mathcal{P}^{\alpha}(U) \cup \mathcal{P}\big(\mathcal{P}^{\alpha}(U)\big)$, $\mathcal{P}^{\lambda}(U) = \bigcup\limits_{\alpha<\lambda} \mathcal{P}^{\alpha}(U)$, if λ is a limit

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ordinal, and $\mathfrak{P}$ is the power set operation.) For our proofs, we just assume, that $U \subseteq SM$ (possibly U is a proper class in SM, e.g. in SM_2), SM satisfies weak foundation with respect to U (i.e. in SM: « $SM = \mathfrak{P}^{On}(U)$ ») and in V the cardinality $|SM|^V = On^{SM}$. Then every ε-automorphism π of SM (we write $\pi \in \text{Aut}\, SM$) can be coded by the permutation $\pi \upharpoonright U$ on U (i.e. $\pi \upharpoonright U \in S(U)$, the symmetric group of all permutations on U in V). We also assume, that $x \subseteq SM$ is a SM class, if and only if $x \in V$; hence in particular SM satisfies the global AC (CAC). It follows, that every $\pi \in S(U)$ can be uniquely extended to a SM-class $\hat{\pi} \in \text{Aut}\, SM$, such that $\hat{\pi} \upharpoonright U = \pi$; recursively $\hat{\pi}x = \{\hat{\pi}y : y \in x\} = \hat{\pi}''x$. (In the future we will not distinguish between π and $\hat{\pi}$.) If $G \lhd S(U)$ ($\lhd$ means subgroup) and $\mathcal{F}$ is a normal filter of subgroups of G (normal means, that $\mathcal{F}$ is closed under inner automorphisms), then $P(G, \mathcal{F}) \subseteq SM$ is the corresponding permutation model: If $x \in V$ and $x \subseteq P$, then x is a class in P if and only if x is symmetric (i.e. for some $F \in \mathcal{F}$ $\text{sym}_G x = \{\pi \in G : \pi''x = x\} \supseteq F$), and $x \in P$ (i.e. x is a set in P), if and only if in addition $x \in SM$. Our assumptions on SM ensure, that permutation models can be described in the usual way (e.g. Felgner [3]) within SM (not invoking V). The need for our assumption, that SM is a set in V arises, when we want to speak about the automorphism group of a model with a proper class of urelements. SM can also be used, to add various cardinality restrictions on the sets of the permutation model (e.g. SM_2). Since we are only interested in the symmetry structure of the models, we will not consider the more general notion of « permutation models », where classes need not be symmetric. Moreover, because we need CAC in SM, our results will not extend, if Cohen's nonstandard models [1] are used for SM. These models permit nontrivial ε-automorphisms, but cannot satisfy AC, since the axiom of foundation holds (a well known observation due to H. Friedman).

2. Symmetry structures.

In this section we discuss topologies on subgroups of Aut $\mathcal{M}$ and we refer the reader to Hewitt and Ross [5] for the definition and elementary properties of topological groups. We note the fact that $\mathcal{M}$ need not be a permutation model.

It is easy to check (see Hewitt and Ross [5], p. 18) that the normal filter $\mathcal{F}$ of a permutation model $P(G, \mathcal{F})$ is a neighborhood base of the identity mapping, *id*, of G and thus induces a group topology $\boldsymbol{G}_{\mathcal{F}}$ on G. The question arises, whether the models induced by a normal filter are the most general « symmetric models ». We will include the easy proof of the positive answer to this question (2.2 (ii)), but first we introduce some notation. In the sequel, when we speak of a model, we will always assume, that it contains all urelements (i.e. for permutation models we have to verify $\mathrm{sym}_G u \in \mathcal{F}$, for all $u \in U$), $\mathcal{M} \subseteq SM$ is a transitive class, $U \subseteq \mathcal{M}$ and $\mathcal{M} \models$ axiom of pairing ($\models$ is the satisfaction relation). For a group G, $\boldsymbol{G}$ will always denote some group topology on G.

2.1. DEFINITION. Let $G \lhd \mathrm{Aut}\,\mathcal{M} = \{g \in \mathrm{Aut}\,SM : g''\mathcal{M} = \mathcal{M}\}$. (This is a set in the real world V but not necessarily in $\mathcal{M}$.)

(i) Let $x \in V$, $x \subseteq \mathcal{M}$; x is $(G, \boldsymbol{G})$-*symmetric*, if there is a $F \in \boldsymbol{G}$ such that $\mathrm{sym}_G x \supseteq F \neq \emptyset$.

(ii) $(G, \boldsymbol{G})$ *generates* $\mathcal{M}$, if $x \in V$ is an $\mathcal{M}$-class if and only if $x \subseteq \mathcal{M}$ and x is $(G, \boldsymbol{G})$-symmetric; and $x \in \mathcal{M}$ if and only if $x \in SM$ and x is an $\mathcal{M}$-class. □

Since groups with a nonempty interior are open in a topological group, x is symmetric if and only if $\mathrm{sym}\, x \in \boldsymbol{G}$ (we suppress the subscripts). $P(G, \mathcal{F})$ is generated by $(\hat{G}, \hat{\boldsymbol{G}}_{\mathcal{F}})$. (Formally this is also true without our permanent assumption $U \subseteq P(G, \mathcal{F})$.) Conversely we have:

2.2. LEMMA. Let $G \lhd \mathrm{Aut}\,\mathcal{M}$ and $\boldsymbol{G}$ a group topology.

(i) $\mathcal{F} = \{\mathrm{sym}_G x : x \in \mathcal{M}\}$ is a normal filter.

The topology induced by $\mathcal{F}$, $\boldsymbol{G}_{\mathrm{nat}}$ (the natural topology), is a zero-dimensional T_2 group topology on G. (In $\boldsymbol{G}_{\mathrm{nat}}$, $\mathcal{F}$ is a neighborhood base of *id*.)

(ii) If $(G, \boldsymbol{G})$ generates $\mathcal{M}$, then also $(G, \boldsymbol{G}_{\mathrm{nat}})$ generates $\mathcal{M}$ and $\mathcal{M} = P(G \restriction U, \mathcal{F} \restriction U)$.

PROOF. Since $\mathrm{sym}\, x \cap \mathrm{sym}\, y = \mathrm{sym}\, \langle x, y \rangle \in \mathcal{F}$, $\mathcal{F}$ is a filter base of groups. $\mathcal{F}$ is normal, since $\pi(\mathrm{sym}\, x)\pi^{-1} = \mathrm{sym}\,(\pi x) \in \mathcal{F}$ for $\pi \in G$ and $x \in \mathcal{M}$. It follows that $\mathcal{F}$ is a neighborhood base of *id* for a group topology $\boldsymbol{G}_{\mathrm{nat}}$. $U \subseteq \mathcal{M}$ gives $\{id\} = \bigcap \{\mathrm{sym}\, u : u \in U\}$, whence $\boldsymbol{G}_{\mathrm{nat}}$ is T_0; therefore it is also completely regular. For $x \in \mathcal{M}$, $\mathrm{sym}\, x$ is open, since it is a group containing *id* in its interior, whence its cosets form an open partition of G. Therefore $\operatorname{sym} x$ is clopen and $\boldsymbol{G}_{\text{nat}}$ is zero dimensional. If $id \in \pi \operatorname{sym} x$, then $\pi \operatorname{sym} x = \operatorname{sym} x$, and so $\mathcal{F}$ consists of exactly the base open neighborhoods of id. If $(G, \boldsymbol{G})$ generates $\mathcal{M}$, then $\boldsymbol{G}_{\text{nat}} \subseteq \boldsymbol{G}$ and $\boldsymbol{G}_{\text{nat}}$-symmetric is equivalent to $\boldsymbol{G}$-symmetric, whence $\boldsymbol{G}_{\text{nat}}$ generates $\mathcal{M}$, also. $\mathcal{M} = P(G \restriction U, \mathcal{F} \restriction U)$ is proved by induction on rank. Since $U \subseteq \mathcal{M}$, it follows, that $U \subseteq P$, and so for $x \in SM$, $x \in \mathcal{M} \leftrightarrow x \in P$ is true for rank $(x) = 0$. If it is true for rank $(y) <$ rank (x), $x \subseteq \mathcal{M} \leftrightarrow x \subseteq P$. If $x \in \mathcal{M}$, then $(\operatorname{sym} x) \restriction U \in \mathcal{F} \restriction U$, whence $x \in P$. If $x \in P$, then $(\operatorname{sym} x) \restriction U = \operatorname{sym}_{G \restriction U}(x) \supseteq F \restriction U$, for some $F \in \mathcal{F}$, whence $\operatorname{sym} x \supseteq F \neq \emptyset$. I.e. x is $\boldsymbol{G}_{\text{nat}}$-symmetric and therefore in $\mathcal{M}$. Since $(\mathcal{F} \restriction U)^\wedge = \mathcal{F}$ and $\hat{\boldsymbol{G}}_{\mathcal{F} \restriction U} = \boldsymbol{G}_{\text{nat}}$, $\mathcal{M}$ and P have the same classes. □

Other interesting topologies besides the natural topology, $\boldsymbol{G}_{\text{nat}}$, are $\boldsymbol{G}_{\text{fin}}$ which is induced by the normal filter of groups generated by $\mathcal{F}_{\text{fin}} = \{\text{fix}\, e\colon e$, a finite subset of $U\}$, and $\boldsymbol{G}_{\text{wo}}$, induced by $\mathcal{F}_{\text{wo}} = \{\text{fix}\, e\colon \mathcal{M} \models$ « $e \subseteq U$ and e is well orderable »$\}$ $(\text{fix}\, e = \{\pi \in G\colon \pi \restriction e = id \restriction e\})$. If $e \subseteq U$ is well ordered by $< \in \mathcal{M}$ and $\pi \in G \lhd \operatorname{Aut} \mathcal{M}$, then πe is well ordered by $\pi(<)$, whence $\mathcal{F}_{\text{wo}}$ is in fact a normal filter of groups. Moreover, $\text{fix}\, e = \operatorname{sym}(<)$, whence $\boldsymbol{G}_{\text{wo}} \subseteq \boldsymbol{G}_{\text{nat}}$. The fact that $\boldsymbol{G}_{\text{fin}} \subseteq \boldsymbol{G}_{\text{wo}}$ is obvious. These topologies correspond to special cases of the permutation model construction. FM-models are defined from a filter $\mathcal{F}_I$ which is induced by a « normal » ideal I of SM-subsets of U (i.e. I is a G-invariant ideal, containing all singletons $\{u\}$, $u \in U$; the latter condition is to ensure, that the model contains all urelements) where $\mathcal{F}_I = \{\text{fix}\, e : e \in I\}$. Finite support models are FM-models, where I is the ideal of all finite subsets of U.

2.3. Corollary. If $\mathcal{M} = P(G, \mathcal{F}_I)$ is a FM-model, then $\boldsymbol{G}_{\text{wo}} = \boldsymbol{G}_{\text{nat}}$. Conversely, if $\boldsymbol{G}_{\text{wo}}$ generates $\mathcal{M}$, then $\mathcal{M}$ is a FM-model. Similarly, $\mathcal{M}$ is a finite support model if and only if $\boldsymbol{G}_{\text{fin}}$ generates $\mathcal{M}$, in which case $\boldsymbol{G}_{\text{fin}} = \boldsymbol{G}_{\text{wo}} = \boldsymbol{G}_{\text{nat}}$.

Proof. Since $\boldsymbol{G}_{\text{wo}}$ generates $P(G, \mathcal{F}_I)$, $\boldsymbol{G}_{\text{nat}} \subseteq \boldsymbol{G}_{\text{wo}} \subseteq \boldsymbol{G}_{\text{nat}}$ by 2.2. Conversely, if $\boldsymbol{G}_{\text{wo}}$ generates $\mathcal{M}$, then $\boldsymbol{G}_{\text{nat}} = \boldsymbol{G}_{\text{wo}}$ from above and $\mathcal{M} = P(G, \mathcal{F})$ by 2.2, whence $\boldsymbol{G}_{\mathcal{F}} = \boldsymbol{G}_{\text{nat}} = \boldsymbol{G}_{\mathcal{F}_I} = \boldsymbol{G}_{\text{wo}}$, where $I = \{e \subseteq U\colon \mathcal{M} \models e$ is well orderable$\}$, and so $\mathcal{M} = P(G, \mathcal{F}_I)$. The same argument works for $\boldsymbol{G}_{\text{fin}}$. □

In view of this observation it is of interest to find topological conditions for $\boldsymbol{G}_{\text{nat}} = \boldsymbol{G}_{\text{fin}}$. Lemma 2.4 gives an answer in a special case. It will also follow from our main result, that $\boldsymbol{G}_{\text{nat}} = \boldsymbol{G}_{\text{fin}}$ holds if $\boldsymbol{G}_{\text{nat}}$ is totally bounded (3.4). A topological group is *totally bounded*, if for each Q, $\emptyset \neq Q \in \boldsymbol{G}$, there is a finite $F \subseteq G$ such that $F \cdot 0 \supseteq G$ (i.e. the left uniformity is totally bounded). For example, a pseudocompact group is totally bounded ([2]).

2.4. LEMMA. Let $G \lhd \operatorname{Aut} \mathcal{M}$, $\mathcal{M}$ not necessarily a permutation model.

(i) If $\boldsymbol{G}_{\text{nat}}$ is compact, then $\boldsymbol{G}_{\text{nat}} = \boldsymbol{G}_{\text{fin}}$. In particular, if a compact $(G, \boldsymbol{G})$ generates $\mathcal{M}$, then $\boldsymbol{G} = \boldsymbol{G}_{\text{nat}} = \boldsymbol{G}_{\text{fin}}$ and $\mathcal{M}$ is a finite support model.

(ii) If $\boldsymbol{G}_{\text{wo}}$ is totally bounded, then $\boldsymbol{G}_{\text{fin}} = \boldsymbol{G}_{\text{wo}}$.

PROOF. (i) Since $U \subseteq \mathcal{M}$, $\{id\} = \cap \{\text{fix}\,\{u\}\colon u \in U\}$, so $\boldsymbol{G}_{\text{fin}}$ is T_2. Since compact T_2 spaces are minimal Hausdorff spaces, it follows from $\boldsymbol{G}_{\text{fin}} \subseteq \boldsymbol{G}_{\text{nat}}$, that $\boldsymbol{G}_{\text{fin}} = \boldsymbol{G}_{\text{nat}}$, if $\boldsymbol{G}_{\text{nat}}$ is compact. If $\boldsymbol{G}$ is compact and generates $\mathcal{M}$, then $\boldsymbol{G}_{\text{fin}} \subseteq \boldsymbol{G}_{\text{nat}} \subseteq \boldsymbol{G}$, whence we again have equality. In this case, $\mathcal{M}$ is a finite support model by 2.3.

(ii) Let $e \subseteq U$ be well orderable in $\mathcal{M}$. Since $\boldsymbol{G}_{\text{wo}}$ is totally bounded, $E \cdot \text{fix}\, e \supseteq G$ for some finite $E = \{\pi_i\colon i \in n\}$. We may assume that the π_i are coset representatives of fix e and that $\pi_0 = \text{id}$. Since $\pi_i \notin \text{fix}\, e$ for $0 < i < n$, there are $u_i \in e$ such that $\pi_i u_i \neq u_i$. We set $f = \{u_i\colon 0 < i < n\}$. Then fix f = fix e. Clearly fix $e \subseteq$ fix f. In the other direction, take any $\pi \in \text{fix}\, f$. Then $\pi = \pi_i \cdot \psi$, where $\psi \in \text{fix}\, e$ and $i \in n$. But $i \neq 0$ is impossible, since then $\pi u_i = \pi_i u_i \neq u_i$. So $\pi = \psi \in \text{fix}\, e$. □

2.5. EXAMPLE. A permutation model $\mathcal{M}_1$ such that $\boldsymbol{G}_{\text{fin}}$ is compact, but $\boldsymbol{G}_{\text{fin}} \neq \boldsymbol{G}_{\text{wo}} = \boldsymbol{G}_{\text{nat}}$.

PROOF. For SM we take SM_1 (defined in the introduction) and index U as $U = \cup \{P_\alpha\colon \alpha \in \omega_1\}$, where the P_α are pairwise disjoint two-element sets. G is the group of all permutations on U which respect each P_α (i.e. $g'' P_\alpha = P_\alpha$ for all $g \in G$ and $\alpha \in \omega_1$), I is generated by $\left\{\bigcup_{\alpha \in \gamma} P_\alpha\colon \gamma \in \omega_1\right\}$ and $\mathcal{M}_1 = P(G, \mathcal{F}_I)$. Then $\boldsymbol{G}_{\text{fin}} \cong 2^{\omega_1}$ (with the product topology) is compact, while $\boldsymbol{G}_{\text{wo}} = \boldsymbol{G}_{\text{nat}}$ is the G_δ-modification of $\boldsymbol{G}_{\text{fin}}$ and therefore a P-space (countable intersections of open sets are open). This implies, that compact subsets are finite in $\boldsymbol{G}_{\text{wo}}$ (anticompact). □

As this example shows, it can be of interest, to study topologies $\boldsymbol{G}$ on $G \lhd \operatorname{Aut} \mathcal{M}$ which do not generate $\mathcal{M}$; e.g. $\boldsymbol{G}_{\text{fin}}$ describes the elements $x \in \mathcal{M}$ which are definable from finitely many urelements.

2.6. EXAMPLE. A finite support model $\mathcal{M}_2$ and a group $H \lhd \operatorname{Aut} \mathcal{M}_2$ such that $\boldsymbol{H}_{\text{fin}} = \boldsymbol{H}_{\text{wo}} \neq \boldsymbol{H}_{\text{nat}}$ and $(H, \boldsymbol{H}_{\text{nat}})$ generates $\mathcal{M}_2$.

PROOF. We set $SM = SM_1$. Let U and G be as in example 2.5; $I = [U]^{<\omega}$, the set of finite subsets of U, and $\mathcal{M}_2 = P(G, \mathcal{F}_I)$. H is the group of all permutations on U which respect all but finitely many P_α (i.e. each $\pi \in H$ is a composite $\pi = \sigma \circ \psi$, $\sigma \in S^{\mathcal{M}}(U)$ and $\psi \in G$). Of course $H \lhd \operatorname{Aut} \mathcal{M}_2$ and $\operatorname{sym}_H \langle P_\alpha \colon \alpha \in \omega_1 \rangle \notin \boldsymbol{H}_{\text{fin}}$. Thus $\boldsymbol{H}_{\text{nat}} \neq \boldsymbol{H}_{\text{fin}}$. Since U is Dedekind-finite, $\boldsymbol{H}_{\text{fin}} = \boldsymbol{H}_{\text{wo}}$. In order to verify that $\boldsymbol{H}_{\text{nat}}$ generates $\mathcal{M}_2$ we use the following general observations: If $G \lhd H \lhd \operatorname{Aut} \mathcal{M}$ and $(G, \boldsymbol{G})$ generates $\mathcal{M}$, then $\boldsymbol{H}_{\text{nat}}$ generates $\mathcal{M}$. For if $x \in \mathcal{M}$, then $\operatorname{sym}_H x \in \boldsymbol{H}_{\text{nat}}$ from the definition, and if $\operatorname{sym}_H x \in$ $\in \boldsymbol{H}_{\text{nat}}$, then $\operatorname{sym}_G x = \operatorname{sym}_H x \cap G \in \boldsymbol{G}_{\text{nat}}$ ($\boldsymbol{G}_{\text{nat}}$ is the subspace topology of $\boldsymbol{H}_{\text{nat}}$), whence by 2.2 $\operatorname{sym}_G x \in \boldsymbol{G}$ and $x \in \mathcal{M}$. □

It follows from 2.6 that two groups which generate the same model need not induce the same support structure on it, not even for clopen subgroups. A property which is more general than « generate the same model » is « *induce the same symmetry structure* »: $G, H \lhd \operatorname{Aut} \mathcal{M}$ and for all $x \subseteq \mathcal{M}$: $\operatorname{sym}_G x \in \boldsymbol{G}$ if and only if $\operatorname{sym}_H x \in \boldsymbol{H}$. If $\boldsymbol{G}$ generates $\mathcal{M}$ and induces the same symmetry structure as $\boldsymbol{H}$, then $\boldsymbol{H}$ generates $\mathcal{M}$. For example (for $H \lhd \operatorname{Aut} \mathcal{M}$), if $G \lhd H$ is open, then G and H induce the same symmetry structure $\mathcal{M}$.

2.7. LEMMA. Let $G \lhd H \lhd \operatorname{Aut} \mathcal{M}$ $\boldsymbol{G}_{\text{nat}} \subseteq \boldsymbol{G}$ and $\boldsymbol{H}_{\text{fin}} \subseteq \boldsymbol{H}$, $\boldsymbol{G}$ the subspace topology from $\boldsymbol{H}$ and G dense in H ($G^- = H$). Then G and H induce the same symmetry structure on $\mathcal{M}$.

PROOF. If $\operatorname{sym}_H x \in \boldsymbol{H}$, then $\operatorname{sym}_G x = \operatorname{sym}_H x \cap G \in \boldsymbol{G}$. For the converse, let $\operatorname{sym}_G x \in \boldsymbol{G}$, $x \subseteq \mathcal{M}$, and choose $Q \in \boldsymbol{H}$, $Q = Q^{-1}$, such that $Q \cap G = \operatorname{sym}_G x$. Then $\operatorname{sym}_H x \supseteq Q \neq \emptyset$ and $\operatorname{sym}_H x \in \boldsymbol{H}$. For otherwise, let x be a counterexample of minimal rank (> 0, since $\boldsymbol{H}_{\text{fin}} \subseteq \boldsymbol{H}$). There is a $\pi \in Q$ and an $a \in x$ such that $b = \pi a \notin x$. (It suffices to show $\pi'' x \subseteq x$; for $\pi'' x \supseteq x$ take $\pi^{-1} \in Q^{-1} = Q$.) Then $W = \pi \cdot \operatorname{sym}_H a \in \boldsymbol{H}$, since $a \in \mathcal{M}$, $\operatorname{sym}_G a \in \boldsymbol{G}$ and rank $a <$ rank x, the minimal rank counterexample; also $\pi \in W \cap Q$. As G is dense, there is a $\varphi \in W \cap Q \cap G$. Because $\varphi \in W$, $b \in (\varphi'' x) \setminus x$ and because $\varphi \in Q \cap G = \operatorname{sym}_G x$, $\varphi'' x = x$, a contradiction. It follows in particular, that $\boldsymbol{H}_{\text{nat}} \subseteq \boldsymbol{H}$. □

As for some applications of these concepts, let us just mention the following observation. Felgner [3] defines the Fraenkel-Halpern model from a countable (in SM) set U of urelements $\boldsymbol{H} = S(U)$ and the finite topology. He also considers the Specker model defined from the dense subgroup G of all permutations on U with finite supports. It follows that these models are identical. While the proof of the Kurepa Antichain Principle in the Specker model is a trivial consequence of the fact that the generating group is a torsion group, it requires some work in the Fraenkel-Halpern model setting.

The observation that the proof can be simplified since these models are equal was the starting point of this paper. It is also interesting to observe that $\mathbb{Z}$ with some totally bounded group topology can generate a model. Just take a compact *monothetic* group (contains a dense cyclic subgroup) like $G = \prod_{p \text{ prime}} Z_p$ and let $\boldsymbol{H}$ be the dense cyclic subgroup of G. Since the finite support model $\mathcal{M}$ generated by G does not satisfy CAC, $\boldsymbol{H}$ is not discrete (3.1) and by 2.7 $\boldsymbol{H}$ generates $\mathcal{M}$. Moreover, by 3.1, $\mathcal{M} \models CMC$, a class form of the multiple choice axiom.

3. Multiple choice axiom.

It follows from section 2 that the property to generate a model is shared by so many topologically different groups that it seems unlikely that we could come up with theorems of the form « $\mathcal{M} \models$ some form of $AC \Leftrightarrow$ a group $(G, \boldsymbol{G})$ generating $\mathcal{M}$ satisfies some topological property » (c.f. 3.2). We show, however, that a class form CMC of the multiple choice axiom can be characterized in that way. In the sequel CAC is $CWO4$ from Rubin and Rubin's new monograph [9] (i.e. in $\mathcal{M}$ there is wellordering of $\mathcal{M}$) and CMC is $CWO\,9$ (in $\mathcal{M}$ there is a family $\langle F_\alpha \colon \alpha \in On^{\mathcal{M}} \rangle$ of finite sets such that $\mathcal{M} = \cup \{F_\alpha \colon \alpha \in On^{\mathcal{M}}\}$).

3.1. LEMMA. Let $(G, \boldsymbol{G})$ generate $\mathcal{M}$.

(i) $\mathcal{M} \models CAC$ if and only if $\boldsymbol{G}$ is discrete.

(ii) $\mathcal{M} \models CMC$ if and only if $\boldsymbol{G}_{\mathrm{nat}}$ is locally bounded (i.e. $\boldsymbol{G}_{\mathrm{nat}}$ has a totally bounded nonempty open set).

PROOF. (i) If $\mathcal{M} \models CAC$, then there is a wellordering $<$ of $\mathcal{M}$ in $\mathcal{M}$ (as a symmetric proper class). Then sym $(<) = \text{fix}\, \mathcal{M} = \{id\} \in$

$\in \boldsymbol{G}_{\text{nat}} \subseteq \boldsymbol{G}$, and $\boldsymbol{G}$ is discrete. Conversely, if $< \in V$ is a wellordering of $\mathcal{M}(SM \models CAC$ and $<$ is a class in $SM)$, then sym $(<) \supseteq \{id\} \in \boldsymbol{G}$, whence $<$ is symmetric; $\mathcal{M} \models CAC$.

(ii) If $\mathcal{M} \models CMC$, then $\boldsymbol{H} = \text{sym}\,\langle F_\alpha\colon \alpha \in On^{\mathcal{M}}\rangle \in \boldsymbol{G}_{\text{nat}}$ it totally bounded. For if $Q \supseteq \pi\,\text{sym}_H x \in \boldsymbol{H}$ for some $x \in F_\alpha$, then $\text{orb}_H x = \{\pi x\colon \pi \in \boldsymbol{H}\} \subseteq F_\alpha$ is finite, say $\text{orb}_H x = \{\pi_i x\colon i \in n\}$, and $E \cdot Q \supseteq \boldsymbol{H}$ for $E = \{\pi_i \circ \pi^{-1}\colon i \in n\}$. Conversely, if $\boldsymbol{G}_{\text{nat}}$ is locally bounded, there is an open, totally bounded group $\boldsymbol{H} \lhd G$. If $x \in \mathcal{M}$, then $\text{orb}_H x = \{\pi x\colon \pi \in \boldsymbol{H}\} = \{\pi x\colon \pi \in E\}$, for any finite E such that $E \cdot \text{sym}_H x \supseteq \boldsymbol{H}$, whence $\text{orb}_H x$ is finite. We use CAC in $\Delta(\boldsymbol{H}) = \{x \in \mathcal{M}\colon \text{sym}\, x \supseteq \boldsymbol{H}\}$ (follows from AC in V) to enumerate the $\boldsymbol{H}$-orbits $\text{orb}_H x$ (they are in $\Delta(\boldsymbol{H})$) as F_α, $\alpha \in On^{\mathcal{M}}$; this proves CMC in $\mathcal{M}$. □

3.2. Example. A permutation model $\mathcal{M}_3$ such that $\mathcal{M}_3 \models AC + RA +$ not CMC (RA: the universe is a wellordered union of sets), whence no locally bounded group can generate $\mathcal{M}$.

Proof. Let SM be SM_2 and in SM index U as $U = \cup \{P_\alpha\colon \alpha \in On\}$, P_α disjoint two element sets. G respect all the P_α and I is the V-ideal of all SM-subsets of U. $\mathcal{M}_3 = P(G, \mathcal{F}_I)$ then satisfies AC; RA holds because $\mathcal{M}_3 = \cup \{R_\alpha\colon \alpha \in On^{\mathcal{M}} = On^{SM}\}$; where $R_\alpha = \{x \in \mathcal{M}\colon TC(x) \cap P_\beta = \emptyset$ for all $\beta > \alpha$ and rank $(x) < \alpha\}$; and $\boldsymbol{G}_{\text{nat}} = \boldsymbol{G}_{\text{wo}}$ is not locally bounded, since $\text{orb}_H x$ is infinite for $\boldsymbol{H} = \text{fix}\, e$ and $x \setminus e$ infinite. It follows from 3.1 (and the fact that totally boundedness is inherited by coarser topologies), that no generating topology can be locally bounded and that $\mathcal{M} \models$ not CMC. □

The following theorem and its corollary are our main results.

3.3. Theorem. Let $\mathcal{M}$ be a permutation model. Then $\mathcal{M} \models CMC$ if and only if Aut $\mathcal{M}$ with the natural topology is locally compact.

Proof. As follows from the proof of 2.6, Aut $\mathcal{M}$ with the natural topology generates $\mathcal{M}$. If it is locally compact, then some sym x is compact and therefore totally bounded (a special case of Comfort, Ross [2]), whence $\mathcal{M} \models CMC$ by 3.1. For the proof of the converse, let $\mathcal{M} \models CMC$, $\langle F_\alpha\colon \alpha \in \Omega\rangle$ be a CMC-covering of $\mathcal{M}$ and set $G = \text{sym}\,\langle F_\alpha\colon \alpha \in \Omega\rangle$ $(\Omega = On^{\mathcal{M}} \in On^V)$. We shall show that G is compact.

First of all, since each $g \in G$ respects F_α, $g \restriction F_\alpha \in S(F_\alpha)$. Therefore G can be represented as a subgroup of $\Pi = \prod_{\alpha \in \Omega} S(F_\alpha)$. In the product topology $\mathbf{\Pi}$ on Π each $S(F_\alpha)$ is discrete. We shall show G is closed;

i.e. for each net $\langle g_i: i \in I\rangle$ in G converging to $g \in \Pi$, we have $g \in G$. Because each $g \in \Pi$ is a bijective map $g: \mathcal{M} \to \mathcal{M}$ such that $g''F_\alpha = F_\alpha$ for each $\alpha \in On$ all that we have to verify is that $g''x = gx$ for each $x \in \mathcal{M}$. (Since $\mathcal{M}$ is a permutation model, it then follows that $g = \hat{g} \restriction \mathcal{M}$ for some $\hat{g} \in \mathrm{Aut}\, SM$.) We note that for each $x \in \mathcal{M}$ there is an $i_x \in I$ such that for $i \geqslant i_x$, $g_i x = gx$. For if $x \in F_\alpha$, then $Q = \{h \in \Pi: h \restriction F_\alpha = g \restriction F_\alpha\}$ is an open neighborhood of g in Π, whence $g_i \in Q$ for all $i \geqslant i_x$, by convergence, and $g_i x = (g_i \restriction F_\alpha)x = (g \restriction F_\alpha)x = gx$. Hence if $y \in x \in \mathcal{M}$ and $i \geqslant i_x, i_y$, then $gy = g_i y \in g_i x$; i.e. $g''x \subseteq gx$. As Π is a topological group, also $g^{-1} \in G^-$, whence $g''x \supseteq x$. Finally we observe that $\boldsymbol{G}_{\mathrm{nat}}$ is the subspace topology from $\boldsymbol{\Pi}$, proving compactness ($\boldsymbol{\Pi}$ is compact, since each $S(F_\alpha)$ is finite). For if $x \in F_\alpha$, $\mathrm{sym}\, x \supseteq \mathrm{fix}\, F_\alpha \in \boldsymbol{\Pi} \restriction G$; and conversely, if $e \subseteq \Omega$ is finite, $P = \bigcap_{\alpha \in e} \mathrm{fix}\, F_\alpha = \{g \in G \subseteq \Pi: g \restriction F_\alpha = id \restriction F_\alpha,\ \alpha \in e\}$ is a $\boldsymbol{\Pi}$-open neighborhood of id, then $P = \bigcap_{\alpha \in e} \bigcap_{x \in F} \mathrm{sym}\, x \in \boldsymbol{G}_{\mathrm{nat}}$. □

3.4. Corollary. If $G \lhd \mathrm{Aut}\, \mathcal{M}$ ($\mathcal{M}$ is not necessarily a permutation model) and $\boldsymbol{G}_{\mathrm{nat}}$ is totally bounded, then $\boldsymbol{G}_{\mathrm{nat}} = \boldsymbol{G}_{\mathrm{fin}}$.

Proof. We set $\mathcal{F} = \{\mathrm{sym}_G\, x: x \in \mathcal{M}\}$ and let $\mathcal{N} = P(G, \mathcal{F})$; since $U \subseteq \mathcal{M}$ we have $U \subseteq \mathcal{N}$. Then $\boldsymbol{G}_{\mathcal{N}\text{-nat}} \subseteq \boldsymbol{G}_{\mathcal{F}} = \boldsymbol{G}_{\mathrm{nat}}$ and if $\mathrm{sym}\, x \in \boldsymbol{G}_{\mathrm{nat}}$, we shall show that there is a $y \in \mathcal{N}$ such that $\mathrm{sym}\, y = \mathrm{sym}\, x$, then $\boldsymbol{G}_{\mathcal{N}\text{-nat}} = \boldsymbol{G}_{\mathrm{nat}}$. It is true if x has rank zero. Suppose it is true for each $a \in x$. Choose in SM $\alpha(a) \in On$ and $b(a) \in \mathcal{N}$ such that $\mathrm{sym}\, b(a) = \mathrm{sym}\, a$ and α is one to one. We set $y = \{\pi\langle\alpha(a), b(a)\rangle: a \in x, \pi \in \mathrm{sym}\, x\}$; $y \subseteq \mathcal{N}$ and since $\mathrm{sym}\, y \supseteq \mathrm{sym}\, x \in \mathcal{F}$, y is an $\mathcal{N}$ class. Because $\mathrm{sym}\, x$ is totally bounded, by 3.1 $\mathrm{orb}_{\mathrm{sym}\, x}(\alpha(a), b(a))$ is finite for each $a \in x$, whence y is a set in SM and $y \in \mathcal{N}$. It remains to be shown that $\mathrm{sym}\, y \subseteq \mathrm{sym}\, x$. If $\pi \in \mathrm{sym}\, y$, and $a \in x$, then $\pi b(a) = \psi b(a)$ for some $\psi \in \mathrm{sym}\, x$ (because α is injective), whence $\psi^{-1}\pi \in \mathrm{sym}\, b(a) = \mathrm{sym}\, a$ and $\pi a = \psi a \in x$; i.e. $\pi''x \subseteq x$, proving $\pi \in \mathrm{sym}\, x$. As $\boldsymbol{G}_{\mathrm{nat}}$ is totally bounded, $\mathcal{N} \models CMC$ by 3.1, whence $\mathrm{Aut}\, \mathcal{N}$ is locally compact by 3.3. $\boldsymbol{H}$ is the closure of G in $\mathrm{Aut}\, \mathcal{N}$ with the natural topology. Since G is dense in $\boldsymbol{H}$ and $\boldsymbol{H}$ is locally compact, $(\boldsymbol{H}, \boldsymbol{H}_{\mathrm{nat}})$ is the Weil completion of G. $\boldsymbol{H}$ is compact, because G is totally bounded (see Weil's monograph [10] for definitions and proofs). Therefore by 2.4. $\boldsymbol{H}_{\mathrm{nat}} = \boldsymbol{H}_{\mathrm{fin}}$. This proves $\boldsymbol{G}_{\mathrm{nat}} = \boldsymbol{G}_{\mathrm{fin}}$ in $\mathcal{N}$ (2.7) and by the above remarks also in $\mathcal{M}$. □

It follows from example 2.6 that $\boldsymbol{G}_{\mathrm{nat}}$ need not be totally bounded even if CMC holds.

We conclude this section with a remark on quotient groups. If G generates $\mathcal{M}$ and $x \subseteq \mathcal{M}$ is a class, then x is wellorderable, if and only if the factor group $\operatorname{sym} x/\operatorname{fix} x$ is discrete and x is a wellorderable union of finite sets, if and only if $\operatorname{sym} x/\operatorname{fix} x$ is locally bounded. As an application, if $\boldsymbol{G}$ is monothetic, then wellorderable families $F = \langle F_\alpha : \alpha \in \varkappa \rangle$, F_α pairwise disjoint and $|F_\alpha| = n$, have a choice function. For, if $\boldsymbol{H}$ is a cyclic group supporting F and generating $\mathcal{M}$ (2.7), then $\boldsymbol{H}_1 = \boldsymbol{H}/\operatorname{fix} \cup F \lhd Z^k_{n!}$ which is a torsion group, whence $\boldsymbol{H}_1$ is finite and $\cup F$ wellorderable, since fix $(\cup F)$, a closed group of finite index, is open.

4. Set forms of the axiom of choice.

It follows from 3.2 that even under the condition RA there is no natural way to translate set forms of AC into topological properties of the natural topology. In this section we collect some results related to this translation problem.

4.1. Lemma. Let $(G, \boldsymbol{G})$ generate $\mathcal{M}$: $\boldsymbol{G}_{\text{nat}}$ is a P-space, if and only if every function $f: \omega \to \mathcal{M}$ of V is in $\mathcal{M}$.

Proof. If $\boldsymbol{G}_{\text{nat}}$ is a P-space, G_δ sets are open by the definition. So if $f: \omega \to \mathcal{M}$ is a function, then $\operatorname{sym} f = \bigcap_{n\in\omega} \operatorname{sym} f(n) \in \boldsymbol{G}_{\text{nat}}$ and $f \in \mathcal{M}$. On the other hand, if $\pi \in \bigcap_{n\in\omega} \pi_n \operatorname{sym} x_n = \bigcap_{n\in\omega} \pi \operatorname{sym} x_n$, then the function $f: \omega \to \mathcal{M}, f(n) = x_n$ is in $\mathcal{M}$ by our assumption, whence $\bigcap_{n\in\omega} \pi \operatorname{sym} x_n = \pi \operatorname{sym} f \in \boldsymbol{G}_{\text{nat}}$. □

If every function $f \in \mathcal{M}^\omega$ is in $\mathcal{M}$, then $\mathcal{M}$ satisfies the axiom DC of dependent choice.

4.2. Example. A permutation model $\mathcal{M}_4$ and a group $(G, \boldsymbol{G})$ generating $\mathcal{M}$ such that $\mathcal{M}_4 \models RA + AC^{\text{wo}}$ (AC for wellorderable families which implies DC) and such that $\boldsymbol{G}_{\text{nat}}$ is metrisable but not discrete—and therefore not a P-space.

Proof. $SM = SM_1$, U is ordered like $\mathbb{R}$, G the group of all order preserving maps, I generated by the intervals $(-\infty, n)$, $n \in \omega$, $\mathcal{M}_4 = P(G, \mathcal{F}_I)$. Since $U \in \mathcal{M}_4$, $\mathcal{M}_4 \models RA$. To prove that $\mathcal{M}_4 \models AC^{\text{wo}}$ see Levy [7]. ($AC^{\text{wo}} \Rightarrow DC$ is due to Jensen - c.f. Felgner [3]). Since I is generated by a countable set, $\boldsymbol{G}_{\mathcal{F}_I} = \boldsymbol{G}_{\text{wo}} = \boldsymbol{G}_{\text{nat}}$ is first countable and hence metrisable, but it is not discrete, whence P fails. □

4.3. Example. A finite support model $\mathcal{M}_5$ and group G such that $(G, \boldsymbol{G}_{\text{fin}})$ generates $\mathcal{M}_5$, $\mathcal{M}_5 \models AC$ but $\boldsymbol{G}_{\text{fin}} = \boldsymbol{G}_{\text{nat}}$ is not P.

Proof. Let $SM = SM_2$, $G = S(U)$, $I = [U]^{<\omega}$ and $\mathcal{M}_5 = P(G, \mathcal{F}_I)$. Then $x \subseteq U$ is a set in $\mathcal{M}_5$, if and only if x is finite, whence $\mathcal{M}_5 \models AC$ (see Felgner, Jech [4]) and $\boldsymbol{G}_{\text{fin}} = \boldsymbol{G}_{\text{wo}} = \boldsymbol{G}_{\text{nat}}$ cannot be P. □

4.4. Lemma. Let $(G, \boldsymbol{G})$ generate $\mathcal{M}$, where G is Abelian and $\mathcal{M} \models RA$.

(i) If $\mathcal{M} \models AC^{\omega}$ (countable choice), then $\boldsymbol{G}_{\text{nat}}$ is P.

(ii) If $\mathcal{M} \models AC^{\text{wo}}$, then $\mathcal{M} \models AC$.

Proof. We first show:

Claim. Let $(G, \boldsymbol{G})$ generate $\mathcal{M}$, and let $\mathcal{M} \models RA$. $\mathcal{M} \models AC^{\omega}$ if and only if for all $f: \omega \to \mathcal{M}$, $f \in V$, and all $\boldsymbol{H} \lhd G$, $\boldsymbol{H} \in \boldsymbol{G}_{\text{nat}}$, there are permutations $\pi_n \in \boldsymbol{H}$ such that $\bigcap_{n\in\omega} \text{sym}\,(\pi_n f(n)) \in \boldsymbol{G}_{\text{nat}}$.

Proof of Claim. Since $\mathcal{M} \models RA$, we can choose $\boldsymbol{H}_1 \lhd \boldsymbol{H}$, $\boldsymbol{H}_1 \in \boldsymbol{G}_{\text{nat}}$, so small that an RA-function $M = \langle M_\alpha : \alpha \in On \rangle$ $\left(\mathcal{M} = \bigcup_{\alpha\in On} M_\alpha\right)$ has the property that $\text{sym}_G M \supseteq \boldsymbol{H}_1$. Then $\text{orb}_{H_1} x$ is a set for all $x \in \mathcal{M}$. If $f \in \mathcal{M}^{\omega}$, define $F: \omega \to \mathcal{M}$ by $F(n) = \text{orb}_{H_1} f(n)$. Then $\text{sym}_G F \supseteq \boldsymbol{H}_1$; so $F \in \mathcal{M}$. By AC^{ω} there is a $g \in \mathcal{M}$ such that $g(n) \in F(n)$ for all $n \in \omega$. Choosing (in V) $\pi_n \in \boldsymbol{H}_1 \subseteq \boldsymbol{H}$ such that $g(n) = \pi_n f(n)$, we get

$$\bigcap_{n\in\omega} \text{sym}\,(\pi_n f(n)) = \text{sym}\, g \in \boldsymbol{G}_{\text{nat}}\,.$$

Conversely, if $F = \langle F_n : n \in \omega \rangle$ is a sequence of nonempty sets with $\text{sym}_G F \supseteq \boldsymbol{H}$ we choose in V, $f(n) \in F_n$ and set $g(n) = \pi_n f(n)$, $\pi_n \in \boldsymbol{H}$ from the claim. Then $g(n) \in F_n$ and $g \in \mathcal{M}$ is a choice function.

Since $\text{sym}\, \pi_n f(n) = \pi_n(\text{sym}\, f(n))\pi_n^{-1}$, if G is Abelian, it follows from the claim that $\bigcap_{n\in\omega} \text{sym}\, f(n) \in \boldsymbol{G}_{\text{nat}}$; so $f \in \mathcal{M}$ and $\boldsymbol{G}_{\text{nat}}$ is a P-space (4.1), proving (i). For (ii) we prove a similar claim for AC^{wo}, from which it follows that SM-indexed intersections of open sets are open, whence $\mathcal{M}^{\alpha} \subseteq \mathcal{M}$ for $\alpha < |SM|^V$ (c.f. 4.1.) and so $X \approx |X|^V$ for $X \in \mathcal{M}$. Here we use $|SM|^V = On^{SM}$ (which also follows from RA). □

One usually writes FMS for the sentences which are true in all permutation models (Fraenkel-Mostowski-Specker models); e.g. $FMS \models$ $\models PW$ (H. Rubin's axiom that the power set of an ordinal is wellorderable. $NBG \models PW \Leftrightarrow AC$, see [9], but $PW \not\Rightarrow AC$ in FMS) and $FMS \models AC_{\text{fin}} \Rightarrow AC_{\text{wo}}$ (Howard [6]; $AC_{\text{fin}}/AC_{\text{wo}}$ the AC for families of finite/wellorderable sets; in NBG this does not hold—see Pincus [8]). If one writes $FMSAb$ for permutation models which are generated by Abelian groups, then 4.4 says $FMS\,Ab \models RA + AC^{\omega} \Rightarrow DC$. This is not true for FMS as was shown by Jensen—see [3]. Example 2.5 shows that $FMSAb \not\models RA + AC^{\omega} \Rightarrow AC$. Also from 4.4, $FMSAb \models$ $\models RA + AC^{\text{wo}} \Rightarrow AC$. Example 4.2 shows that this for is not true FMS.

We next derive another FMS theorem, whose proof relates to the methods developed here. We do not know if it is true in NBG^0. MC is the set form of CMC due to Levy ($MC \Rightarrow PW$, c.f. [9]).

4.5. THEOREM. $FMS \models AC^{\omega} + MC \Leftrightarrow AC$.

PROOF. «$\Leftarrow$» is clear and for «$\Rightarrow$» we need only AC^{ω}_{fin} (countable choice for finite sets). We prove Levy's form $WO\,5$ of AC: For every set X there is an $n \in \omega$ such that $X = \cup\{F_\alpha\colon \alpha \in \varkappa\}$ for some function $F = \langle F_\alpha\colon \alpha \in \varkappa\rangle$, $\varkappa \in On$, such that $|F_\alpha| \leqslant n$; see Rubin and Rubin [9]. Assume not and let X be a counterexample in the permutation model $\mathcal{M}$. By MC, $X = \bigcup_{\alpha\in\varkappa_1} F^{(1)}_\alpha$ and $X^{\omega} = \bigcup_{\alpha\in\varkappa_2} F^{(2)}_\alpha$, $F^{(i)}_\alpha$ finite and $\varkappa_i \in On$. Let $H \lhd \operatorname{sym}\langle F^{(1)}, F^{(2)}\rangle$, $H \in \mathbf{Aut}\,\mathcal{M}_{\text{nat}}$, support these functions. Since $WO5$ is false for X, for each $n \in \omega$ there is an $x \in X$ such that $|\operatorname{orb}_H x| > n$; but for $x \in F^{(1)}_\alpha$, $|\operatorname{orb}_H x| \leqslant |F^{(1)}_\alpha|$ is finite. Using AC in V we choose a sequence $Q = \langle O_n\colon n \in \omega\rangle$ of these orbits such that $|O_n| > n$ (since $\operatorname{sym} Q \supseteq H$, $Q \in \mathcal{M}$). By AC^{ω}_{fin} there is a choice function $f \in X^{\omega}$; $f(n) \in O_n$. Since $f \in F^{(2)}_\alpha$, for some $\alpha \in On$, $m = |\operatorname{orb}_H f| \leqslant |F^{(2)}_\alpha|$ is finite. If $n > m$, then $G\colon \operatorname{orb}_H f \to O_n = \operatorname{orb}_H f(n)$, $G(g) = g(n)$ is in $\mathcal{M}$ ($\operatorname{sym} G \supseteq H$) and G is an onto function. Hence $n < |O_n| \leqslant |\operatorname{orb}_H f| = m$, a contradiction. □

Also, $FMS \models AC^{\omega}_{fin} + CMC \Rightarrow CAC$. This follows from 3.3, since compact P-spaces are discrete (P-property: Let $H \lhd \operatorname{Aut}\mathcal{M}$ be compact and $G = \cap\{\operatorname{sym}_H x_n\colon n \in \omega\}$ a $G\delta$. By AC^{ω}_{fin} $X = \cup\{\operatorname{orb} x_n\colon n \in \omega\}$ is well orderable, hence $G \supseteq H \cap \operatorname{fix} X$ is open).

We conclude this paper with the observation that the concept of a permutation model is a geometric one, since it depends on the representation of the generating group, as is shown next.

4.6. EXAMPLE. There are topologically and algebraically isomorphic groups generating finite support models in the same standard model SM which are not elementarily equivalent.

PROOF. U is countable, $SM = SM_1$ and $\mathcal{M}_6$ is the Fraenkel-Halpern model generated from $G = S(\omega)$ with the finite topology (topology of pointwise convergence). In SM we can index U as

$$U = \{u_{n,m}\colon (n, m) \in \omega^2\}.$$

We set

$$H = \{g \in S(U)\colon \exists h \in S(\omega), \forall n, m \in \omega\colon gu_{n,m} = u_{n,h(m)}\}.$$

H with the topology of pointwise convergence generates $\mathcal{M}_7$, $\boldsymbol{H} = \boldsymbol{H}_{\text{fin}}$. Then G and H are topologically and algebraically isomorphic. But in $\mathcal{M}_6$ U is amorphous (infinite subsets are cofinite), while in $\mathcal{M}_7$ U is Dedekind-infinite. Since U is defined by the NBG^0 sentence $U = \{x\colon x = \{x\}\}$, $\mathcal{M}_6$ and $\mathcal{M}_7$ are not elementarily equivalent. □

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