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Combining Stability with Symmetry Properties in Bifurcation Problems.

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0. Introduction.

It is well known that the existence of bifurcation phenomena is intimately connected to stability properties of the system [1, 9, 12-16]. Loosely speaking, a typical result is that a transition from asymptotic stability to complete instability, produced by some small perturbation, corresponds to the appearance of a stable bifurcation. We refer to [12-15] for any detail about these ideas; we will adopt also the main definitions and notations from these references.

In many cases, the problem under consideration exhibits a « covariance » property under some symmetry group G, usually deriving from precise physical properties (see e.g. [6, 10, 11, 17-20]). We will try to combine the above mentioned approach to bifurcation theory with some ideas taken from group-theoretical arguments. We shall show that some refinements in the results can be obtained in this way; we will present also some explicit examples as illustration of the possible applications.

1. Preliminary group-theoretical statements.

In order to be as general as possible, we shall start by giving the definition of « covariance » for a generic dynamical system

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$u = u(t, t_0, u_0)$, where $u: I \times I \times R^n \to R^n$, and $I = R$ or Z (respectively for continuous or discrete systems):

DEFINITION 1. The dynamical system $u = u(t, t_0, u_0)$ is said to be covariant with respect to a topological group G if there is a real continuous representation D of G, acting on R^n, such that, for all $g \in G$,

$$u(t, t_0, D(g)u_0) = D(g)u(t, t_0, u_0) \qquad (u, u_0 \in R^n;\ t, t_0 \in I)\,. \tag{1}$$

In this Definition, D may be reducible or not; the only assumption we shall make for convenience is that, if D is reducible, it is also completely reducible (of course, this is guaranteed if G is finite or compact). In order to avoid confusion, we shall call t-orbit of a point $u_0 \in R^n$ the usual « dynamical » notion of orbit (with initial condition $u(t_0) = u_0$), and G-orbit of $u \in R^n$ the set

$$\{v \in R^n: v = D(g)u,\ g \in G\}\,.$$

Similarly, one can introduce the notion of G-orbit of any subset of R^n. Note that, in view of (1), time action commutes with group action. One could give also to time variations $t \to t + t'$ a group-theoretical meaning: e.g. in periodic problems time variation can be viewed equivalent to the action of the circle group S^1 [8, 19]. Apart from the last lines of this paper, we will not consider here this type of « internal » covariance.

Given a point $u \in R^n$, let G_u denote the isotropy subgroup (or little-group) of u, i.e. the set of those $g \in G$ for which $D(g)u = u$. From (1), it is clear that if $g \in G_{u_0}$, then $g \in G_u$, with $u = u(t, t_0, u_0)$, for all $t \in I$; and also that—in general—the isotropy subgroup of any element u cannot be changed (i.e. enlarged) with the time. Similarly, when the symmetry is described by a continuous Lie group, one cannot expect that a t-orbit of a generic point u_0 is asymptotically stable: in fact, the t-orbit of a point near u_0 and belonging to its G-orbit (apply an infinitesimal transformation of G to u_0), will not—in general—approach arbitrarily near the t-orbit of u_0. Therefore, we can expect asymptotic stability only—at most—for G-orbits.

If the dynamical system is defined by means of differential equations:

$$\dot{u} = f(t, u) \tag{2}$$

the above Definition 1 becomes, with the same notations,

$$D(g)\dot{u} = f\big(t, D(g)u\big) = D(g)f(t, u)\,. \tag{3}$$

In autonomous problems, if one is interested in stationary solutions ($\dot{u} = 0$), the last property of f in (3) is the usual definition of covariance for nonlinear maps (see [3, 4, 6, 17]).

In the following, we shall consider autonomous problems, expressed by families of ordinary differential equations, depending on a real parameter λ, of the type

$$\dot{u} = f(\lambda, u) \qquad u \in R^n,\ \lambda \in R \tag{4}$$

where $f\colon R \times R^n \to R^n$ and $f(\lambda, 0) = 0$. We shall assume for simplicity that f is an analytic function: the possible changes in the results below can be easily devised if this hypothesis is modified (see [12-15]). Another important simplification we shall adopt is the following: we assume that the problem (4) is already restricted in such a way that at the critical point $\lambda = \lambda_0$ (we choose $\lambda_0 = 0$) all eigenvalues of the Jacobian $\partial_u f(0, 0)$ have zero real part. This is not a severe restriction, in fact: i) Covariance property is inherited, as well known [4, 17], from the general to this restricted case; ii) Existence theorems are easily extended to the general case by means of standard procedures (Lyapunov-Schmidt, invariant center manifold); iii) All facts concerning stability remain substantially valid in the general case with the additional hypothesis that all other eigenvalues have negative real part [1, 9, 15] (see however also [21] and ref. therein).

The main assumption concerning symmetry properties of our system (4) can be stated in the following form:

(G) Let the system (4) be covariant under a group G, acting on R^n through a representation D, and assume that there exists a nontrivial subgroup H of G such that the subspace $X \subset R^n$ of all vectors which are left fixed by H is not trivial (i.e. $X \neq 0$ and $X \neq R^n$):

$$Tx = x \qquad \text{where } T \equiv D(h),\ h \in H,\ x \in X\,. \tag{5}$$

This means, in other words, that in the decomposition of the representation D into subrepresentations of H, one or more trivial (identity) representations appear. The simplest case occurs when there is just one trivial representation of H: then X is one-dimensional, H its isotropy subgroup, and (G) reads:

(G_1) There is in R^n a special direction x such that the subspace of all vectors in R^n which are left fixed by the isotropy subgroup $G_x = H$ of x is spanned just by x.

The basic result is the following.

LEMMA 1. If (G) is verified, one has for the restriction $f|_X$ that

$$f|_X: R \times X \to X$$

then one can consider the restricted problem

$$\dot{x} = f(\lambda, x)\,; \qquad x \in X \tag{6}$$

and if $x(t_0) \in X$, then $x(t) \in X$ for all t.

PROOF. From covariance and (G):

$$f(\lambda, x) = f(\lambda, Tx) = T \cdot f(\lambda, x)$$

which implies, by (G), $f(\lambda, x) \in X$.

This very simple result is the basis of some theorems on bifurcation theory in the presence of symmetry (see [3-8, 10, 11, 17-20]); the situation is very interesting also from a group-theoretical point of view [8, 18, 20]. Assumption (G) can be put in variously modified or enlarged ways, e.g.:

($\tilde{G}$) Assuming covariance of eq. (4) under a group G with a representation D, let H be a subgroup of G such that decomposing D into direct sum of subrepresentations of H:

$$\begin{cases} D = T_0 \oplus T_1 \oplus \ldots \oplus T_r & T_i = T_i(h)\,, \quad h \in H\,, \\ & \qquad\qquad i = 0, 1, \ldots, r \\ R^n = X_0 \oplus X_1 \oplus \ldots \oplus X_r & T_i: X_i \to X_i \end{cases} \tag{7}$$

there is one subrepresentation, say T_0, with the property that all tensor products

$$T_0^{\otimes p} \quad (p = 1, 2, ...)$$

do not intertwine with the other representations $T_1, ..., T_r$ appearing in the above decomposition (7).

The same conclusion of Lemma 1 follows also (with $X = X_0$) from $(\tilde{G})$: it suffices to write $f(\lambda, x)$ as an expansion of p-linear terms, which transform under H according to $T_0^{\otimes p}$ Note also that (G) clearly satisfies $(\tilde{G})$, being T a sum of trivial representations of H.

2. Stability and symmetry arguments.

Together with the above group-theoretical arguments, we shall refer to the following assumption, of entirely different nature:

(*S*) Assume that for $\lambda = \lambda_0 = 0$ the solution $u \equiv 0$ of the given dynamical system (4) is asymptotically stable, but for $\lambda > 0$ it becomes completely unstable (i.e. asymptotically stable in the past).

It is known that (*S*) implies, in a very precise way, the appearance of a bifurcation [1, 12-15]. We want finally show that combining both types of assumptions some hints in the problem of finding bifurcations can be obtained.

Precisely, we distinguish two cases (Propositions 1 and 2, respectively).

PROPOSITION 1. Let (4) be covariant with respect to a group G, and let (G) (or $(\tilde{G})$) be verified. Assume that (*S*) is *not* verified in R^n by the original problem (4), but it is verified by its restriction (6) to X. Then:

(i) There is a bifurcation for $\lambda > 0$ of eq. (4), lying in X;

(ii) The bifurcated set is asymptotically stable with respect to initial conditions belonging to X;

(iii) The bifurcated set is stable (in general, not asymptotically stable) with respect to initial data belonging to the G-orbit of the set itself (in this case, it is assumed that G is a continuous Lie group);

(iv) All the G-orbit of the bifurcated set describes a bifurcation set for eq. (4), which is asymptotically stable with respect to initial data belonging to the G-orbit of X.

PROOF. Straightforward, after the above preparation. Note in particular that it was precisely assumption (G) $\big($or $(\tilde{G})\big)$ that allowed the possibility that property (S) holds in X, even if it is not true in R^n; so a bifurcation with properties (i) and (ii) exists. For (iii), let $x(t_0) \in X$ and $u_g(t_0) = D(g)x(t_0)$ be a point in the G-orbit of $x(t_0)$ near $x(t_0)$, therefore $x(t) \in X$ and $u_g(t) = D(g)x(t)$. Then, the simple inequalities

$$\gamma\|x(t)\| \leqslant \|u_g(t) - x(t)\| \leqslant \Gamma\|x(t)\|$$

where γ and Γ depend only on G and X (and, in general, $\gamma > 0$ unless $g \in H$) show (iii) if $x(t_0)$ belongs to the (bounded) bifurcation set. Finally, (iv) is an easy consequence of covariance, which gets

$$\frac{d}{dt}\big(D(g)x\big) = f\big(\lambda, D(g)x\big)$$

and ensures that all stability properties hold in $D(g)X$ exactly as in X.

Example 1 below will illustrate these results. Clearly, as a consequence of the fact that (S) was not assumed to hold in the whole R^n, one cannot expect asymptotic stability for the bifurcation set for generic initial data. Nevertheless, for the practical point of view, the above results could be equally important, if there is some symmetry constraint (e.g. of physical nature) which can confine the possible initial conditions to the restricted subspace X $\big($or $D(g)X\big)$. It can be noted also that it frequently happens that, when (G) holds, there is *no* bifurcation other than the one lying in X (and its G-orbit), discussed here.

The next result concerns the case that property (S) is already verified by system (4) in the whole space R^n: therefore, the existence of a bifurcated set is guaranteed. Including a group-theoretical hypothesis can get some additional amount of information. First, one can obtain a better localization of the bifurcation, and then possibly deduce some indications about the existence of other solutions. We get (the proof is now completely natural):

PROPOSITION 2. Assume that property (S) is verified by the problem (4) in R^n; assume also covariance and that property (G) $\big($or $(\tilde{G})\big)$ holds.

Then, there exists a bifurcated set in X, asymptotically stable with respect to initial conditions belonging to X, and:

(i) if this set, together with its G-orbit, is the unique bifurcated set of (4), then it is asymptotically stable also for generic initial conditions;

(ii) if the bifurcated set found in X, with its G-orbit, is not asymptotically stable for generic initial data, then there is at least another bifurcation.

3. Two explicit examples.

EXAMPLE 1. This is an example for Proposition 1. Let us consider the space R^9 of real 4×4 symmetric traceless matrices u, and the group $G = SO_4$ acting on this space according to the rule (which describes an irreducible representation of G)

$$u \to gug^t\,, \qquad g \in G. \tag{8}$$

A G-invariant norm is given by

$$\|u\|^2 = \mathrm{Tr}(u^2)\,. \tag{9}$$

Consider now the following G-covariant problem

$$\dot{u} = \lambda u + \partial_u(\det u)\,. \tag{10}$$

Using the covariance under (8), one can first greatly simplify the problem by transforming the matrices u into a diagonal form (this simply amounts to take a convenient point in the G-orbit), and choosing the following basis for the subspace of matrices of this form

$$e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \qquad e_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & 0 \end{bmatrix},$$

$$e_3 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{bmatrix}.$$

Considering the Lyapunov function, with $u = u_1 e_1 + u_2 e_2 + u_3 e_3$,

$$V = \tfrac{1}{2}\|u\|^2 = \tfrac{1}{2}(u_1^2 + u_2^2 + u_3^2)$$

one has for its derivative along the solutions of (10)

$$\dot{V} = \lambda\|u\|^2 + 4 \det u$$

which shows, according to Lyapunov stability criteria, that the solution $u = 0$ of (10) is completely unstable for $\lambda > 0$, but for $\lambda = \lambda_0 = 0$ the quantity $\dot{V}$ can be $\gtrless 0$, and condition (S) is not verified. However, note that the vector e_3 satisfies (G_1), whereas neither e_1 nor e_2 satisfy it; this in fact makes possible the restriction to the 1-dimensional subspace X spanned by e_3, and one sees that, being now $\det x = \det(u_3 e_3) = -u_3^4/48$, the condition (S) is satisfied in X, and all assumptions of Proposition 1 are verified. The bifurcation one finds is the stationary solution of (10) given by

$$x = u_3 e_3 \quad \text{with } u_3^2 = 12\lambda$$

and direct calculations can confirm all other conclusions.

EXAMPLE 2. An example for the case (i) of Proposition 2 could be easily constructed: observe in fact that a possible situation where it occurs is clearly given if there is only one type (apart from 0) of orbits under G; i.e. when $D(g)X$ fill the whole space R^n for any X. This is the case e.g. if one chooses $G = SO_n$, operating on R^n through its fundamental representation, and X any 1-dimensional subspace.

We construct then the following example for the case (ii) of Proposition 2. Let R^5 be the space of real 3×3 symmetric traceless matrices u, and let $G = SO_3$ act irreducibly on this space according to the same rule as in eq. (8). The following equation

$$\dot{u} = \lambda u - \partial_u\big(\det(u^2)\big) - u\big(\mathrm{Tr}(u^2)\big)^2 \tag{11}$$

is covariant with respect to this group action. As in example 1, let us put matrices, using covariance, in diagonal form, and choose

$$e_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}, \qquad e_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix}.$$

From the Lyapunov function

$$V = \tfrac{1}{2}\|u\|^2 = \tfrac{1}{2}(u_1^2 + u_2^2)\,, \qquad u = u_1 e_1 + u_2 e_2$$

one obtains

$$\dot{V} = \lambda\|u\|^2 - 6(\det u)^2 - \|u\|^6$$

which shows that condition (S) is satisfied in any case for $\lambda > 0$. Now, one can see that the vector e_2 satisfies condition (G_1) (but not e_1), therefore, the problem can be restricted to the one-dimensional subspace X generated by e_2. In fact, one finds the stationary bifurcating solution

$$x = u_2 e_2 \qquad \text{with } u_2 = (9\lambda/10)^{\frac{1}{4}}$$

but it is easily seen that this solution (with its G-orbit) is not stable. This implies that another (asymptotically stable) solution must exist: there is in fact the stationary solution

$$u = u_1 e_1 \qquad \text{with } u_1 = \lambda^{\frac{1}{4}}$$

with the expected stability properties. One can also see that, apart from these two solutions (and their G-orbits), problem (11) does not admit other bifurcations.

4. An important particular case.

The situation considered in this section can be viewed, in a sense, as a particular case of Propositions 1 and 2, but it deserves a separate statement for its special interest. For simplicity, we shall consider only the case similar to that covered by Proposition 2 (there is no difficulty in dealing with the other possibility). The group-theoretical assumption is now the following:

(G_2) Let R^n be even dimensional ($n = 2m$) and let the group G describing the covariance of the system act on R^n through a representation D which splits into the direct sum of two equivalent irreducible representations:

$$D = D_1 \oplus D_2 \qquad (D_1 \simeq D_2) .$$

Denoting by Y_i the m-dimensional subspace acted upon by D_i ($i = 1, 2$; $Y_1 \oplus Y_2 = R^n$; $Y_1 \simeq Y_2$), assume that there is a vector x_1 in Y_1 (and then a vector x_2 in Y_2) satisfying (G_1) with respect to the subrepresentation D_1 of G.

Assumption (G_2) implies that the original problem (4) can now be reduced to a 2-dimensional one by restricting the space R^n to the subspace X generated by x_1 and x_2. Actually, this situation has been already considered (especially from the group-theoretical point of view) in [8]. Restricting to the subspace X allows us the resort to all known theorems on R^2. For instance, we get:

Proposition 3. (i) Assume properties (G_2) and (S); then, if eq. (4) has no singular points other than the origin $u = 0$ (at least in some neighbourhood of $u = 0$, $\lambda = \lambda_0 = 0$), there are bifurcating sets, which are G-orbits of annular regions (possibly cyclic t-orbits) contained in X;

(ii) Assume (G_2) and that the two eigenvalues (necessarily having multiplicity m) of the Jacobian matrix $\partial_u f(0, 0)$ are complex conjugate and satisfy the usual Hopf transversality condition. Then, there is in X a bifurcating periodic solution of Hopf type, and its G-orbit is made up of periodic solutions. Stability properties of these solutions easily follow from known theorems (see [2, 12-15]) and previous remarks.

As a final remark, concerning (ii) above, we note that, as a consequence of covariance, the matrix $L(\lambda) = \partial_u f(\lambda, 0)$ is forced to commute with D, and the property of D assumed in (G_2) implies that $L(\lambda)$ must have the following form

$$L(\lambda) = \begin{pmatrix} \alpha_1 I_m & \beta_1 I_m \\ \beta_2 I_m & \alpha_2 I_m \end{pmatrix}$$

where α_i, β_i are real functions of λ and I_m is the m-dimensional identity matrix. It is then possible to give conditions on α_i, β_i in order that transversality is satisfied; alternatively, as shown in [8], one can impose an additional circle-symmetry $S^1 = SO_2$ to the problem (corresponding to the time translations $t \to t + t'$, mod 2π), and the same result is obtained.

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