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On the Minima of Functionals with Linear Growth.

G. ANZELLOTTI (*)

SUMMARY - We study the trace on the singular support of Du of various geometrically meaningful vectorfields associated to u, where $u \in BV(\Omega)$ is a minimum point for a functional with linear growth. For example, we consider the vector field $N = Du/\sqrt{1 + |Du|^2}$ and its averages $N_\varrho(x)$ in balls of radius ϱ and center x, and we show that one has $N_\varrho \to Du/|Du|$ in integral mean with respect to the measure $|Du|$, in the zone where Du is singular. This means that the vector $Du/\sqrt{1 + |Du|^2}$ is in some weak sense continuous across the singularities of u.

0. Introduction.

Let Ω be an open bounded set in $\mathbf{R}^n$, $n \geqq 2$, assume that $\partial\Omega$ is Lipschitz continuous and denote by $\nu_\Omega(x)$ the outward normal to $\partial\Omega$ at x.

We shall consider non-parametric integrands $f(x, p)$: $\bar{\Omega} \times \mathbf{R}^n \to$ $\to [0, +\infty)$ such that

(0.1) $\quad f$ is convex in p for each fixed $x \in \bar{\Omega}$

(0.2) $\quad |p| \leqq f(x, p) \leqq M(1 + |p|) \qquad x, p \in \bar{\Omega} \times \mathbf{R}^n$

(0.3) $\quad$ for any $\varepsilon < 0$ there exists a number $\delta < 0$ such that for all $x, y \in \bar{\Omega}$ with $|x - y| < \delta$ one has $|f(x, p) - f(y, p)| \leqq \varepsilon\sqrt{1 + |p|^2}$ for all $p \in \mathbf{R}^n$

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and for each such f we set

$$f^0(x, p) = \lim_{t \to 0^+} f\left(x, \frac{p}{t}\right) t \tag{0.4}$$

For any function $u \in BV(\Omega)$ we consider the Lebesgue decomposition $Du = (Du)^a(x)\,dx + (Du)^s$ and the vector field $\nu_u(x) = (Du/|Du|)(x)$. If $u \in BV(\Omega)$ we set

$$f(x, Du) = \int_\Omega f(x, (Du)^a(x))\,dx + \int_\Omega f^0(x, \nu_u(x))\,|Du|^s$$

and we consider the functional [11], [7], [1]

$$F(u) = \int_\Omega f(x, Du) + \int_\Omega H(u)u(x)\,dx + \int_{\partial\Omega} f^0(x, [g(x) - u(x)]\,\nu_\Omega(x))\,d\mathcal{H}^{n-1} \tag{0.5}$$

where $H \in L^n(\Omega)$ is given and $g \in L^1(\partial\Omega)$ is the trace on $\partial\Omega$ of a function $g^* \in W^{1,1}(\Omega)$ which is also given. Here we shall make use of the notation and the results in [5], where one can find also some further bibliographic references on the subject.

Under suitable conditions for H [10], [13], [1], there exist solutions $u \in BV(\Omega)$ to the problem

$$\begin{cases} F(u) \to \min \\ u \in BV(\Omega), \end{cases} \tag{0.6}$$

and it is a natural question to ask whether these solutions enjoy some regularity properties. In a few cases [12], [14], [9], [11] the answer is positive, in most cases the answer is not known. In particular, it must be noticed that many functionals of the type considered may have minima which are discontinuous along a $(n-1)$-dimensional surface in Ω. For example, consider the set $\Omega = \{x \in \mathbb{R}^2 : |x| < 2\}$ and the functional

$$F(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_\Omega Hu + \int_{\partial\Omega} |u - g|\,d\mathcal{H}^{n-1} \tag{0.7}$$

where we take $g \equiv 0$ and

$$H(x) = \begin{cases} 0 & \text{if } 1 < |x| < 2 \\ -2 & \text{if } |x| < 1. \end{cases}$$

Then the function

$$u(x) = \begin{cases} c \log \dfrac{2+\sqrt{3}}{|x| + \sqrt{|x|^2 - 1}} & \text{if } 1 < x < 2 \\ 2 + \sqrt{3} + a + \sqrt{1 - |x|^2} & \text{if } |x| < 1 \end{cases}$$

where a is any fixed positive number, is a solution to problem (0.6) for the functional (0.7), and it has a jump of height a along the circle $|x| = 1$. On the other hand, it is easily seen that the vector $(Du/\sqrt{1 + |Du|^2})(x)$, which is defined for $|x| \neq 1$, and can be viewed as the projection on the base of the unit normal to the graph of u, can be continuously extended across the discontinuity of u, where it coincides to the normal to the jump.

This fact is not just a coincidence. In fact, under suitable quite general assumption for f, we shall show that if u is a solution to problem (0.6) then the averages

$$N_\varrho(x) = \fint_{B_\varrho(x)} \frac{Du}{\sqrt{1 + |Du|^2}}(\xi)\, d\xi$$

converge to $(Du/|Du|)(x)$ in $L^p(\Omega, |Du|^s)$ for $1 \leqq p < +\infty$. In section 3 we observe that if one takes the averages of $(Du/\sqrt{1 + |Du|^2})$ on suitable cylinders, one has a convergence $|Du|^s$-almost everywhere in Ω.

The results stated above are obtained through similar trace results for the vector field $f_p(x, (Du)^a(x))$ (theorems 2.4, 2.7, 3.1, 3.2) which hold under even weaker assumptions.

Similar trace results hold at the boundary of $\partial\Omega$, in the zone where the minimum u does not attain the prescribed value g.

In the appendix we discuss various examples that illustrate the different results obtained in sections 2 and 3.

The main tools that we shall use in the paper are the Euler equation derived in [4] and the divergence theorem (fact 1.1 below) proved in [3], [5]. We point out that a similar approach has been used in [2] to study the energy functional in Hencky plasticity.

1. First results.

We shall make use of the following assumptions

(1.1) for each fixed $x \in \bar{\Omega}$, $f(x, p)$ is differentiable in p at all $p \in \mathbf{R}^n$, and $f^0(x, p)$ is differentiable in p at all $p \neq 0$.

(1.2) there exists a continuous function $\omega(t): \mathbf{R}^+ \to \mathbf{R}^+$ such that $\lim_{t\to 0^+} \omega(t) = 0$, and for all $x, y \in \bar{\Omega}$ one has

$$|f_p(x, \alpha) - f_p(y, \alpha)| \leqq \omega(|x - y|) \qquad \forall \alpha \in \mathbf{R}^n$$

$$|f_p^0(x, \alpha) - f_p^0(y, \alpha)| \leqq \omega(|x - y|) \qquad \forall \alpha \in \mathbf{R}^n, \ \alpha \neq 0$$

For each $x \in \bar{\Omega}$ we consider the set

$$K_x = \{f_p(x, \alpha) : \alpha \in \mathbf{R}^n\}$$

In view of assumption (0.1), from the standard theory of convex functions [15], [8], it is known that K_x is convex and one has

$$\partial K_x = \{f_p^0(x, \alpha) : \alpha \in \mathbf{R}^n, \ \alpha \neq 0\} \tag{1.4}$$

moreover, for each x, for all $\alpha \in \mathbf{R}^n$, $\alpha \neq 0$ one has

$$K_x \subset \{\xi \in \mathbf{R}^n : \xi \cdot \alpha \leqq f_p^0(x, \alpha) \cdot \alpha\} \tag{1.5}$$

i.e. the set $\{\xi : \xi \cdot \alpha = f_p^0(x, \alpha) \cdot \alpha\}$ is a supporting hyperplane of K_x at its boundary point $f_p^0(x, \alpha)$. Form assumption (0.2) it follows that for each $x \in \bar{\Omega}$ one has

$$\sup_{\alpha \in K_x} |\alpha| \leqq M . \tag{1.6}$$

From assumption (1.2) it follows that for all $x, y \in \bar{\Omega}$ one has

$$\operatorname{dist}(K_x, K_y) \leqq \omega(|x - y|) \tag{1.7}$$

where

$$\operatorname{dist}(K_x, K_y) = \max\left\{\sup_{\alpha\in K_x} \operatorname{dist}(\alpha, K_y),\ \sup_{\beta\in K_y} \operatorname{dist}(K_x, \beta)\right\}$$

in fact, for any $\alpha \in K_x$ one has $\alpha = f_p(x, \xi)$ for some $\xi \in \mathbf{R}^n$ and $\operatorname{dist}(\alpha, K_y) \leq \operatorname{dist}(f_p(x, \xi), f_p(y, \xi)) \leq \omega(|x-y|)$, and similarly for $\beta \in K_y$.

For later reference, we recall the following result, which is proved in [2: section 1 and theorem 2.4], [5: theorems 2.2 and 3.6].

FACT. 1.1. *If $\psi \in L^\infty(\Omega, \mathbf{R}^n)$ and $\operatorname{div}\psi \in L^n(\Omega)$, then there exist a real valued function*

$$[\![\psi\cdot\alpha]\!]\,(x)$$

defined for (x, α) belonging to some set $G \in \Omega\times S^{n-1}$ and a function

$$[\psi\cdot\nu_\Omega]\,(x) \in L^\infty(\partial\Omega, \mathcal{H}^{n-1})$$

such that for any function $u \in BV(\Omega)$ the function

$$x \to [\![\psi\cdot\nu_u(x)]\!]\,(x)$$

is defined $|Du|$-a.e. in Ω, it is $|Du|$-measurable, and one has the integration by parts formula

$$\int_\Omega u(x)\operatorname{div}\psi(x)\,dx = \int_{\partial\Omega}[\psi\cdot\nu_\Omega]\,(x)\,d\mathcal{H}^{n-1} - \int_\Omega[\![\psi\cdot\nu_u(x)]\!]\,(x)\,|Du| \tag{1.8}$$

The function $[\![\psi\cdot\alpha]\!]\,(x)$ is defined by a pointwise limit of averages of ψ as follows

$$[\![\psi\cdot\alpha]\!]\,(x) = \lim_{\varrho\to0^+}\lim_{r\to0^+}\frac{1}{2r\omega_{n-1}\varrho^{n-1}}\int_{C_{r,\varrho}(x,\alpha)}\psi(\xi)\cdot\alpha\,d\xi \tag{1.9}$$

where

$$C_{r,\varrho}(x, \alpha) = \{\xi\in\mathbf{R}^n\colon |(\xi-x)\cdot\alpha| < r,\ |(\xi-x)-[(\xi-x)\cdot\alpha]\alpha| < \varrho\}$$

and a similar representation [5: *proposition* 2.2] *holds for $[\psi\cdot\nu_\Omega]\,(x)$.*

Moreover one has

$$(1.10)\qquad [\![\psi\cdot\nu_u(x)]\!] = \psi(x)\cdot\nu_u(x) \qquad |Du|^a\text{-a.e. } in\ \Omega\,.$$

We shall need the following simple property of convex functions [15].

LEMMA 1.2. *Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a differentiable convex function such that*

$$(1.11)\qquad |p| \leqq f(p) \leqq M(1+|p|) \qquad \forall p\in\mathbf{R}^n$$

and set $f^0(p) = \lim\limits_{s\to+\infty} s^{-1}f(sp)$, then one has

$$(1.12)\qquad f^0(\beta) \geqq f_p(\alpha)\cdot\beta \qquad \forall \alpha,\beta\in\mathbf{R}^n, \qquad \beta\neq 0\,.$$

Here is our first result, compare [2].

THEOREM 1.3. *Assume that f satisfies* (0.1), (0.2), (0.3), (1.1), (1.2). *Let $u\in BV(\Omega)$ be a minimum point for F and set*

$$\psi(x) = f_p\big(x, (Du)^a(x)\big)$$

Then one has $\psi\in L^\infty(\Omega,\mathbf{R}^n)$, $div\psi = H\in L^n(\Omega)$ and

$$(1.15)\qquad [\![\psi\cdot\nu_u(x)]\!]\,(x) = f^0\big(x,\nu_u(x)\big) \qquad |Du|^s\text{-a.e. in } \Omega$$

$$(1.16)\qquad \sigma(x)\,[\psi\cdot\nu_\Omega]\,(x) = f^0\big(x,\sigma(x)\nu_\Omega(x)\big)$$
$$\mathcal{H}^{n-1}\text{-a.e. } in\ the\ set\ T = \{x\in\partial\Omega\colon g(x)\neq u(x)\}$$

where $\sigma(x) = \operatorname{sign}\big(g(x)-u(x)\big)$.

PROOF. As u is a minimum point for F, for all the functions $\varphi\in BV(\Omega)$ suc that

$$(1.17)\qquad |D\varphi|^s \ll |Du|^s, \qquad \varphi(x) = 0\ \ \mathcal{H}^{n-1}\text{-a.e. in } T$$

one has [4: theorem 3.7]

$$(1.18)\qquad \int_\Omega f_p\big(x,(Du)^a(x)\big)(D\varphi)^a(x)\,dx + \int_\Omega f_p^0\big(x,\nu_u(x)\big)\cdot\nu_\varphi(x)\,|D\varphi|^s +$$
$$+\int_\Omega H(x)\varphi(x)\,dx + \int_{\partial\Omega} f_p^0\big(x,\sigma(x)\,\nu_\Omega(x)\big)\cdot\nu_\Omega(x)\varphi(x)\,d\mathcal{H}^{n-1} = 0\,.$$

Obviously, conditions (1.17) are satisfied by the function $\varphi = u - g$ and one has $(D\varphi)^s = (Du)^s$ and $\nu_\varphi(x) = \nu_u(x)$ $|Du|^s$-a.e. For the choice $\varphi = g - u$, condition (1.18) becomes

$$\int_\Omega \psi(x)\big(D(u-g)\big)^a(x)\,dx + \int_\Omega f^0\big(x, \nu_u(x)\big)\,|Du|^s + \int_\Omega H(x)\,(u-g)\,(x)\,dx + \int_\Omega f^0\big(x, \sigma(x)\nu_\Omega(x)\big)\,|g-u|\,(x)\,d\mathcal{H}^{n-1} = 0\,.$$

On the other hand, conditions (1.17) are satisfied by all $\varphi \in C_0^\infty(\Omega)$ and yield $\operatorname{div}\psi = H \in L^n(\Omega)$, hence we may apply Fact. 1.1 to the vector field ψ and the function $v - g$ and we get

$$(1.20)\qquad \int_\Omega \psi(x)\big(D(u-g)\big)^a(x)\,dx + \int_\Omega [\![\psi\cdot\nu_u(x)]\!]\,(x)\,|Du|^s + \int_\Omega H(x)(u-g)(x)\,dx + \int_{\partial\Omega} [\psi\cdot\nu_\Omega]\,(x)(g-u)(x)\,d\mathcal{H}^{n-1} = 0\,.$$

Subtracting (1.20) from (1.19) we obtain

$$(1.21)\qquad \int_\Omega \big\{f_0\big(x, \nu_u(x)\big) - [\![\psi\cdot\nu_u(x)]\!]\,(x)\big\}\,|Du|^s + \int_{\partial\Omega} \big\{f^0\big(x, \sigma(x)\nu_\Omega(x)\big) - \sigma(x)\,[\psi\cdot\nu_\Omega]\,(x)\big\}\,|g-u|\,d\mathcal{H}^{n-1} = 0$$

and (1.15), (1.16) will follow as soon as we prove the inequalities

$$(1.22)\qquad f^0\big(x, \nu_u(x)\big) \geqq [\![\psi\cdot\nu_u(x)]\!]\,(x) \qquad |Du|^s\text{-a.e. in } \Omega$$

$$(1.23)\qquad f^0\big(x, (g-u)(x)\nu_\Omega(x)\big) \geqq [\psi\cdot\nu_\Omega]\,(x)(g-u)(x) \qquad \mathcal{H}^{n-1}\text{-a.e. in } T$$

To prove (1.22) we notice that by Lemma 1.2 one has

$$\psi(\xi)\cdot\nu_u(x) = f_p\big(\xi, (Du)^a(\xi)\big)\cdot\nu_u(x) \leqq f^0\big(\xi, \nu_u(x)\big)$$

hence, by assumption (1.2),

$$\int\!\!\!\!\!-_{C_{r,\varrho}(x,\,\nu_u(x))} \psi(\xi)\,d\xi\cdot\nu_u(x) \leqq f^0(x,\nu_u(x)) + \omega\big((\varrho^2+r^2)^{\frac{1}{2}}\big)$$

and (1.22) follows by (1.9). Similarly one gets (1.23). q.e.d.

The requirements on the differentiability in p of the integrand may be relaxed, provided one makes a supplementary assumption on u.

THEOREM 1.4. *Assume that f satisfies* (0.1), (0.2), (0.3) *and the following differentiability conditions*:

(1.24) *for each fixed $x \in \Omega$, $f(x,p)$ and $f^0(x,p)$ are differentiable in p at all $p \neq 0$, moreover, if $f(x,\cdot)$ is not differentiable at $p = 0$, then one has $f(x,0) = 0$.*

(1.25) *there exists a continuous function $\omega(t)\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that $\lim_{t\to 0^+} \omega(t) = 0$ and such that for all $x, y \in \Omega$ and for all $\alpha \in \mathbf{R}^n - \{0\}$ one has*

$$|f_p(x,\alpha) - f_p(x,\alpha)| \leqq \omega(|x-y|)$$

$$|f_p^0(x,\alpha) - f_p^0(y,\alpha)| \leqq \omega(|x-y|)\,.$$

Let $u \in BV(\Omega)$ be a minimum point for the functional F and assume moreover that

(1.26) $$(Du)^a(x) \neq 0 \qquad \mathfrak{L}^n\text{-a.e. in } \Omega$$

then one has again that

(1.27) $$[\![\psi\cdot\nu_u(x)]\!](x) = f^0(x,\nu_u(x)) \qquad |Du|^s\text{-a.e. in } \Omega$$

(1.28) $$\sigma(x)\,[\psi\cdot\nu_\Omega](x) = f^0(x,\sigma(x)\nu_\Omega(x))$$
$$\mathcal{H}^{n-1}\text{-a.e. in the set } T = \{x \in \partial\Omega\colon u(x) \neq g(x)\}\,.$$

PROOF. The proof works exactly as the proof of theorem 1.3, the only difference is that in this case, due to the lesser differentiability of f, the Euler equation [4: theorem 3.9] holds only for the functions

$\varphi \in BV(\Omega)$ such that

$$(1.29)\qquad \begin{cases} |D\varphi|^s \ll |Du|^s & \\ (D\varphi)^a(x) = 0 & \mathcal{L}^n\text{-a.e. in the set} \\ & \quad T_1 = \{x \in \Omega\colon (Du)^a(x) = 0\} \\ \varphi(x) = 0 & \mathcal{H}^{n-1}\text{-a.e. in the set} \\ & \quad T_2 = \{x \in \partial\Omega\colon u(x) = g(x)\}\,. \end{cases}$$

However, by assumption (1.26), one has again that conditions (1.29) are satisfied by all $\varphi \in C_0^\infty(\Omega)$ and by $\varphi = u - g$, and one obtains again (1.21). From this on, the proof is the same. q.e.d.

REMARK 1.5. The assumptions of theorem 1.4 are satisfied in particular by functionals of the type $\int_\Omega |Du|$ which do not satisfy condition (1.1) and for which the statement of theorem 1.3 is not valid.

2. The convergence of the averages on balls.

In this section we shall use theorem 1.3 to prove various results on the convergence of the averages

$$\psi_\varrho(x) = \fint_{B_\varrho(x)} \psi(\xi)\, d\xi$$

of the vector field $\psi(\xi) = f_p(\xi, (Du)^a(\xi))$, when $p \to 0$.

Let us illustrate the very simple underlying idea. For general ψ and u one has some weak convergence of $\psi_\varrho(x)\cdot\nu_u(x)$ to the function $[\![\psi\cdot\nu_u]\!]\,(x)$, which in our case, by theorem 1.3, concides to $f^0(x, \nu_u(x)) = f_p^0(x, \nu_u(x))\cdot\nu_u(x)$. For simplicity, let us assume that f does not depend on x so that $K_x \equiv K$ for all $x \in \bar\Omega$. Now, the vectors $\psi_\varrho(x)$ belong to K while $f_p^0(\nu_u(x)) \in \partial K$ and $\nu_u(x)$ is a normal vector to ∂K at $f_p^0(\nu_u(x))$, hence it is intuitively clear that suitable assumptions of strict convexity on K will imply strong convergence of the vectors $\psi_\varrho(x)$ to $f^0(\nu_u(x))$.

Later we shall see that additional assumptions of uniform convexity on K and f imply more precise results and yield also the convergence of the averages of the vector field $\left(Du/\sqrt{1 + |Du|^2}\right)(x)$.

PROPOSITION 2.1. *Let f, u, ψ be as in theorem 1.3, then, for all open sets $A \subset\subset \Omega$ one has*

$$\psi_\varrho(x)\cdot\nu_u(x) \to f_0(x, \nu_u(x)) \quad \text{in } L^1(A, |Du|^s)\,. \tag{2.1}$$

PROOF. For any open set $A \subset\subset \Omega$ one has

$$\psi_\varrho \rightharpoonup \psi \qquad \text{in } L^\infty(A, \mathbb{R}^n) - w^*$$

$$\operatorname{div}\psi_\varrho \to \operatorname{div}\psi \qquad \text{in } L^n(A)$$

hence, by [4: propositions 2.1, 2.3], one has

$$\psi_\varrho(x)\cdot\nu_u(x) \rightharpoonup [\![\psi\cdot\nu_u]\!](x) \qquad \text{in } L^\infty(A, |Du|) - w^*$$

and, by theorem 1.3 we get in particular that

$$\psi_\varrho(x)\cdot\nu_u(x) \rightharpoonup f^0(x, \nu_u(x)) \qquad \text{in } L^1(A, |Du|^s) - w \tag{2.2}$$

By (1.7), for any fixed $x \in A$ and for $0 < \varrho < \operatorname{dist}(\bar{A}, \partial\Omega)$ we have

$$\psi(\xi) \in K_x + B_{\omega(\varrho)}(0) = \{z \,|\operatorname{dist}(z, K_x) < \omega(\varrho)\}$$

for all $\xi \in B_\varrho(x)$, but $K_x + B_{\omega(\varrho)}(0)$ is convex and it follows that also

$$\psi_\varrho(x) \in K_x + B_{\omega(\varrho)}(0)$$

hence we may write $\psi_\varrho(x) = v_\varrho(x) + \omega_\varrho(x)$, where $v_\varrho(x) \in K_x$ and $|\omega_\varrho(x)| < \omega(\varrho)$. Recalling (1.5) it follows that

$$\psi_\varrho(x)\cdot\nu_u(x) \leqq v_\varrho(x)\cdot\nu_u(x) + \omega(\varrho) \leqq f^0(x, \nu_u(x)) + \omega(\varrho) \tag{2.3}$$

and (2.3) together with (2.2) gives (2.1). q.e.d.

DEFINITION 2.2. *We shall say that a convex set $K \subset \mathbb{R}^n$ is strictly convex if for any point $x \in \partial K$ and every supporting hyperplane S of K at x one has $K \cap S = \{x\}$.*

Here is the key fact for the following theorem 2.4.

LEMMA 2.3. *Let K be a bounded strictly convex subset of $\mathbb{R}^n$, take a point $v \in \partial\Omega$, choose any fixed normal vector α to K at v (i.e. α be such that $(\xi - v)\cdot\alpha \leqq 0$ for all $\xi \in K$), and let $v_j \in K$ be such that*

$$v_j \cdot \alpha \to v \cdot \alpha \tag{2.4}$$

then one has also

$$v_j \to v\,. \tag{2.5}$$

PROOF. As K is bounded, for any subsequence w_h of v_j there exists a subsequence w_{h_i} that converges to some vector $w \in \bar{K}$. By (2.4) one must have $v\cdot\alpha = w\cdot\alpha$ and the strict convexity of K implies that $v = w$. q.e.d.

THEOREM 2.4. *Let f be as in theorem* 1.3., *then for all open sets $A \subset\subset \Omega$, one has*

$$\psi_\varrho(x) \to f^0_p(x, \nu_u(x)) \quad \text{in } L^1(A, |Du|^s)\,. \tag{2.6}$$

PROOF. As we did in the proof of proposition 2.1, we write $\psi_\varrho(x) = v_\varrho(x) + w_\varrho(x)$, where $v_\varrho(x) \in K_x$, $|w_\varrho(x)| < \omega(\varrho)$. The convergence (2.6) is equivalent to say that

$$v_\varrho(x) \to f^0_p(x, \nu_u(x)) \quad \text{in } L^1(A, |Du|^s) \tag{2.7}$$

and (2.6) is equivalent to say that for any sequence $\{\varrho_i\}_{i\in\mathbb{N}}$ with $\varrho_i \to 0$ there exists a subsequence ϱ_{i_j} such that

$$\lim_{j\to\infty} v_{\varrho_{i_j}} = f^0_p(x, \nu_u(x)) \quad \text{in } L^1(A, |Du|^s)\,. \tag{2.8}$$

Now, let $A \subset\subset \Omega$ be fixed. By proposition (2.1) we have in particular that $v_\varrho(x)\cdot\nu_u(x) \to f^0_p(x, \nu_u(x))\cdot\nu_u(x)$ in $L^1(A, |Du|^s)$. Let a sequence $\varrho_i \to 0$ be given. Then there exists a subsequence ϱ_{i_j} such that

$$\lim_{j\to\infty} v_{\varrho_{i_j}}(x)\cdot\nu_u(x) = f^0_p(x, \nu_u(x))\cdot\nu_u(x) \tag{2.9}$$

for $|Du|^s$-almost all $x \in A$. Here we notice that the set K_x defined in (1.3) is strictly convex for each $x \in \bar{\Omega}$, because the function $p \to f^0(x, p)$ is differentiable and has a unique supporting hyperplane at each point different from the origin, hence if $x \in A$ is such that (2.9) holds, we may use lemma 2.3 $\big($with $v = f(x, \nu_u(x))$, $\alpha = \nu_u(x)\big)$ to say that

$$\lim_{j\to\infty} v_{\varrho_{t_j}}(x) = f_\nu^0(x, \nu_u(x)) . \tag{2.10}$$

Now (2.10) holds $|Du|^s$-a.e. in A and by the dominated convergence theorem we get (2.7). q.e.d.

REMARK 2.5. The functions $\psi_\varrho(x)$ are equibounded, hence by theorem 2.4 we get immediately that $\psi_\varrho(x) \to f_\nu(x, \nu_u(x))$ in $L^p(A, |Du|^s)$ for $1 \leqq p < +\infty$.

If we assume some uniform convexity of f, we get a little better result.

DEFINITION 2.6. *We shall say that a bounded convex set $K \subset \mathbf{R}^n$, with a class C^1-boundary, is c-uniformly convex if there exists a number $c > 0$ such that for any $\alpha \in K$, $\beta \in \partial K$ and for any unit outward normal vector ν to ∂K at β one has*

$$|\alpha - \beta|^\sigma \leqq c(\beta - \alpha)\cdot\nu .$$

REMARK. A bounded convex set K in $\mathbf{R}^n$ is c-uniformly convex if and only if for all $\bar\alpha$, $\beta \in \delta K$ and for any outward unit normal vector ν to K at β, one has

$$|\bar\alpha - \beta|^2 \leqslant c_1(\beta - \bar\alpha)\cdot\nu$$

for some positive number c_1.

THEOREM 2.7. *Let f, u, ψ be as in theorem 1.3, moreover assume that there exist a number $c < 0$ such that the set K_x is c-uniformly convex for each $x \in \Omega$. Then for any $q \in [1, +\infty)$ and for any open set $A \subset\subset \Omega$ one has*

$$\lim_{\varrho\to 0^+} \int_A \Big\{ \fint_{B_\varrho(x)} |\psi(\xi) - f_\nu^0(x, \nu_u(x))|^q d\xi \Big\} |Du|^s = 0 .$$

PROOF. We notice that, by the boundedness of the involved functions, it is sufficient to prove the theorem for just one value of $q \in [1, +\infty)$. For every $\xi \in \Omega$ such that $(Du)^a(\xi)$ is defined, we may write $\psi(\xi) = a(\xi) + b(\xi)$, where $a(\xi) \in K_x$ and $|b(\xi)| < \omega(|x - \xi|)$. Hence, using also the uniform convexity of K_x, we have

$$|\psi(\xi) - f_p^0(x, \nu_u(x))|^2 \leqq c_1\{|a(\xi) - f_p^0(x, \nu_u(x))|^2 + |b(\xi)|^2\} \leqq$$
$$\leqq c_2\{[f^0(x, \nu_u(x)) - a(\xi)\cdot\nu_u(x)] + |b(\xi)|^2\}$$

and it follows that

$$\fint_{B_\varrho(x)} |\psi(\xi) - f_p^0(x, \nu_u(x))|^2 d\xi \leqq c_2 \left[f^0(x, \nu_u(x)) - \fint_{B_\varrho(x)} a(\xi)\, d\xi \cdot \nu_u(x)\right] + \omega(\varrho)^2$$

where

$$\fint_{B_\varrho(x)} a(\xi)\, d\xi \cdot \nu_u(x) \to f^0(x, \nu_u(x)) \qquad \text{in } L^1(A)$$

by proposition 2.1 and because

$$|\psi_\varrho(x) - \fint_{B_\varrho(x)} a(\xi)\, d\xi| \leqq \omega(\varrho) .$$

This way we have proved the theorem for $q = 2$, and the proof is concluded. q.e.d.

I think it would be nice to know simple and general conditions on f sufficient to guarantee that the set K_x is uniformly convex. A few examples are discussed in the appendix.

A certain kind of uniform convexity of f yields the convergence of the averages of the vector field $\nu_u(x)$. In fact one has the following result, whose proof is similar to the proof of theorem 2.7.

THEOREM 2.8. *Let f, u, ψ be as in theorem 2.7: moreover assume that there exist numbers $c > 0$, $\gamma < 1$ such that for all $x \in \Omega$ one has*

$$|f_p(x, tp_1) - f_p^0(x, p_2)| \geqq c|p_1 - p_2|^\gamma \tag{2.12}$$

for all $t < 0$, $p_1, p_2 \in \mathbf{R}^n$, $|p_1| = |p_2| = 1$. Then, for any open set $A \subset\subset \Omega$ and for all $q \in [1, +\infty)$, one has

$$\lim_{\varrho \to 0^+} \int_A \left\{ \fint_{B_\varrho(x)} |\nu_u(x) - \nu_u(\xi)|^q d\xi \right\} |Du|^s = 0 . \tag{2.13}$$

If we assume some sort of uniform convexity of $f(x, p)$ at $|p| \to \infty$, then from proposition 2.1 we obtain some information on the convergence of the averages of $|Du|/\sqrt{1 + |Du|^2}$.

THEOREM 2.9. *Let f, u, ψ be as in proposition* 2.1: *moreover, assume that there exists a continuous strictly decreasing function $h(t)$*: $(0, +\infty) \to$ $\to (0, +\infty)$ *that converges to zero for $t \to +\infty$ and such that*

$$f^0(\xi, \alpha) - f_p(\xi, t\beta)\cdot\alpha \geqq h(t) \tag{2.14}$$

for all $\xi \in \Omega$, $t < 0$, $\alpha, \beta \in \mathbf{R}^n$ with $|\alpha| = |\beta| = 1$. Then, for any open set $A \subset\subset \Omega$ one has

$$\lim_{\varrho\to 0^+} \int_A \left\{1 - \fint_{B_\varrho(x)} \frac{|Du|}{\sqrt{1 + |Du|^2}}(\xi)\, d\xi\right\} |Du|^s = 0\,. \tag{2.15}$$

PROOF. Using (2.14), one has

$$f^0(x, \nu_u(x)) - f_p(\xi, (Du)^a(\xi))\cdot\nu_u(x) = [f^0(x, \nu_u(x)) - f^0(\xi, \nu_u(x))] +$$
$$+ [f^0(\xi, \nu_u(x)) - f_p(\xi, |Du|^a(\xi)\nu_u(\xi))\cdot\nu_u(x)] \geqq h(|Du|^a(\xi)) - \omega(|x - \xi|)$$

hence, by proposition 2.1 we get

$$\lim_{\varrho\to 0^+} \int_A \left\{\fint_{B_\varrho(x)} h(|Du|^a(\xi))\, d\xi\right\} |Du|^s = 0\,. \tag{2.16}$$

Now consider the function $h_1(t) = 1 - t/\sqrt{1 + t^2}$, defined for $t > 0$. We shall prove that (2.16) implies

$$\lim_{\varrho\to 0^+} \int_A \left\{\fint_{B_\varrho(x)} h_1(|Du|^a(\xi))\, d\xi\right\} |Du|^s = 0 \tag{2.17}$$

and, clearly, (2.17) implies (2.15). To prove (2.17) it is sufficient to show that for any sequence $\varrho_j \to 0^+$ there exists a subsequence, that we denote τ_i, such that

$$\lim_{i\to\infty} \int_A \left\{\fint_{B_{\tau_i}(x)} h_1(|Du|^a(\xi))\, d\xi\right\} |Du|^s = 0\,. \tag{2.18}$$

Let $\varrho_j \to 0^+$ be fixed. Then, by (2.16), there exists a subsequence τ of ϱ_j such that

$$\lim_{i\to+\infty} \fint_{B_{\tau_i}(x)} h(|Du|^a(\xi))\, d\xi = 0$$

and, by lemma 2.10 below, we have also

$$\lim_{i\to+\infty} \fint_{B_{\tau_i}(x)} h_1(|Du|^a(\xi))\, d\xi = 0$$

and (2.18) follows by the dominated convergence theorem. q.e.d.

LEMMA 2.10. *Let $h(t)$, $h_1(t)$: $(0, +\infty) \to (0, +\infty)$ be strictly decreasing continuous functions that convergence to zero fot $t \to +\infty$, let g: $\Omega \to [0, +\infty)$ be a summable function and let $x \in \Omega$. Then, if one has*

$$\lim_{\varrho\to 0^+} \fint_{B_\varrho(x)} h(g(\xi))\, d\xi = 0 \tag{2.19}$$

one has also

$$\lim_{\varrho\to 0^+} \fint_{B_\varrho(x)} h_1(g(\xi))\, d\xi = 0\,. \tag{2.20}$$

PROOF. For any number $L > 0$ one has

$$\int_{B_\varrho(x)} h_1(g(\xi)) \leqq h_1(0) \operatorname{mis}\{\xi \in B_\varrho(x) \colon g(\xi) < L\} + h_1(L)\,\omega_n \varrho^n \tag{2.21}$$

on the other hand

$$\operatorname{mis}\{\xi \in B_\varrho \colon g(\xi) < L\} = \operatorname{mis}\{\xi \in B_\varrho(x) \colon h(g(\xi)) > h(L)\} \leqq$$
$$\leqq \frac{1}{h(L)} \int_{B_\varrho(x)} h(g(\xi))\, d\xi$$

and, by (2.19) one gets

$$\lim_{\varrho\to 0^+} \frac{1}{\omega_n \varrho^n} \operatorname{mis}\{\xi \in B_\varrho \colon g(\xi) < L\} = 0\,. \tag{2.22}$$

Taking the limit in (2.21) for $\varrho \to 0^+$ one gets

$$\max\lim_{\varrho\to 0^+} \fint_{B_\varrho(x)} h_1(g(\xi))\, d\xi \leqq h_1(L)$$

where L is arbitrary, and (2.20) follows. q.e.d.

Finally, using all the assumptions of uniform convexity we obtain the convergence of the vector field $Du/\sqrt{1+|Du|^2}$.

THEOREM 2.11. *Let f, u be as in proposition* 2.1, *moreover assume that f satisfies to all the assumption made in theorems* 2.7, 2.8, 2.9. *Then for all open sets $A \subset\subset \Omega$, one has*

$$\lim_{\varrho\to 0^+} \int_A \left| \fint_{B_\varrho(x)} \frac{Du}{\sqrt{1+|Du|^2}}(\xi)\, d\xi - \nu_u(x) \right| |Du|^s = 0 .$$

PROOF. The proof follows immediately from theorems 2.8, 2.9. q.e.d.

3. Pointwise convergence of special averages.

Theorems 1.3 and 1.9 imply that for $|Du|^s$-almost all $x \in \Omega$ one has

$$\lim_{\varrho\to 0^+}\lim_{r\to 0^+} \frac{1}{2r\,\omega_{n-1}\varrho^{n-1}} \int_{C_{r,\varrho}(x,\nu_u(x))} \psi(\xi)\, d\xi \cdot \nu_u(x) = f^0(x, \nu_u(x)) . \tag{3.1}$$

Similarly to what we did in section 2, we can see that (3.1), together with various convexity assumption, implies that the averages of ψ, ν_u and $Du/\sqrt{1+|Du|^2}$ on cylinders $C_{r,\varrho}$ converge $|Du|^s$-a.e. in Ω. Here we bound ourselves to state two of the results that one can prove.

THEOREM 3.1. *Let f, u, ψ be as in theorem* 1.3. *Then, for $|Du|^s$-almost all $x \in \Omega$ one has*

$$\lim_{\varrho\to 0^+} \max\lim_{r\to 0^+} \left| f^0_p(x, \nu_u(x)) - \fint_{C_{r,\varrho}(x,\nu_u(x))} \psi(\xi)\, d\xi \right| = 0 .$$

THEOREM 3.2. *Let f, u, ψ be as in theorem* 1.3; *moreover assume that there exists a number $c < 0$ such that the set K_x is c-uniformly convex* (*definition* 2.6) *for all $x \in \Omega$. Then, for $|Du|^s$-almost all $x \in \Omega$, one has*

$$\lim_{\varrho \to 0^+} \operatorname{maxlim}_{r \to 0^+} \fint_{C_{r,\varrho}(x, \nu_u(x))} |\psi(\xi) - f_p^0(x, \nu_u(x))| \, d\xi = 0$$

4. Boundary behaviour.

In sections 2 and 3 we have seen that relation (1.15) entails the existence of the trace on the singular support of $|Du|$ of various vector fields related to u. In a completely similar way, one can show that relation (1.16) entails the existence of the trace of the same vector fields on the boundary of Ω, in the set where u does not attain the prescribed boundary datum g. We shall not enter in the details.

5. Appendix.

A typical integrand $f(x, p)$ that satisfies the assumptions of all the theorems in sections 2 and 3 is

$$f(x, p) = \sqrt{1 + \sum_{i,j=1}^{n} a_{ij}(x) p_i p_j} \tag{5.1}$$

where $a_{ij}(x) \colon \bar{\Omega} \to \mathbf{R}$ are continuous functions on $\bar{\Omega}$ such that for each $x \in \bar{\Omega}$ one has

$$|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leqq M_1 |\xi|^2 \qquad \forall \xi \in \mathbf{R}^n$$

Conditions (0.1), (0.2), (0.3) are easily checked, moreover one has

$$f^0(x, p) = \left(\sum_{i,j=1}^{n} a_{ij}(x) p_i p_j \right)^{\frac{1}{2}}.$$

and conditions (1.1), (1.2) also follow readily. For each $x \in \bar{\Omega}$ one has

$$K(x) = \left\{\alpha \in \mathbb{R}^n : \sum_{i,j=1}^{n} A_{ij}(x)\alpha_i\alpha_j = 1\right\}$$

where $A(x) = (a(x))^{-1}$, and K_x is clearly strictly convex and also C-uniformly convex for some constant C that depends on the greatest eigenvalue of $a(x)$. Finally, condition (2.14) is satisfied with $h(t) = 1 - t/\sqrt{1+t^2}$, while condition (2.11) is satisfied with $\gamma = 1$ and a constant c depending again on M_1.

More generally, one can consider integrands of the type

$$f(x, p) = \beta(x, \gamma(x, p)) \tag{5.2}$$

where the functions

$$\beta(x, t) : \bar{\Omega} \times [0, +\infty) \to [0, +\infty)$$

$$\gamma(x, p) : \bar{\Omega} \times \mathbb{R}^n \to [0, +\infty)$$

satisfy suitable assumptions.

For example, choosing $\beta(x, t) = (1 + t^k)^{1/k}$, $\gamma(x, p) = |p|$ one obtains the functional

$$\int_{\Omega} \sqrt[k]{1 + |Du|^k}$$

that satisfies to all the assumptions of theorem 2.10.

Another example, of interest in Hencky plasticity [6], [17], is obtained taking $\gamma(x, p) = |p|$ and

$$\beta(t) = \begin{cases} \frac{1}{2}t^2 + \frac{1}{2} & \text{if } t \leqq 1 \\ t & \text{if } t \geqq 1 . \end{cases} \tag{5.4}$$

In this case the assumptions of theorem 2.7, 2.8 are satisfied but condition (2.13) is not satisfied and the conclusion of theorem 2.9 actually does not hold. In fact, consider the open set $\Omega = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\}$, and the function $u \in BV(\Omega)$ defined as

$$u(x) = \begin{cases} x_1 & \text{if } x_1 < \frac{1}{2} \\ x_1 + 1 & \text{if } x_1 > \frac{1}{2} . \end{cases}$$

Clearly one has $D_2 u \equiv 0$ in Ω and

$$\int_\Omega \beta(|Du|) = \int_\Omega |D_1 u| = 2$$

On the other hand, for all $v \in BV(\Omega)$ such that $v(0, x_2) = 0$, $v(1, x_2) = 2$ one has

$$\int_\Omega \beta(|Du|) \geqq \int_\Omega |Dv| \geqq \int_\Omega |D_1 v| \geqq \int_\Omega D_1 v = 2$$

and it follows that u is a minimum for $F(u) = \int_\Omega \beta(|Du|)$, while (2.14) does not hold for u.

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