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Notes on mixed groups II

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Introduction.

Part I of this paper appeared in the Proceedings of the Udine Conference on Abelian Groups and Modules, published by C.I.S.M., Udine, 1984. The section numbering of Part II follows that of Part I and the bibliography refers to both parts, but otherwise this paper is completely self contained.

All groups are abelian, so the adjective is usually ommitted. The notations and definitions are those of the standard reference [3]; in particular, a group is mixed if it contains non-zero elements of both finite and infinite order.

Section 4 is a classification of mixed groups which have a maximal rank torsion-free subgroup A such that A has finite rank and A/pA has rank $\leqslant 1$ for all primes p.

Section 5 is a study of a class of mixed groups which can be classified by linear transformations of rational vector spaces, and which exhibit some pathological decompositions.

4. Mixed extensions of Murley groups.

Let G be a mixed group with torsion subgroup t, let $W = G/t$ and let $\varrho: G \to W$ be the natural homomorphism. Let A be a full torsion-free subgroup of G, so $U = G/A$ is torsion, and let $\pi: G \to U$

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be the natural homomorphism. Since $t \cap A = 0$, the inclusions of A and t in G induce monomorphisms $\varkappa\colon t \to U$ and $\sigma\colon A \to W$ such that $U/\varkappa(t) \cong W/\sigma(A) \cong G/\langle t, A\rangle$ and the diagram of Figure 1 is exact and commutative.

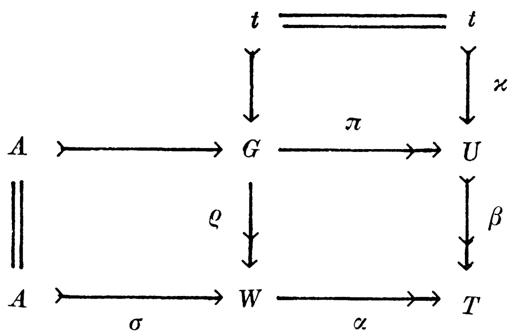

Figure 1

Murley [7] introduced the class $\mathcal{E}$ of torsion-free groups A of finite rank such that for all primes p, $r_p(A) = \operatorname{rank}(A/pA) \leqslant 1$. $\mathcal{E}$ is an important class because it has a satisfactory classification theory. Let us call a mixed group G a *mixed Murley group* if it contains a full subgroup from $\mathcal{E}$, and call such a subgroup a *Murley subgroup* of G.

THEOREM 4.1. Let G with pullback diagram Figure 1, be a mixed Murley group with Murley subgroup A. Then T is locally cyclic and W is a Murley group.

Conversely, let

$$A \overset{\sigma}{\rightarrowtail} W \overset{\alpha}{\twoheadrightarrow} T \text{ and } t \overset{\varkappa}{\rightarrowtail} U \overset{\beta}{\twoheadrightarrow} T$$

be short exact sequences with A and W Murley groups and t, U and T torsion groups. Then the pullback G of α and β is a mixed Murley group; moreover G is determined by the monomorphisms σ and $\varkappa$.

PROOF. The exact sequence $\mathbb{Z} \rightarrowtail \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ induces an exact sequence $\mathbb{Z} \otimes A \rightarrowtail \mathbb{Q} \otimes A \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \otimes A \cong \bigoplus_p \bigoplus_{r_p(A)} \mathbb{Z}(p^\infty)$. Let R be the set of primes p for which $r_p(A) = 1$, so $\mathbb{Q}/\mathbb{Z} \otimes A \cong \bigoplus_{p \in R} \mathbb{Z}(p^\infty)$.

Since A is full in G and $W/\sigma(A)$ is a torsion group, W is isomorphic to a subgroup of $\mathbb{Q}\otimes A$, so $T \simeq W/A \leqslant \bigoplus_{p\in R}\mathbb{Z}(p^\infty) \leqslant \mathbb{Q}/\mathbb{Z}$.

Next, the exact sequence $A \rightarrowtail W \twoheadrightarrow T$ induces for each prime p an exact sequence.

$$(*)\qquad \mathrm{Tor}\,(\mathbb{Z}(p), T) \rightarrowtail \mathbb{Z}(p)\otimes A \to \mathbb{Z}(p)\otimes W \twoheadrightarrow \mathbb{Z}(p)\otimes T\,.$$

If $T_p \simeq 0$ or $\mathbb{Z}(p^\infty)$, the last term is zero, so $\mathbb{Z}(p)\otimes W \simeq \bigoplus_{r_p(W)}\mathbb{Z}(p)$ is a homomorphic image of $\mathbb{Z}(p)\otimes A$, so $r_p(W)\leqslant 1$.

If $T_p \simeq \mathbb{Z}(p^n)$ for some n, $\mathrm{Tor}\,(\mathbb{Z}(p), T) \simeq \mathbb{Z}(p)$, so $\mathbb{Z}(p)\otimes A \simeq \mathbb{Z}(p)$ and the first map in the exact sequence $(*)$ is an isomorphism. Hence the last map is also an isomorphism, so $\mathbb{Z}(p)\otimes W \simeq \mathbb{Z}(p)$ and $r_p(W) = 1$. Thus W is a Murley group.

Conversely, suppose the short exact sequences are given, and let $G = \{(w, u) \in W \oplus U\colon \alpha(w) = \beta(u)\}$. With the obvious identifications of A and t with subgroups of G, Figure 1 is a pullback diagram for G, so G is a mixed Murley group and hence T is locally cyclic.

Consequently, every automorphism of T is a multiplication by a unit of $\hat{\mathbb{Z}}$, the completion of $\mathbb{Z}$ in the natural topology; since U is torsion, any such multiplication can be lifted to an automorphism of U. Hence by [4, Theorem 1], all pullbacks of W and U with kernels A and t respectively are isomorphic.

Theorem 4.1 classifies mixed Murley groups, but does not answer the question: which extensions of a torsion group by a torsion-free group are mixed Murley? This question is answered by the following:

THEOREM 4.2. Let G be an extension of a torsion group by a finite rank torsion-free group W. Then G is mixed Murley if and only if W is Murley and has a full subgroup A such that

(*a*) $T = W/A$ is locally cyclic and

(*b*) $pW = W$ for all primes p for which $T_p \simeq \mathbb{Z}(p^\infty)$.

In this case, G is determined up to isomorphism by A.

PROOF. ($\Rightarrow$) Let A be a full Murley subgroup of G, so by Theorem 4.1 W is Murley and T is locally cyclic. Let p be a prime for which $T_p \simeq \mathbb{Z}(p^\infty)$. Either $pW = W$ or $W/pW \simeq \mathbb{Z}(p)$, so assume the latter. If $A \leqslant pW$, then W/pW is a homomorphic image of T_p, a contradiction, so A is not a subgroup of pW, and hence $A \neq pA$. Thus $A/pA \simeq$

$\cong \mathbb{Z}(p)$; since $(A \cap pW)/pA \leqslant A/pa$ and $A \cap pW \neq A$, $A \cap pW = pA$ so A is p-pure in W. Consequently, $T_p = 0$, a contradiction. Thus $pW = W$ as required.

($\Rightarrow$) First consider a prime p for which $T_p \cong \mathbb{Z}(p^\infty)$ and $pW = W$. Then $A \leqslant pW$ so A/pA is a subgroup of the locally cyclic group pW/pA, so A/pA has rank $\leqslant 1$.

Next, let p be a prime for which $T_p = \mathbb{Z}(p^k)$ for some $0 \leqslant k < \infty$. If A/pA is not cyclic, W/pA has a p-component which is not cyclic, but is bounded by p^{k+1}. Hence W has a finite homomorphic image which is not cyclic, contradicting [7, Lemma 8]. Hence, A/pA has rank $\leqslant 1$.

By Theorem 4.1, G is mixed Murley, and determined by A.

It is easy to see that if G and G' are mixed Murley groups determined by the pairs of short exact sequences:

$$A \rightarrowtail W \twoheadrightarrow T \quad \text{and} \quad t \rightarrowtail U \twoheadrightarrow T$$

and

$$A' \rightarrowtail W' \twoheadrightarrow T' \quad \text{and} \quad t \overset{\varkappa'}{\rightarrowtail} U' \twoheadrightarrow T', \qquad \text{respectively},$$

then there is an isomorphism $\theta\colon G \to G'$ mapping A onto A' whenever $\varkappa$ is equivalent to $\varkappa'$ in the sense of [14, Section 3]. However, it is not known whether distinct embeddings of A in G produce t-equivalent groups U and U'. Consequently it is not known whether a theorem analogous to Theorem 3.1 of [14] holds for mixed Murley groups.

5. Orthogonally mixed groups.

In this section we characterize a mixed group G with reduced torsion subgroup as a pullback of its torsion-free factor W and a subgroup of the cotorsion hull $c(t)$ of its torsion group t. This construction is due to Harrison [5]. In the special case that $\mathrm{Hom}(W, t) = 0$, we use his construction to classify these groups up to congruence. The following notation is used:

For any group G, $t(G)$ is the torsion subgroup of G, $f(G) = G/t(G)$, $c(G) = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ is the cotorsion hull of G, and $d(G)$ is the divisible hull of G. Properties of these functors are described in [3, 5 and 11]. Mader [13] has a similar construction.

THEOREM 5.1. [5, Proposition 2.4]. Let t be a reduced torsion group and W a torsion-free group. Let $\beta: c(t) \to fc(t)$ be the natural homomorphism.

Every extension of t by W is a pullback $G(\varphi)$ of β and φ for some $\varphi: W \to fc(t)$.

If φ and φ' are elements of Hom $(W, fc(t))$, then $G(\varphi)$ is congruent to $G(\varphi')$ if and only if $\varphi - \varphi'$ lifts to Hom $(W, c(t))$.

PROOF. The exact sequence $t \rightarrowtail c(t) \overset{\beta}{\twoheadrightarrow} fc(t)$ induces an exact sequence

$$\mathrm{Hom}\,(W, t) \rightarrowtail \mathrm{Hom}\,(W, c(t)) \to \mathrm{Hom}\,(W, fc(t)) \twoheadrightarrow \mathrm{Ext}\,(W, t)$$

where the epimorphism δ is defined as follows:

For all $\varphi \in \mathrm{Hom}\,(W, fc(t))$, let $G(\varphi)$ be the pullback of φ and β, so $G(\varphi)$ has the pullback diagram shown in Figure 2.

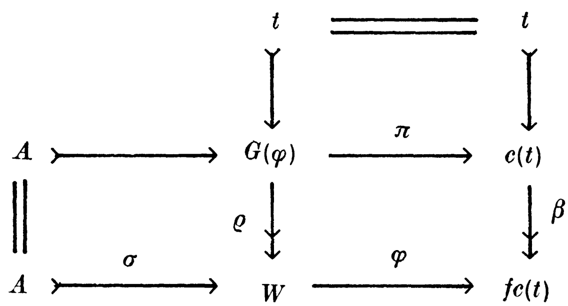

Figure 2

Then $\delta(\varphi)$ is defined to be the congruence class of $G(\varphi)$.

If $\varphi' \in \mathrm{Hom}\,(W, fc(t))$, then $G(\varphi)$ is congruent to $G(\varphi')$ if and only if $\varphi - \varphi' \in \mathrm{Ker}\,\delta$ if and only if $\varphi - \varphi'$ lifts to an element of Hom $(W, c(t))$.

COROLLARY. $G(\varphi)$ splits if and only if φ lifts to an element of Hom $(W, c(t))$.

Stratton [12] used a similar theorem to show that $G(\varphi)$ is isomorphic to a subgroup of $c(t)$ if and only if φ is monic.

Note that π and φ in Figure 2 are not necessarily epimorphisms, but if one is, so is the other. Since $fc(t)$ is torsion-free divisible [11, Proposition 2.3], $\sigma(A)$ is pure in W, and Hom $(W, fc(t)) \cong$ Hom $(d(W), fc(t))$, a group of rational linear transformations. Thus Theorem 5.1 is a classification of extensions of t by W modulo the group $\mathrm{Hom}(W, c(t))$. Therefore it is worthwhile to consider a special case in which this group is trivial.

LEMMA 5.2. Let t be a torsion group and W a torsion-free group. The following are equivalent:

(1) $\mathrm{Hom}(W, t) = 0$;

(2) t is reduced and for all relevant primes p, $pW = W$;

(3) t is reduced and $\mathrm{Hom}(W, c(t)) = 0$.

PROOF. Without loss of generality we may assume t and W are non-zero. Let R be the set of primes relevant for t, and let $p \in R$.

(1) $\Rightarrow$ (2). Since $\mathrm{Hom}(W/pW, t_p) \leqslant \mathrm{Hom}(W, t) = 0$, $pW = W$. Let F be a full free subgroup of W; then W/F is a p-divisible torsion group. Since $\mathrm{Hom}(W/F, t_p) \leqslant \mathrm{Hom}(W, t) = 0$, t_p is reduced.

(2) $\Rightarrow$ (3). Since t is reduced, $c(t_p)$ is p-reduced [12, (2.6)], so $\mathrm{Hom}(W, c(t)) \simeq \prod_{p \in R} \mathrm{Hom}(W, c(t_p)) = 0$.

(3) $\Rightarrow$ (1). $\mathrm{Hom}(W, t) \leqslant \mathrm{Hom}(W, c(t)) = 0$.

DEFINITION. Let t be a reduced torsion group and W a torsion free group. The pair (W, t) is called an *orthogonal pair* if $\mathrm{Hom}(W, t) = 0$, and an extension of t by W is called *orthogonally mixed.*

COROLLARY. Let (W, t) be an orthogonal pair.

(1) There is a $1-1$ correspondence between $\mathrm{Hom}(W, fc(t))$ and (congruence classes of) extensions of t by W.

(2) $G(\varphi)$ splits if and only if $\varphi = 0$.

(3) Every extension of t by W splits if and only if t is bounded.

In the remainder of this section, we determine some structure theorems for orthogonally mixed groups. From now on (W, t) always represents a non-trivial orthogonal pair, $\varphi \in \mathrm{Hom}(W, fc(t))$ and A, $G(\varphi)$, β, ϱ, σ and π are as in Figure 2.

DEFINITION. A mixed group G is *adjusted* if it has no non-trivial torsion-free summand.

LEMMA 5.3. $G(\varphi)$ is not adjusted if and only if $\sigma(A)$ contains a non-trivial direct summand of W.

PROOF. ($\Rightarrow$) Let $G(\varphi) = H \oplus K$ where K is torsion-free. Since $t \leqslant H$, $W = U \oplus V$, where $U = H/t$. Hence V is a summand of W. Since $\mathrm{Hom}\,(W, c(t)) = 0$, V is contained in $\sigma(A)$.

($\Leftarrow$) Suppose $W = U \oplus V$ with $V \leqslant \sigma(A)$. Let $H = \varrho^{-1}(U)$, $J = \varrho^{-1}(V)$, so $G(\varphi) = \langle H, J \rangle$ and $t = H \cap J$. Using the representation $G(\varphi) = \{(w, s) \in W \oplus c(t) : \varrho(w) = \beta(s)\}$, $J = \{(w, s) \in V \oplus c(t) : \beta(s) = 0\}$, so $J = K \oplus t$, say, with $K \simeq V$. Hence $G(\varphi) = H \oplus K$, with K torsion-free.

COROLLARY. (1) Every non-splitting extension of t by W is adjusted if and only if W is indecomposable.

(2) No extension of t by W is adjusted if and only if $\mathrm{rank}(W) > \mathrm{rank}\,(fc(t))$ and every pure subgroup of W contains a direct summand of W.

DEFINITION. A mixed group G is *p-mixed* if its torsion subgroup is a p-group and *Σ-p-mixed* if G is a direct sum of p-mixed groups for distinct primes p.

Oppelt [8] introduced these concepts and found necessary and sufficient conditions on t and a completely decomposable W for every extension of t by W to be Σ-p-mixed, and necessary and sufficient conditions for every summand of a Σ-p-mixed group to be Σ-p-mixed.

Griffith [6] found necessary and sufficient conditions on a torsion-free W for every extension of any torsion group t by W to be Σ-p-mixed.

Given an orthogonal pair (W, t), we shall now determine conditions on $\varphi \in \mathrm{Hom}\,(W, fc(t))$ which ensure that the extension $G(\varphi)$ is Σ-p-mixed. We also demonstrate various « pathologies » of decompositions of a mixed group into p-mixed summands.

THEOREM 5.4. Let (W, t) be an orthogonal pair and let R be the set of primes relevant for t. Let $\varphi \in \mathrm{Hom}\,(W, fc(t))$. Then $G(\varphi)$ is Σ-p-mixed if and only if $W = \bigoplus_{p \in R} W^p$ such that, for all $p \in R$, $0 \neq \varphi(W^p) \leqslant fc(t_p)$.

PROOF. ($\Rightarrow$) Suppose $G(\varphi) = \bigoplus_{p \in R} G^p$ with $t_p \leqslant G^p$; define $W^p = \varrho(G^p)$ for all $p \in R$, so $W = \bigoplus_{p \in R} W^p$.

Let $p \in R$; since $c(t_p)$ is a direct summand of $c(t)$ containing t_p, $fc(t_p)$ is a summand of $fc(t)$, say $c(t) = c(t_p) \oplus X$ and $fc(t) = fc(t_p) \oplus Y$

Let $\gamma: c(t) \to X$ and $\delta: fc(t) \to Y$ be the projections determined by these decompositions, so $\beta\circ\gamma = \delta\circ\beta$.

If $\gamma\pi(G^p) = C \neq 0$, let $D = \beta(C)$; since $t_p \cap X = 0$, $\beta\restriction C$ is monic, so has an inverse η on D. Then $\beta\circ\gamma\circ\pi = \delta\circ\beta\circ\pi = \delta\circ\varphi\circ\varrho$, so $\gamma\circ\pi\restriction G^p = \eta\circ\delta\circ\varphi\circ\varrho\restriction G^p \neq 0$. Hence $\eta\circ\delta\circ\varphi: W^p \to X$ can be extended to a non-zero homomorphism of W into $c(t)$, contradicting orthogonality.

Thus π maps G^p into $c(t_p)$, so φ maps W^p into $fc(t_p)$.

($\Leftarrow$) For all $p \in R$, let G^p be the pullback of $\beta\restriction c(t_p)$ and $\varphi\restriction W^p$. Thus $t_p \leqslant G^p$ and $G^p \cap \langle G^q: q \neq p\rangle = 0$, so it remains to show that the G^p generate $G(\varphi)$. Let $x \in G(\varphi)$, and suppose $\varrho(x) = \sum w^p$, where the sum is over a finite subset S of R, and $w^p \in W^p$ for all $p \in S$. Since ϱ is surjective, there exists $x^p \in G$ such that $\varrho(x^p) = w^p$ for all $p \in S$. Let $\pi(x^p) = s^p \in c(t)$. Since $\beta(s^p) = \varphi(w^p) \in fc(t_p)$, $s^p \in \langle c(t_p), t\rangle$. Hence $(w^p, s^p) \in \langle G^p, t\rangle$. Now $\varrho(\Sigma(w^p, s^p)) = \varrho(x)$, so $x - \Sigma(w^p, s^p) \in t$ and hence $x \in \langle G^p: p \in R\rangle$ as required.

Finally we present some examples of «paradoxical» decompositions of Σ-p-mixed groups, which reflect similar properties of torsion-free groups.

EXAMPLE 1. Decomposition of an adjusted mixed group into p-mixed summands need not be unique:

Fuchs and Loonstra [3, Theorem 90.3] constructed the following example of non-isomorphic indecomposable rank 2 torsion-free groups A and C such that $A \oplus A = C \oplus C$:

Let x_1, y_1, x_2 and y_2 be independent generators, 5, p and q distinct primes and P_1 and P_2 disjoint infinite sets of primes not containing 5, p or q. Let $X_i = \langle p_1^{-1}x_i: p_1 \in P_1\rangle$, $Y_i = \langle p_2^{-1}y_i: p_2 \in P_2\rangle$, $A_i = \langle X_i \oplus Y_i, 5^{-1}(x_i + y_i)\rangle$ and $C_i = \langle X_i \oplus Y_i, 5^{-1}(x_i + 2y_i)\rangle$ for $i = 1$ and 2. Fuchs and Loonstra showed that $A_1 \cong A_2$, $C_1 \cong C_2$, A_1 is not isomorphic to C_1 and $A_1 \oplus A_2 = C_1 \oplus C_2$.

To construct W, we modify this construction slightly. Let $Z(p, q)$ be the rank 1 ring generated over $\mathbb{Z}$ by p^{-1} and q^{-1} and let $W_1^p = Z(p, q) \otimes A_1$, $W_1^q = Z(p, q) \otimes A_2$, $W_2^p = Z(p, q) \otimes C_1$ and $W_2^q = Z(p, q) \otimes C_2$; so $W_1^p \cong W_1^q$, $W_2^p \cong W_2^q$, W_1^p is not isomorphic to W_2^p and $W_1^p \oplus W_1^q = W_2^p \oplus W_2^q = W$.

Now let $B_p^i = \bigoplus_{k\geqslant 1} \mathbb{Z}(p^k)$ and $B_q^i = \bigoplus_{k\geqslant 1} \mathbb{Z}(q^k)$ for $i = 1$ and 2, and let $t = B_p^1 \oplus B_p^2 \oplus B_q^1 \oplus B_q^2$, so (W, t) is an orthogonal pair. In the notation of Figure 2, let $e_p^i \in c(B_p^i)\setminus B_p^i$, $e_q^i \in c(B_q^i)\setminus B_q^i$, $f_p^i = \beta(e_p^i)$ and $f_q^i = \beta(e_q^i)$ for $i = 1$ and 2.

Define $\varphi: W \to fc(t)$ by $x_1 \mapsto f_p^1$, $x_2 \mapsto f_q^1$, $y_1 \mapsto f_p^2$ and $y_2 \mapsto f_q^2$.

Thus φ is monic and hence by Lemma 5.3, $G = G(\varphi)$ is adjusted. Furthermore $\varphi(W_i^p) \subseteq fc(t_p)$ and $\varphi(W_i^q) \subseteq fc(t_p)$ for $i = 1$ and 2.

Now let G_i^p be the pullback of $\varphi \restriction W_i^p$ and β, and let G_i^q be the pullback of $\varphi \restriction W_i^q$ and β for $i = 1$ and 2. By Theorem 5.4, $G = G_1^p \oplus G_1^q = G_2^p \oplus G_2^q$ are decompositions of G as direct sums of p-mixed groups for distinct primes p and q. But there can be no isomorphism of G_1^p onto G_2^p, because it would induce an isomorphism of W_1^p onto W_2^p.

Example 2. An adjusted direct summand of a Σ-p-mixed group need not be Σ-p-mixed:

Corner [3, Theorem 90.2] constructed the following example of a torsion-free rank 4 group $A = A_1 \oplus A_2 = A^1 \oplus A^2$ where A_1 and A_2 are non-isomorphic indecomposable rank 2 groups, A^1 is rank 1 and A^2 is an indecomposable rank 3 group:

Let u_1, u_2, x_1 and x_2 be independent generators and p_1, p_2, p_3, q_1 and q_2 distinct primes.

$$\text{Let } A = \langle p_1^{-\infty} u_1, p_1^{-\infty} u_2, p_2^{-\infty} x_1, p_3^{-\infty} x_2, q_1^{-1}(u_1 + x_1), q_2^{-1}(u_1 + x_2)\rangle,$$

$$A_1 = \langle p_1^{-\infty} v_1, p_2^{-\infty} x_1, q_1^{-1}(v_1 + x_1)\rangle$$

$$A_2 = \langle p_1^{-\infty} v_2, p_3^{-\infty} x_2, q_2^{-1}(v_2 + x_2)\rangle$$

$$A^1 = \langle p_1^{-\infty} v_1'\rangle$$

$$A^2 = \langle p_1^{-\infty} v_2', p_2^{-\infty} x_1, p_3^{-\infty} x_2, q_1^{-1}(v_2' + x_1), q_2^{-1}(v_2' + x_2)\rangle,$$

where v_i and v_i' are suitably chosen integral combinations of u_1 and u_2 for $i = 1$ and 2.

Now let p and q be primes different from p_1, p_2, p_3, q_1 and q_2, let $Z(p, q)$, B_p^1, B_q^1, e_p^1, e_q^1, f_p^1 and f_q^1 be defined as in Example 1, and let $W = Z(p, q) \otimes A$, $W_i = Z(p, q) \otimes A_i$ and $W^i = Z(p, q) \otimes A^i$ for $i = 1$ and 2. Then $W = W_1 \oplus W_2 = W^1 \oplus W^2$, W_1 and W_2 are non isomorphic indecomposable rank 2 groups, W^1 is rank 1 and W^2 is an indecomposable rank 3 group. With $t = B_p^1 \oplus B_q^1$, (W, t) is an orthogonal pair.

Define $\varphi\colon W \to fc(t)$ by $u_1 \mapsto 0$, $u_2 \mapsto 0$, $x_1 \mapsto f_p^1$ and $x_2 \mapsto f_q^1$. Thus $\varphi(W_1) \leqslant fc(t_p)$ and $\varphi(W_2) \leqslant fc(t_p)$, so $G = G(\varphi)$ is Σ-p-mixed. W^1 is a direct summand of W in the kernel of φ, so W^1 lifts to a torsion-free summand of G. Let K be a complement of W^1 in G, so ϱ maps K onto the indecomposable group W^2. Hence K is an adjusted direct summand of G which is not Σ-p-mixed.

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