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On the Application of Measure of Noncompactness to Existence Theorems.

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1. Introduction.

The notion of measure of noncompactness was introduced by Kuratowski [12]. For any bounded subset X of a metric space the measure of noncompactness—denoted $\alpha(X)$—is defined to be the infimum of positive numbers ε such that X can be covered by a finite number of sets of diameter $\leqslant \varepsilon$.

The first who used the index α to the fixed point theory was Darbo [7]. Later its result has been generalized by Sadovskii [16]. The following theorem is a modified version of the Darbo-Sadovskii result:

THEOREM 1. Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let G be a continuous mapping of D into itself. If the implication

$$(1) \qquad V = \operatorname{conv} G(V) \text{ or } V = G(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then G has a fixed point.

PROOF [19]. Define a sequence (y_n) by $y_0 = 0$, $y_{n+1} = G(y_n)$ $(n = 0, 1, 2, \ldots)$. Let $Y = \{y_n : n = 0, 1, 2, \ldots\}$. As $Y = G(Y) \cup \{0\}$, from (1) it follows that Y is relatively compact in D. Denote by Z

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the set of all limit points of (y_n). It can be easily verified that $Z = G(Z)$. Let us put $R(X) = \operatorname{conv} G(X)$ for $X \subset D$, and let Ω denote the family of all subsets X of D such that $Z \subset X$ and $R(X) \subset X$. Clearly $D \in \Omega$. Denote by V the intersection of all sets of the family Ω. As $Z \subset V$, V is nonempty and $Z = G(Z) \subset R(Z) \subset R(V)$. Since $R(V) \subset R(X) \subset X$ for all $X \in \Omega$, $R(V) \subset V$ and therefore $V \in \Omega$. Moreover, $R(R(V)) \subset \subset R(V)$, and hence $R(V) \in \Omega$. Consequently $V = R(V)$, i.e. $V = \operatorname{conv} G(V)$. In view of (1), this implies that $\overline{V}$ is a compact subset of D. Applying now the Schauder fixed point theorem to the mapping $G|\overline{V}$, we conclude that G has a fixed point.

Let us remark that our proof is simpler than that in [16].

Throughout this paper we shall assume that $I = [0, a]$ is a compact interval in R, E is a real Banach space, $B = \{x \in E: \|x - x_0\| \leqslant b\}$ and μ is the Lebesgue measure in R. Moreover, for a given set V of functions $I \to E$ let us denote $V(t) = \{x(t): x \in V\}$ $(t \in I)$ and $V(I) = \{x(s): x \in V,\ s \in I\}$.

In recent years there appeared a lot of papers using α to existence theorems for the Cauchy problem

$$x' = f(t, x), \qquad x(0) = x_0, \tag{2}$$

in Banach spaces (see e.g. Ambrosetti [1], Szufla [18], Goebel-Rzymowski [9], Cellina [4], Pianigiani [14] and Deimling [8]). The best result of this kind has been proved in [20]:

Assume that $h: I \times R_+ \to R_+$ is a Kamke function, i.e. h satisfies the Caratheodory conditions and for any c, $0 < c \leqslant a$, $u = 0$ is the only absolutely continuous function on $[0, c]$ which satisfies $u'(t) = h(t, u(t))$ almost everywhere on $[0, c]$ and such that $u(0) = 0$.

THEOREM 2. Lef f be a function from $I \times B$ into E which satisfies the Caratheodory conditions. If for any $\varepsilon > 0$ and for any subset X of B there exists a closed subset I_ε of I such that $\mu(I \setminus I_\varepsilon) < \varepsilon$ and

$$\alpha(f(T \times X)) \leqslant \sup_{t \in T} h(t, \alpha(X)) \tag{3}$$

for each closed subset T of I_ε, then there exists at least one solution of (2) defined on a subinterval of I

In this paper we shall extend the method of proving Theorem 2 to more complicated equations More precisely, we shall give new existence theorems for integral equations, boundary value problems and quasilinear differential equations in Banach spaces Our considerations base on the following

LEMMA. Let f be a function from $I\times B$ into a Banach space F which satisfies the Caratheodory conditions and (3), and let K be a bounded strongly measurable function from I^2 into the space of bounded linear mappings $F\to E$. If V is an equicontinuous set of functions $I\to B$, then

$$\alpha\left(\left\{\int_T K(t,s)f(s,x(s))\,ds : x\in V\right\}\right)\leqslant\int_T\|K(t,s)\|h\big(s,\alpha(V(s))\big)\,ds$$

for any measurable subset T of I and any $t\in I$.

We omit the proof of Lemma, because it is similar to the proof of inequality (8) in [20].

2. Hammerstein integral equations.

Consider the integral equation

$$x(t)=p(t)+\lambda\int_I K(t,s)f(s,x(s))\,ds\,, \tag{4}$$

where

1°) p is a continuous function from I into E;

2°) $(s,x)\to f(s,x)$ is a function from $I\times E$ into a Banach space F which satisfies the following conditions:

(i) f is continuous in x and strongly measurable in s;

(ii) for any $r>0$ there exists an integrable function $m_r\colon I\to R_+$ such that $\|f(s,x)\|\leqslant m_r(s)$ for all $s\in I$ and $\|x\|\leqslant r$.

3°) K is a continuous function from I^2 into the space of bounded linear mappings $F\to E$.

THEOREM 3. Assume in addition that there exists an integrable function $h\colon I\to R_+$ such that for any $\varepsilon>0$ and any bounded subset X

of E there exists a closed subset I_ε of I such that $\mu(I\setminus I_\varepsilon) < \varepsilon$ and

$$\alpha\big(f(T\times X)\big)\leqslant\sup_{s\in T} h(s)\alpha(X) \tag{5}$$

for each closed subset T of I_ε.

Then there exists $\varrho > 0$ such that for any $\lambda\in R$ with $|\lambda| < \varrho$ the equation (4) has at least one continuous solution.

PROOF. Denote by $C(I, E)$ the Banach space of continuous functions $I\to E$ with the usual supremum norm $\|\cdot\|_c$. Let $r(H)$ be the spectral radius of the integral operator H defined by

$$Hu(t) = \int_I \|K(t, s\|h(s)\,u(s)\,ds \qquad \big(u\in C(I, R),\ t\in I\big)\,,$$

and let

$$\varrho = \min\left(\sup_{r>0}\frac{r-\|p\|_c}{\sup\limits_{t\in I}\int_I\|K(t, s)\|m_r(s)\,ds},\qquad \frac{1}{r(H)}\right).$$

Fix $\lambda\in R$ with $|\lambda| < \varrho$, and choose $b > 0$ in such a way that

$$\|p\|_c + |\lambda|\sup\int_I\|K(t, s)\|m_b(s)\,ds\leqslant b\,. \tag{6}$$

Let $D = \{x\in C(I, E)\colon \|x\|_c\leqslant b\}$. It is well known that the assumptions 1°-3°, plus (6), imply that the operator G, defined by

$$G(x)(t) = p(t) + \lambda\int_I K(t, s)f\big(s, x(s)\big)\,ds \qquad (x\in D,\ t\in I)\,,$$

maps continuously D into itself and the set $G(D)$ is equicontinuous. Now we shall show that G satisfies (1). Let V be a subset of D such that

$$V\subset\overline{\operatorname{conv}}\big(G(V)\cup\{0\}\big)\,. \tag{7}$$

Then V is equicontinuous and therefore the function $t\to v(t) = \alpha\big(V(t)\big)$ is continuous on I. Moreover, by (7), (5) and Lemma, for any $t\in I$

we have

$$v(t) \leqslant \alpha\big(G(V)(t)\big) \leqslant |\lambda| \int_I \|K(t,s)\| h(s) v(s)\, ds\,.$$

Since $|\lambda| r(H) < 1$, it follows that $v(t) = 0$ for $t \in I$. In view of the Ascoli theorem, this implies that V is relatively compact in $C(I, E)$. Applying now Theorem 1 we conclude that there exists $x \in D$ such that $x = G(x)$, which ends the proof of Theorem 3.

3. Boundary value problems for nonlinear ordinary differential equations of second order.

In this section we consider the differential equation

$$x'' = f(t, x, x') \tag{8}$$

with the boundary conditions

$$x(0) = x(a) = 0\,. \tag{9}$$

We assume that

1°) $(t, x, y) \to f(t, x, y)$ is a function from $I \times E^2$ into E such that

(i) f is a strongly measurable in t and continuous in (x, y);

(ii) there exists an integrable function $m\colon I \to R_+$ such that

$$\|f(t, x, y)\| \leqslant m(t) \text{ for all } t \in I \text{ and } x, y \in E\,.$$

2°) h is an integrable function from I into R_+ and P, Q are positive numbers such that for all $t \in I$

$$\int_I |G(t,s)| h(s)\, ds \leqslant P \quad \text{and} \quad \int_I \left|\frac{\partial G}{\partial t}(t,s)\right| h(s)\, ds \leqslant Q\,,$$

where

$$G(t,s) = \begin{cases} (t-a)s/a & \text{if } 0 \leqslant s \leqslant t \leqslant a \\ (s-a)t/a & \text{if } 0 \leqslant t \leqslant s \leqslant a\,. \end{cases}$$

3°) $(X, Y) \to d(X, Y)$ is a nonnegative function defined for bounded subsets X, Y of E such that

$$d(X, Y) < p\alpha(X) + q\alpha(Y)$$

for all bounded $X, Y \subset E$ with $\max\big(\alpha(X), \alpha(Y)\big) > 0$.

THEOREM 4. If $pP + qQ \leqslant 1$ and, for any bounded subsets X, Y of E and any $\varepsilon > 0$, there exists a closed subset I_ε of I such that $\mu(I \setminus I_\varepsilon) < \varepsilon$ and

$$\alpha\big(f(T \times X \times Y)\big) \leqslant \sup_{t \in T} h(t) \cdot d(X, Y) \tag{10}$$

for each closed subset T of I_ε, then there exists at least one function x which has absolutely continuous derivative and satisfies (8)-(9) almost everywhere on I.

REMARK. For the case $E = R^n$ Theorem 4 reduces to the Scorza-Dragoni theorem [17].

PROOF. It is well known (cf. [10]) that the problem (8)-(9) is equivalent to the equation

$$x(t) = \int_I G(t, s) f\big(s, x(s), x'(s)\big)\, ds .$$

We introduce the following denotations:

C_1: the space of continuously differentiable function $x: I \to E$ with the norm $\|x\|_1 = \max(\|x\|_c, k\|x'\|_c)$, where $k = P/Q$;

$$M = \sup\{|G(t, s)|: t, s \in I\}, \qquad N = \sup\left\{\left|\frac{\partial G}{\partial t}(t, s)\right|: t, s \in I\right\},$$

$$r_0 = M\int_I m(s)\, ds, \qquad r_1 = N\int_I m(s)\, ds, \qquad r = \max(r_0, kr_1);$$

D: the set of all $x \in C_1$ which satisfy the following inequalities:

$$\|x\|_1 \leqslant r, \ \ \|x'(t) - x'(\tau)\| \leqslant \Big|\int_\tau m(s)\, ds\Big|, \ \ \|x(t) - x(\tau)\| \leqslant r_1|t - \tau| (t, \tau \in I) .$$

Further, for any integrable function $y\colon I \to E$ denote by $L(y)$ the unique solution of

$$x'' = y(t),\ x(0) = x(a) = 0 .$$

As

$$L(y)(t) = \int_I G(t, s)\, y(s)\, ds \qquad \text{for } t \in I ,$$

we get

$$\|L(y_1) - L(y_2)\|_1 \leqslant \max(M, kN) \int_I \|y_1(s) - y_2(s)\|\, ds \tag{11}$$

for any integrable $y_1, y_2\colon I \to E$.

Consider now the mapping F defined by

$$F(x)(t) = \int_I G(t, s)\, f\big(s, x(s), x'(s)\big)\, ds \qquad (x \in C_1,\ t \in I) .$$

Since for any $x \in C_1$ the function

$$t \to u(t) = \int_I G(t, s)\, f\big(s, x(s), x'(s)\big)\, ds$$

satisfies the inequalities

$$\|u''(t)\| \leqslant m(t),\ \ \|u'(t)\| \leqslant r_1,\ \ \|u(t)\| \leqslant r_0$$

and consequently, by the mean value theorem,

$$\|u'(t) - u'(\tau)\| \leqslant \Big|\int_\tau^t m(s)\, ds\Big| \ \text{ and } \ \|u(t) - u(\tau)\| \leqslant r_1 |t - \tau|$$

for $t, \tau \in I$, we see that

$$F(C_1) \subset D . \tag{12}$$

Moreover, by (11),

$$\|F(x_1) - F(x_2)\|_1 \leqslant \max(M, kN) \int_I \|f\big(s, x_1(s), x_1'(s)\big) - f\big(s, x_2(s), x_2'(s)\big)\|\, ds$$

for $x_1, x_2 \in C_1$. By 1° and the Lebesgue dominated convergence theorem it follows that F is a continuous mapping $C_1 \to D$.

Let V be a subset of D such that

$$V \subset \overline{\operatorname{conv}}\,(F(V) \cup \{0\}) .$$

Then, owing to (12), V and $V' = \{x' : x \in V\}$ are uniformly bounded and equicontinuous subsets if $C(I, E)$, and for any $t \in I$

$$V(t) \subset \overline{\operatorname{conv}}\left\{\int_I G(t, s)\, f\big(s, x(s), x'(s)\big)\, ds : x \in V\right\}$$

and

$$V'(t) \subset \overline{\operatorname{conv}}\left\{\int_I \frac{\partial G}{\partial t}(t, s) f\big(s, x(s), x'(s)\big)\, ds : x \in V\right\}.$$

By (10) and Lemma, from this it follows that

$$\alpha\big(V(t)\big) \leqslant \int_I |G(t, s)|\, h(s)\, ds \cdot d\big(V(I), V'(I)\big) \leqslant Pd\big(V(I), V'(I)\big)$$

and

$$\alpha\big(V'(t)\big) \leqslant \int_I \left|\frac{\partial G}{\partial t}(t, s)\right| h(s)\, ds \cdot d\big(V(I), V'(I)\big) \leqslant Qd\big(V(I), V'(I)\big)$$

for all $t \in I$.

On the other hand, in view of Ambrosetti's lemmas [1; Th. 2.2, 2.3] we have

$$(13) \qquad \begin{cases} \alpha_c(V) = \alpha\big(V(I)\big) = \sup\limits_{t \in I} \alpha\big(V(t)\big) \\ \alpha_c(V') = \alpha\big(V'(I)\big) = \sup\limits_{t \in I} \alpha\big(V'(t)\big) , \end{cases}$$

where α_c is the Kuratowski measure of noncompactness in $C(I, E)$. Hence

$$(14) \qquad \max\big(\alpha_c(V), k\alpha_c(V')\big) \leqslant Pd\big(V(I), V'(I)\big) .$$

Suppose that $\alpha_c(V) > 0$ or $\alpha_c(V') > 0$. Then, by 3° and (13),

$$d\big(V(I),\, V'(I)\big) < p\alpha_c(V) + q\alpha_c(V') \,. \tag{15}$$

As $P = kQ$, from (14) and (15) we deduce that

$$\max\big(\alpha_c(V),\, k\alpha_c(V')\big) < (pP + qQ)\max\big(\alpha_c(V),\, k\alpha_c(V')\big)$$

which is impossible, since $pP + qQ \leqslant 1$. This proves that $\alpha_c(V) = 0$ and $\alpha_c(V') = 0$, i.e. V is relatively compact in C_1.

Now we can apply Theorem 1 which yields the existence of $x \in D$ such that $x = F(x)$. It is clear that x is a solution of (8)-(9).

EXAMPLE. Let E be the space of real continuous functions on I with the usual supremum norm, and let us put

$$a(x, y)(s) = \frac{p|x(s)|}{1 + |x(s)|} + \frac{q|y(s)|}{1 + |y(s)|} \quad \text{for } x, y \in E \text{ and } s \in I \,.$$

It can be easily verified that

$$\alpha\big(a(X \times Y)\big) \leqslant \frac{p\alpha(X)}{1 + \alpha(X)} + \frac{q\alpha(Y)}{1 + \alpha(Y)}$$

for each bounded subsets X, Y of E.

Therefore, for a given completely continuous function $b\colon I \times E^2 \to E$, the function f defined by

$$f(t, x, y) = h(t)a(x, y) + b(t, x, y)$$

satisfies (10) with

$$d(X, Y) = \frac{p\alpha(X)}{1 + \alpha(X)} + \frac{q\alpha(Y)}{1 + \alpha(Y)} \,.$$

Notice also that for any p_1, q_1 such that $p_1 \leqslant p$, $q_1 \leqslant q$ and $p_1 + q_1 < p + q$ there exist subsets X, Y of E for which $d(X, Y) > p_1\alpha(X) + q_1\alpha(Y)$.

4. Quasilinear differential equations.

In this section we give an existence theorem for solutions of the Cauchy problem

$$(16) \qquad x' = A(t)x + f(t, x), \qquad x(0) = x_0 ,$$

where A is a Bochner integrable function from I into the space $L(E)$ of bounded linear operators in E and f is a function from $I \times B$ into E satisfying the Caratheodory conditions, i.e. f is strongly measurable in t and continuous in x, and there exists an integrable function $m: I \to R_+$ such that $\|f(t, x)\| \leqslant m(t)$ for $(t, x) \in I \times B$. Let $\Delta = \{(t, s): 0 \leqslant s \leqslant t \leqslant a\}$, and let $U: \Delta \to L(E)$ be an evolution operator for the equation $x' = A(t)x$. Then

($U1$) the function $(t, s) \to U(t, s)$ is continuous on Δ;

($U2$) $U(t, s)\, U(s, r) = U(t, r)$ and $U(t, t) = I$ for all (t, s), $(s, r) \in \Delta$;

($U3$) there is an integrable function $a: I \to R_+$ such that

$$\| U(t, s) \| \leqslant \exp\left[\int_s^t a(r)\, dr\right] \text{ for all } (t, s) \in \Delta .$$

Let us recall (cf. [3]) that a function $x: [0, d] \to E$ is called a solution of (16) if x is continuous and satisfies

$$(17) \qquad x(t) = U(t, 0)\, x_0 + \int_0^t U(t, s) f\big(s, x(s)\big)\, ds \text{ for all } t \in [0, d] .$$

THEOREM 5. Assume in addition that

1°) h is a function from $I \times R_+$ into R_+ such that $(t, r) \to a(t)r + h(t, r)$ is a Kamke function;

2°) for any subset X of B and any $\varepsilon > 0$ there exists a closed subset I_ε of I such that $\mu(I \setminus I_\varepsilon) < \varepsilon$ and

$$\alpha\big(f(T \times X)\big) \leqslant \sup_{t \in T} h\big(t, \alpha(X)\big)$$

for each closed subset T of I_ε.

Then there exists at least one solution of (16) defined on a subinterval of I.

PROOF. Let us put $K(t, s) = \exp\int_s^t a(r)\,dr$ and $A = \sup\{k(t, s)\colon (t, s) \in \in \Delta\}$. We choose a number d such that $0 < d \leqslant a$ and

$$\|U(t, 0)x_0 - x_0\| + \int_0^t k(t, s)\,m(s)\,ds \leqslant b \text{ for all } t \in [0, d]\,. \tag{18}$$

Let $J = [0, d]$, and let D be the set of those functions from $C(J, E)$ with values in B. Consider the mapping F defined by

$$F(x)(t) = U(t, 0)x_0 + \int_0^t U(t, s) f\big(s, x(s)\big)\,ds \qquad (x \in D,\ t \in J)\,.$$

From $(U1')$, (18) and the inequalities

$$\|F(x)(t) - F(x)(\tau)\| \leqslant$$

$$\leqslant \|U(t, 0)x_0 - U(\tau, 0)x_0\| + \int_0^\tau \|U(t, s) - U(\tau, s)\|\,m(s)ds + A\int_\tau^t m(s)ds\,.$$

and

$$\|F(x)(t) - x_0\| \leqslant \|U(t, 0)x_0 - x_0\| + \int_0^t k(t, s)m(s)ds \qquad (x \in D,\ 0 \leqslant \tau \leqslant t \leqslant d)$$

it follows that $F(D)$ is an equicontinuous subset of D.

On the other hand, if x_n, $x \in D$ and $\lim\limits_{n\to\infty} \|x_n - x\|_c = 0$, then by

$(U1')$ and the Lebesgue dominated convergence theorem we get

$$\lim_{n\to\infty} F(x_n)(t) = F(x)(t) \quad \text{for } t \in J .$$

From this we infer that F is a continuous mapping $D \to D$.

For any positive integer n we define a function u_n by

$$u_n(t) = \begin{cases} x_0 & \text{if } 0 \leqslant t \leqslant d/n \\ U(t-d/n, 0)\, x_0 + \int\limits_0^{t-d/n} U(t-d/n, s)\, f\big(s, u_n(s)\big)\, ds & \text{if } d/n \leqslant t \leqslant d . \end{cases}$$

Then $u_n \in D$ and

$$\lim_{n\to\infty} \|u_n - F(u_n)\|_c = 0 . \tag{19}$$

Put $V = \{u_n : n = 1, 2, \ldots\}$ and $W = F(V)$. It is clear from (19) that the sets W, V are equicontinuous and uniformly bounded, and

$$\alpha\big(V(t)\big) = \alpha\big(W(t)\big) \quad \text{for } t \in J . \tag{20}$$

Thus the function $t \to v(t) = \alpha\big(V(t)\big)$ is continuous on J. By Lemma from 2° it follows that

$$\alpha\Big(\Big\{\int\limits_t^\tau U(\tau, s) f\big(s, x(s)\big) ds : x \in V\Big\}\Big) \leqslant \int\limits_t^\tau k(\tau, s) h\big(s, v(s)\big) ds \qquad (0 \leqslant t \leqslant \tau \leqslant d) .$$

On the other hand, as

$$F(x)(\tau) = U(\tau, t) F(x)(t) + \int\limits_t^\tau U(\tau, s) f\big(s, x(s)\big) ds \qquad (x \in D,\ 0 \leqslant t \leqslant \tau \leqslant d) ,$$

we have

$$\alpha\big(F(V)(\tau)\big) \leqslant \|U(\tau, t)\| \alpha\big(F(V)(t)\big) + \alpha\Big(\Big\{\int\limits_t^\tau U(\tau, s) f\big(s, x(s)\big) ds : x \in V\Big\}\Big) .$$

Hence, owing to $(U3)$ and (20), we get

$$v(\tau) \leqslant \exp\left[\int_t^\tau a(s)\,ds\right] v(t) + \int_t^\tau \exp\left[\int_s^\tau a(r)dr\right] h(s, v(s))ds\,,$$

i.e.

$$v(\tau) - v(t) \leqslant \left(\exp\left[\int_0^\tau a(s)ds\right] - \exp\left[\int_0^t a(s)ds\right]\right) \exp\left[-\int_0^t a(s)ds\right] v(t) +$$

$$\exp\left[\int_0^\tau a(s)ds\right] \int_t^\tau \exp\left[-\int_0^s a(r)dr\right] h(s, v(s))ds \qquad \text{for } 0 \leqslant t \leqslant \tau \leqslant d\,.$$

From this we deduce that the function v is absolutely continuous on J and

$$v'(t) \leqslant a(t)v(t) + h(t, v(t)) \qquad \text{for almost every } t \in J\,.$$

In view of 1° and the theorem on differential inequalities, it follows that $v(t) = 0$ for all $t \in J$. Consequently, by Ascoli's theorem, the set V is relatively compact in $C(J, E)$. Hence we can find a subsequence (u_{n_j}) of (u_n) which converges uniformly to a limit u. Now, using (19) and the continuity of F, we obtain $u = F(u)$. It is clear that u is a solution of (17).

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