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ZBIGNIEW SLODKOWSKI

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Introduction.

Applications of potential theory to spectral theory began in 1968 with the following theorem of E. Vesentini [11].

If $z \to T_z$ is an analytic operator valued function in $G \subset \mathbf{C}$ then $\log r(T_z)$, where $r(\cdot)$ denotes the spectral radius, is subharmonic in G.

J. Wermer [12] has obtained a similar result in uniform algebras:

If f, g belong to a uniform algebra A then the function log $\max|\hat{g}(\hat{f}^{-1}(z))|$ is subharmonic in $G = f(M_A) \setminus f(\partial_A)$.

These results were extended by Aupetit-Wermer [2], and the author [7]. Eventually the author has proved the following (cf. [7], Cor. 3.3 and Cor. 3.4):

If $K_z = \sigma(T_z)$ or $K_z = \hat{g}(\hat{f}^{-1}(z))$ (with the above denotations), then the correspondence $z \to K_z$: $G \to 2^{\mathbf{C}}$ is analytic.

(A set-valued correspondence $z \to K_z$ is *analytic* if it is upper semicontinuous and the set $\{(z, w) \in G \times \mathbf{C} : w \notin K_z\}$ is pseudoconvex (cf. [4], [8], [1], [5])).

It appears that the analogy between operator theory and uniform algebras, suggested by these results is not accidental: they are special cases of the next theorem (announced in Slodkowski [8*a*], Sec. 7).

THEOREM 1. Let X, Y be Banach spaces, $G \subset \mathbf{C}$ and V: $G \to L(X, Y)$, T: $G \to L(Y)$ be analytic operator-valued functions. Assume that

Indirizzo dell'A.: University of California, Department of Mathematics, Los Angeles, CA 90024, U.S.A.

Current address: The University of Illinois at Chicago, Department of Mathematics, Statistics and Computer Sciences, Chicago, Illinois 60680.

all V_z are topological embeddings and for each $z \in G$, $\operatorname{Im} V_z$ is an invariant subspace for T_z. Let $\tilde{T}_z$ denote the induced operator acting in the quotient space $Y/\operatorname{Im} V_z$. Then

i) $z \to \log r(T_z)$ is a subharmonic function in G,

ii) the set-valued function $z \to \sigma(\tilde{T}_z): G \to 2^{\mathbf{C}}$ is analytic.

It is not the aim of this paper to give the shortest possible proof of this theorem, but rather to present the method together with, implicit to it, notion of analytic families of Banach spaces. From among the variety of possible definitions the author has singled out two, namely of subanalytic (Def. 1.2) and of analytic families of Banach spaces (Def. 1.3). The family of quotient spaces $Y/\operatorname{Im} V_z$ is a special case of the latter notion.

It is perhaps worthwhile to stress that the spaces in the analytic family do not have to be isomorphic (unlike fibers of locally trivial Banach bundle) nor they have to be given as subspaces of some ambient space (ax it is in Shubin [10]).

We will state now the main results, but we have to refer the reader to indicated places of this article for undefined terms.

THEOREM 2. If $X = \{X_z\}_{z\in M}$, M open in $\mathbf{C}$, is a subanalytic family of Banach spaces (Def. 1.2) and $T_z \in L(X_z)$ is an analytic family of bounded operators (Def. 1.12) with locally uniformly bounded norm, then the function $z \to \log r(T_z)$ is quasi-subharmonic (Def. 1.8).

THEOREM 3. If $X = \{X_z\}$, $M \subset \mathbf{C}$, is an analytic family of spaces (Def. 1.3) and $T_z \in L(X_z)$ is an analytic family of bounded operators, then the set valued function $z \to \sigma(T_z)$ is analytic.

Theorems 1 and 2 are proved in Sec. 1. For the proof of Th. 3, which is given in Sec. 3, several properties of analytic sections are required. They are studied in Sec. 2.

All Banach spaces considered in this note are assumed to be complex.

1. Subanalytic families of Banach spaces.

DEFINITION 1.1. A *family of Banach spaces* with parameter space M is a collection $X = \{X_z, \|\cdot\|_z\}_{z\in M}$, of Banach spaces and norms, together with sets of sections $F(U)$, $F^a(V)$, where U and V are arbitrary open subsets of M. If $x \in F(U)$ or $y \in F^a(V)$, then $x: U \to \bigcup \{X_z: z \in U\}$ and $y: V \to \bigcup \{X_z^*: z \in V\}$, so that $x(z) \in X_z$ and $y(z) \in X_z^*$ for every $z \in U$ or V.

This notion plays an auxiliary role only. We denote also $\|x\|_U = \sup \{\|x(z)\|_z: z \in U\}$, and, by $C(z, U)$ (where $z \in U \subset M$), the infimum of constants C (or $+\infty$) such that for every $x_0 \in X_z$ and $y_0 \in X_z^*$, with $\|x_0\|_z = 1$ and $\|y_0\|_z = 1$ exist $x \in F(U)$ and $y \in F^a(U)$ such that $x(z) = x_0$, $y(z) = y_0$, $\|x\|_U \leqslant C$ and $\|y\|_U \leqslant C$.

DEFINITION 1.2. A family of Banach spaces is called *subanalytic* if (i) M is an analytic space; (ii) for every U and V, open subsets of M, and for every $x \in F(U)$, $y \in F^a(V)$, the function $z \to \langle x(z), y(z) \rangle$ is analytic in $U \cap V$, and (iii) for every $z \in M$ there is a neighbourhood U of z such that $C(z, U) < \infty$.

DEFINITION 1.3. A family of Banach spaces is called analytic if (i) and (ii) of the last definition hold together with

(iii)′ there is an open covering $\boldsymbol{U}$ of M such that for every $U \in \boldsymbol{U}$ the constant $C(U) = \sup \{C(z, U): z \in U\}$ is finite.

Of course analytic families are subanalytic. By Ex. 1.13, and Th. 3 the converse is not true.

EXAMPLE 1.4. Let X and Y be Banach spaces and $z \to T_z: M \to L(X, Y)$ be analytic. Assume that all T_z, $z \in M$, have closed ranges. Set $X_z = \operatorname{Im} T_z$. For $U \subset M$ let $F(U)$ be the set of sections $z \to T_z x$, where $x \in X$, and $F^a(U)$ be the set of sections $z \to \varphi | X_z$, where φ is an arbitrary functional in Y^*. It is easy to observe that $\{X_z\}$, together with $F(U)$ and $F^a(U)$, and $\|\cdot\|_z$ restriction of $\|\cdot\|$ in Y to X_z, form a subanalytic family of Banach spaces. If, moreover, $z \to \gamma(T_z)$ (reduced minimum modulus of T_z) is locally uniformly bounded from below, then the family is actually analytic. Under the latter assumption it can be proved, by the methods of [9], that also the collection of quotient spaces $Y/\operatorname{Im} T_z$ is, with a natural structure, an analytic family of Banach spaces. Here we contend ourselves with *a*, seemingly special, case of this situation.

EXAMPLE 1.5. Let X, Y, G, V be as in Th. 1. Set $X_z = Y/\operatorname{Im} V_z$ and let $\|\cdot\|_z$ be the quotient norm. For $U \subset G$, let $F(U)$ be the set of functions $z \to [y + \operatorname{Im} V_z]$, where $z \in U$ and $y \in Y$ is arbitrary, and $F^a(U)$ be the set of functions $\varphi: U \to Y^*$ such that $V_z^* \varphi_z = 0$.

(Note that $\ker V_z^* = (Y/\operatorname{Im} V_z)^*$). The family thus defined is analytic by the next lemma, applied to $S(z) = V_z^*$.

LEMMA 1.6. Let E and F be Banach spaces and $S\colon D(0,R) \to L(E,F)$ be a holomorphic function. Assume that $S(z) = \sum_{n=0}^{\infty} S_n z^n$ where $\|S_n\| \leqslant CR^{-n}$, and $\gamma(S_0) > 1/M$. Then for every $y \in \ker S_0$, with $\|y\| = 1$, there exist analytic function $y\colon D(0,r) \to E$, where $r = R/(1+CM)$, such that (i) $y(0) = y$; (ii) $y(z) \in \ker S(z)$ if $|z| < r$, and (iii) $\|y_n\| \leqslant CM)(1 + CM^{n-1}R^{-n}$, $n \geqslant 1$, where $y(z) = \sum_{n=0}^{\infty} y_n z^n$.

PROOF. Condition (ii) requires that all the equations $\sum_{i=0}^{n} S_i y_{n-i} = 0$, $n = 0, 1, 2, \ldots$, hold. We solve them by induction on n: we start from $y_0 = y$, and then make sure, that (iii) be fulfilled at each step. Assume that $y_0, y_1, \ldots, y_{n-1}$, satisfying (iii) and the first n equations are already constructed. Since $\gamma(S_0) > 1/M$, there is y_n such that $Sy_n = -\sum_{i=1}^{n} S_i y_{n-i}$ and $\|y_n\| \leqslant M\|\sum_{i=1}^{n} S_i y_{n-i}\|$. Using (iii), we can estimate the latter number by

$$M\sum_{i=1}^{n} \|S_i\|\,\|y_{n-i}\| \leqslant \frac{MC}{1+MC}\left((1+MC)/R\right)^n,$$

and so (iii) holds for y_n. It is now lear that $y(z) = \sum_{n=0}^{\infty} y_n z^n$ converges in $D(0,r)$ and has the required properties. Q.E.D.

There are several ways to define analyticity for families of operators in the subanalytic case.

DEFINITION 1.7. Let $T_z \in L(X_z)$, where $\{X_z\}_{z\in M}$ is a subanalytic family. We say that $\{T_z\}_{z\in M}$ is preanalytic if for every $U, V \subset M$ and $x \in F(U)$, $y \in F^d(V)$, the function $z \to \langle T_z x(z), y(z)\rangle$ is analytic in $U \cap V$.

DEFINITION 1.8. We say that $\varphi\colon G \to [-\infty, +\infty)$ is quasi-subharmonic if for every $z \in G$ there is $r_0 > 0$ such that for every $r \in (0, r_0)$

$$\varphi(z) \leqslant \frac{1}{2\pi}\int^{*} \varphi(z + re^{it})\,dt,$$

where $\overset{*}{\int}$ denotes the upper integral (= supremum of integrals of Borel measurable minorants of φ).

LEMMA 1.9. Let $T_z \in L(X_z)$, where $\{X_z\}_{z\in G}$ is a subanalytic family of spaces. Assume that $\|T_z\|$ is locally bounded on G and that for every $n = 1, 2, \ldots, z \to T_z^n$ is a preanalutic family. Then $z \to \log r(T_z)$ is quasi-subharmonic in G.

By Def. 1.12 and the remark after it, the lemma contains Th. 2. We do not know whether the norm condition is not redundant; it is however easy to show that preanalytic families of operators are not preserved by taking powers. Combining the proof below with the methods of [7], one can conclude that the lemma holds also for functions $z \to \log \delta_n(\sigma(T_z))$, $z \to \log \operatorname{cap} \sigma(T_z)$, $n \geqslant 0$.

PROOF OF LEMMA 1.9. Fix $z \in G$ and choose $r_0 > 0$, so that $\|T_z\|$ is bounded (by C^1) on $D = D(z, r_0)$, and $C = C(z, D) < \infty$ (cf. Def. 1.2). For each $n \geqslant 1$ choose $x_n \in F(D)$, $y_n \in F^d(D)$, such that $\|x_n(z)\|_z = 1 = \|y_n(z)\|_z$ and $|\langle T_z^n x_n(z), y_n(z)\rangle| \geqslant \|T_z^n\|(1 - 1/n)$ and moreover $\|x\|_D \leqslant C$, $\|y\|_D \leqslant C$. Clearly the functions $z' \to 1/n \log |\langle T_z^n x_n(z'), y_n(z')\rangle|$ are locally uniformly bounded from the above in D and so their limes superior, which we denote $\psi(z')$, is Borel measurable and quasisubharmonic in D. Since $\limsup 1/n \log \|T_{z'}^n\| = \log r(T_{z'})$, we get $\psi(z') \leqslant \log r(T_{z'})$ in D, and so

$$\log r(T_z) = \psi(z) \leqslant \frac{1}{2\pi}\int \psi(z + re^{it})\,dt \leqslant \frac{1}{2\pi}\overset{*}{\int} \log r(T_{z+re^{it}})\,dt\,. \qquad \text{Q.E.D.}$$

LEMMA 1.10. Let $T_z \in L(X_z)$, where $\{X_z\}_{z\in G}$, $G \subset C$ is a subanalytic family of spaces. Assume that $z \to \sigma(T_z)\colon G \to 2^{\mathbf{C}}$ is upper semi-continuous, and that all powers of $(z, w) \to (T_z - wI)^{-1}$ are preanalytic in $\{(z, w)\colon z \in G,\ w \notin \sigma(T_z)\}$. Then the multifunction $K_z = \sigma(T_z)$ is analytic in G.

PROOF. By [8, Th. 3.2] it suffices to show that for every $a, b \in C$ the function $\varphi(z) = \log \max \{|w - za - b|^{-1}\colon w \in K_{az+b}\}$ is subharmonic in $\{z\colon az + b \notin K_z\}$. Of course $\varphi(z) = \log r(S_z)$, where $S_z = (T_z - (za + b)I)^{-1} \in L(X_z)$. Since $z \to S_z^n$ is preanalytic for $n \geqslant 1$, $\varphi(z)$ is quasi-subharmonic; seeing that K is upper semi-continuous, so is $\varphi(z)$. Thus $y(z)$ is subharmonic. Q.E.D.

Example 1.13 will show that assumptions of the above, rather technical lemma, are difficult to check in subanalytic families of spaces.

The remaining sections of this paper are mostly devoted to showing that the required assumptions hold for an analytic family T_z in an analytic family of Banach spaces. We will now observe that the lemma yields Theorem 1 rather easily.

PROOF OF THEOREM 1 (*Sketch*). We keep the notation of Example 1.5. We omit easy proof that $z \to \sigma(T_z)$, and so $r(T_z)$, are upper semicontinuous. If T is analytic, then for every n, $z \to \langle T_z^n y + \operatorname{Im} V_z, \varphi_z \rangle$, is analytic provided $y \in Y$, φ_z analytic and $\varphi_z \in \ker V_z^*$. Thus $z \to T_z^n$ is preanalytic for $n \geqslant 1$. By Lemma 1.9 function $z \to \log r(T_z)$ is subharmonic.

We have to chack yet that $(z, w) \to (\tilde{T}_z - wI)^{-n}$ are preanalytic, *i.e.*, that with y, φ_z as above, $(z, w) \to \langle (\tilde{T}_z - wI)^{-n} y, \varphi_z \rangle$ is analytic. To this end it is enough to find a polydisc neighbourhood P of given (z_0, w_0), $w_0 \notin \sigma(\tilde{T}_{z_0})$, such that for every $f \in H(P, Y) =$ (holomorphic functions with values in Y), there are $g \in H(P, Y)$ and $h \in H(P, X)$ such that

$$(1.1) \qquad (T_z - wI) g(z, w) + V_z h(z, w) = f(z, w) \,, \qquad (z, w) \in P \,.$$

Then $(\tilde{T}_z - wI)^{-1} [f(z, w) + \operatorname{Im} V_z] = [g(z, w) + \operatorname{Im} V_z]$, and, by iteration, one gets the same for $(T_z - wI)^{-n}$. Concerning (1.1), it can be solved by modifying slightly the method of Lemma 1.6, or else by application of a more general result of Leiterer [3, Th. 5.1]. We omit further details. Q.E.D.

Before closing this section, we introduce analytic families of operators of a new kind and compare them with preanalytic ones. We will also discuss examples indicating some pathologies of the subanalytic case. First define analytic sections.

DEFINITION 1.11. Let $\{X_z\}_{z \in M}$ (M is an analytic space) be a subanalytic family of Banach spaces. Let $f \colon U \to \bigcup X_z$ or $g \colon U \to \bigcup X_z^*$, $U \subset M$, and $f(z) \in X_z$, $g(z) \in X_z^*$, $z \in U$. Function f (or g) is said to be analytic if for every $V \in M$ and for every $y \in F^d(V)$ (or $x \in F(V)$) the function $z \to \langle f(z), y(z) \rangle$ (or $z \to \langle x(z), g(z) \rangle$) is analytic in $U \cap V$. We write $f \in H(U, X)$ (or $g \in H(U, X^*)$).

REMARK. One can also define spaces of strongly analytic sections, $H_s(U, X)$ and $H_s(U, X^*)$ by pairing f, g with arbitrary y, x in $H(V, X^*)$, $H(V, X)$ respectively, similarly as in the last definition. It will turn out later that for analytic families of spaces classes H_s and H are equal.

DEFINITION 1.12. Let $T_z \in L(X_z)$, $z \in M$, where $\{X_z\}_{z\in M}$ is a subanalytic family of Banach spaces. We call $z \to T_z$ analytic if for every $f \in H(U, X)$, $U \subset M$, the section $z \to T_z f(z)$ is analytic.

REMARK. It is obvious that if $\{T_z\}$ and $\{S_z\}$ are analytic families of operators then $z \to T_z \circ S_z$ is also analytic. Moreover analytic families of operators are preanalytic. By this and Lemma 1.9, Theorem 2 holds.

EXAMPLE 1.13. Define subanalytic family of spaces setting $X_z = \mathbf{C} \oplus \mathbf{C}$, $z \in \mathbf{C} \setminus \{0\}$, and $X_0 = \mathbf{C} \oplus (0)$; $X_z^* = X_z$, with the natural pairing. We set $F(U) = F^a(U) = H(U) \oplus H(U)$ if $0 \notin U$, and $F(U) = F^a(U) = H(U) \oplus zH(U)$, if $0 \in U$. One can check that $H(U, X) = H(U, X^*) = H(U) \oplus \chi_{|U \setminus \{0\}|} H(U)$, where χ denotes the characteristic function. Define $T_z \in L(X_z)$, $z \in \mathbf{C}$, by $T_z(x, y) = (x, zy)$ for $z \neq 0$ and $T_0 = Id_{X_0}$. It is clear that $z \to T_z$ is an analytic family of invertible operators. However, neither $z \to T_z^{-1}$ is analytic, nor $z \to \sigma(T_z)$ is uppersemicontinuous (for $\sigma(T_z) = \{1, z\}$ if $z \neq 0$ and $\sigma(T_0) = \{1\}$).
This example suggests that, in the general case, Lemma 1.10 cannot be much improved.

2. Sections of analytic families of Banach spaces.

In this section we study properties of analytic sections needed for the proof of Th. 3. The key result is Th. 2.5, which will enable us to represent locally an analytic section by means of a «power series» with respect to a given, sufficiently rich, set of sections.

LEMMA 2.1. Let $X = \{X_z\}_{z\in G}$, $G \subset \mathbf{C}$ be an analytic family of Banach spaces. Let $U \subset G$ be open such that $C = C(U)$ is finite. Then for every K compact in U and for every $g \in H(U, X^*)$

$$\|g\|_K \leqslant C \|g\|_{\partial K} .$$

Sections in $H(U, X)$ have the same property.

PROOF. Let z be an arbitrary point of K. By Def. 1.3 exists $x \in F(U)$ such that $\|x(z)\| = 1$, $|\langle x(z), g(z)\rangle| \geqslant (1 - \varepsilon)\ \|g(z)\|_z$ and $\|x\|_U \leqslant C$. Since

$\langle x, g\rangle$ is analytic in U,

$$(1-\varepsilon)\|g(z)\|_z \leqslant \max_{u\in\partial K} |\langle x(u), g(u)\rangle| \leqslant C\|g\|_{\partial K}.$$

Seeing that $\varepsilon > 0$ and $z \in K$ are arbitrary, the required estimate follows. The second statement is obtained in the same way. Q.E.D.

COROLLARY 2.2. Let $X = \{X_z\}_{z\in M}$, where M is a complex manifold, be an analytic family of Banach spaces. Then for every open $U \subset M$ the spaces $H(U, X)$ and $H(U, X^*)$, considered with seminorms $\|\cdot\|_K$, K compact in U, are complete locally convex spaces.

LEMMA 2.3. Let X be analytic family of Banach spaces, $g \in H(U, X^*)$, and $a \in U \subset \mathbf{C}$. Assume that U is connected and that for every $n = 1, 2, \ldots$ there exist g_n in $H(U, X^*)$ such that $g(z) = (z-a)^n g_n(z)$, $z \in U$. Then $g(z) = 0$ in U. The same property holds for $H(U, X)$.

PROOF. Denote by R the interior of the set $\{z \in U : g(z) = 0\}$. It is enough to show that R is nonempty and open. Choose connected $V \subset U$ such that $C = C(V) < \infty$. Consider $b \in V$ and suppose $g(b) \neq 0$. Then there is $x \in F(V)$ such that $\langle x(b), g(b)\rangle \neq 0$. (Note that the bound C does not play any role here). Since $\langle x(z), g(z)\rangle = (z-a)^n \cdot \langle x(z), g_n(z)\rangle$, $n \geqslant 1$, the analytic function $\langle x(z), g(z)\rangle$ vanishes in V and so at b. Thus R is nonempty. If $z_0 \in \bar{R}$, choose connected V with $C(V) < +\infty$. For every $x \in F(V)$, and for every $z \in V$, $\langle x(\cdot), g(\cdot)\rangle$ vanishes on $V \cap R$ and so at z. Arguing as above we get $g(z) = 0$ for $z \in V$ and so $z_0 \in R$. Set R being both closed and open is equal to U. Q.E.D.

LEMMA 2.4. Let X be an analytic family over U, $a \in U$ and $g \in H(U, X^*)$. Assume that $g(a) = 0$. Then there is $h \in H(U, X^*)$ such that $g(z) = (z-a)h(z)$ in U.

PROOF. Assume without loss of generality that $a = 0$. Choose V such that $0 \in V \subset U$ and $C = C(V) < \infty$. Since $\langle x(z), g(z)\rangle$ is analytic for every $x \in F(V)$, the limit

$$\lim_{z\to 0} z^{-1}\langle x(z), g(z)\rangle \tag{2.1}$$

always exists. We intend to show that the limit depends only on $x(0)$. For this we need an estimate of $z^{-1}g(z)$, similar to Lemma 2.1 but more delicate. Consider compact $K \subset V$ such that $0 \in \operatorname{Int} K$. Let $z \in K\setminus\{0\}$. For $\varepsilon > 0$, choose $f \in F(V)$ such that $|\langle f(z), g(z)z^{-1}\rangle| \geqslant (1-\varepsilon)\, \|g(z)z^{-1}\|_z$ and $\|f\|_V \leqslant C$. Since the function $\langle f(u), g(u)\rangle u^{-1}$ has a removable singularity at $u = 0$,

$$|\langle f(z), g(z)z^{-1}\rangle| \leqslant \sup_{u\in\partial K} |\langle f(u), g(u)u^{-1}\rangle|\,,$$

and, since $\varepsilon > 0$ is arbitrary, we get

$$\sup_{z\in K\setminus\{0\}} \|g(z)z^{-1}\|_z \leqslant C \sup_{z\in\partial K} \|g(z)z^{-1}\|_z\,. \tag{2.2}$$

In a similar way one can prove that for $f \in H(V, X)$, such that $f(0) = 0$,

$$\sup_{z\in K\setminus\{0\}} \|f(z)z^{-1}\|_z \leqslant C \sup_{z\in\partial K} \|f(z)z^{-1}\|_z\,.$$

From the last two estimates it follows in particular that if $f(0) = 0$, $g(0) = 0$, $f \in H(V, X)$, and $g \in H(V, X^*)$, then $\lim_{z\to 0} \langle f(z), g(z)\rangle z^{-1} = 0$. Thus if $x_1, x_2 \in F(V)$ and $x_1(z) = x_2(z)$ then $\lim_{z\to 0} z^{-1}\langle x_1(z) - x_2(z), g(z)\rangle = 0$. We conclude that the limit (2.1) depends only on $x(0)$. Since each vector in X_0 is equal to $x(0)$ for some x in $F(V)$, we have thus defined a function, say φ, on X_0. We omit easy proof that φ is linear, and note that if $\|x\|_V \leqslant C\|x(0)\|$, then

$$|\varphi(x(0))| = \lim_{z\to 0} |z^{-1}\langle x(z), g(z)\rangle| \leqslant \sup_{z\in K\setminus\{0\}} \|x(z)\|_z\, \|g(z)z^{-1}\|_z \leqslant$$

$$\leqslant C\|x(0)\| \sup_{z\in\partial K} \|g(z)z^{-1}\|_z \qquad (\text{by } (2.2))\,.$$

Thus $\varphi \in X_0^*$. It is now evident that if we set $h(z) = g(z)z^{-1}$ for $z \in U\setminus\{0\}$ and $h(0) = \varphi$, we obtain an analytic section h of X^* over U. Of course $g(z) = zh(z)$, for $z \in U$. Q.E.D.

THEOREM 2.5. Let X be an analytic family of Banach spaces over $U \subseteq \mathbf{C}$. Let $a \in U$ and let G be a subset of $H(U, X^*)$. Assume that $\{g(a) : g \in G\}$ contains the unit sphere of X_a^* and that G is uniformly bounded on U. Then there is $r > 0$ such that $D = D(a, r) \subset U$ and for every $f \in H(U, X^*)$ there exist sequences $\{g_n\} \subset G$ and $\{\alpha_n\} \subset \mathbf{C}$ such that

$$f(z) = \sum_{n=0} \alpha_n g_n(z)(z-a)^n, \quad z \in D,$$

where $\sum_n |\alpha_n| r^n < \infty$.

PROOF. Assume without loss of generality that $a = 0$. Let $D' = D(0, R)$ be such that $\bar{D}' \subset V$, where $V \subset U$ with $C(V) < \infty$. Using properties of G and Lemma 2.4., we can represent any f in $H(U, X^*)$ as $g(z) + zf_1(z)$, where $\alpha \in \mathbf{C}$, $g \in G$ and $\|g(0)\| = 1$. Let $C = C(V)$ and $L = \sup\{\|g\|_{\bar{D}'} : g \in G\}$. Then we have the estimates $|\alpha| \leqslant \|f\|_{\bar{D}'}$, $\|zf_1(z)\|_{\partial D'} \leqslant \|f\|_{\bar{D}'} (1 + L)$, and, by Lemma 2.1,

$$\|f_1\|_{\bar{D}'} \leqslant CR^{-1}(1 + L)\, \|f\|_{\bar{D}'}. \tag{2.3}$$

Using this we can construct, starting from arbitrary $f \in H(U, X^*)$, sequences $\{f_n\}$ in $H(U, X^*)$, $\{g_n\}$, $n \geqslant 0$, in G, and $\{\alpha_n\}$ in $\mathbf{C}$ so that $f_0 = f$, $\|g_n(0)\|_0 = 1$, and $f_n(z) = \alpha_n g_n(z) + zf_{n+1}(z)$, $n \geqslant 0$, $z \in U$. By (2.3) $\|f_n\|_{\bar{D}'} \leqslant CR^{-1}(1 + L)\, \|f_{n-1}\|_D$ and so by induction $\|f_n\|_{\bar{D}'} \leqslant \big(C(1 + L)/R\big)^n \|f\|_{\bar{D}'}$. Since $\alpha_n = \|f_n(0)\|_0$, $\sum_n |\alpha_n| r^n < \infty$ for $r < R\big(C(1+L)\big)^{-1}$. For such r function $\tilde{f}(z) = \sum_{n=0}^{\infty} \alpha_n g_n(z) z^n$ is well defined in $D = D(0, r)$ and by Corollary 2.2 belongs to $H(D, X^*)$. It follows from our construction that

$$(f - \tilde{f})(z) = z^n\Big(f_n(z) - \sum_{k=n}^{\infty} \alpha_k z^{k-n} g_n(z)\Big), \quad \text{for } n = 1, 2, \ldots,$$

and so by Lemma 2.3. $f - \tilde{f} \equiv 0$ in D. Q.E.D.

COROLLARY 2.6. Let X be an analytic family of Banach spaces over $U \subset C$. Then for every $f \in H(U, X)$ and for every $g \in H(U, X^*)$ the function $z \to \langle f(z), g(z) \rangle$ is analytic in U.

PROOF. For arbitrary $a \in U$ choose $V \subset U$ such that $C(V) < +\infty$ and let $G = \{y \in F^d(V) : \|y\| V \leqslant C(V) + 1\}$. By Th. 2.5 there is a disc $D \subset V$, centered at a, such that each $g \in H(U, X^*)$ can be represented by means of an absolutely convergent series $\sum_{n=0}^{\infty} \alpha_n g_n(z)(z - a)^n$ with $g_n \in G$. Since $g_n \in H^d(V)$, all functions $z \to \langle f(z), g_n(z) \rangle$ are holomorphic in D, and so is $z \to \langle f(z), g(z)\rangle$. Since a was arbitrary, the function $\langle f(z), g(z)\rangle$ is holomorphic in U. Q.E.D.

REMARK 2.7. This result implies that for an analytic family X of Banach spaces $H(U, X) = H_s(U, X)$ and $H(U, X^*) = H_s(U, X^*)$.

3. Families of invertible operators.

THEOREM 3.1. Let $\{X_z\}_{z\in U}$, $U \subset \mathbf{C}$, be an analytic family of Banach spaces and $T_z \in L(X_z)$, $z \in U$. Assume that all T_z are invertible and the family $\{T_z\}_{z\in U}$ is analytic. Then

i) the set-valued function $z \to \sigma(T_z)\colon U \to 2^{\mathbf{C}}$ is upper semicontinuous;

ii) the family $\{T_z^{-1}\}_{z\in U}$ is analytic.

This fact is all we need to derive Theorem 3 from Lemma 1.10. We prove the statements (i) and (ii) separately. To handle the first one it is convenient to introduce continuous families of Banach spaces.

DEFINITION 3.2. A family of spaces as in Def. 1.1 is called *continuous* if M is locally compact and admits an open covering $\boldsymbol{U}$ such that

i) for every $U, V \in \boldsymbol{U}$ the collection of functions $z \to \langle x(z), y(z)\rangle$, $z \in U \cap V$, $x \in F(U)$, $y \in F^d(V)$ is *equicontinuous*;

ii) for every $U \in \boldsymbol{U}$ there is a constant C such that for every $z \in U$, $x \in X_z$, $y \in X_z^*$ with $\|x_0\|_z \leqslant 1$, $\|y_0\|_z \leqslant 1$, there exist $x \in F(U)$, $y \in F^d(U)$ such that $x(z) = x_0$ and $y(z) = y_0$.

A family of operators $T_z \in L(X_z)$ is called *continuous* if for every $U \in \boldsymbol{U}$ the set of functions

$$z \to \langle T_z x(z), y(z)\rangle, \quad z \in U, \quad x \in F(U), \quad y \in F^d(U),$$

is *equicontinuous*.

PROPOSITION 3.3. (i) An analytic family of Banach spaces has natural continuous strucutre.

(ii) Analytic family of operators in an analytic family of spaces is continuous.

PROOF. (i) we have to define the continous structure. Let $\mathbf{U}$ be the collection of all open subsets U of M such that $C(U) < \infty$. Define the new sets of sections $F_c(U) = \{x \in F(U): \|x\|_U \leqslant C(U) + 1\}$; $F_c^d(U) = \{y \in F^d(U): \|y\|_U \leqslant C(U) + 1\}$. It is clear that condition (ii) of Def. 3.2 holds. Concerning condition (i), if U and V belong to $\mathbf{U}$, the set of all functions $z \to \langle x(z), y(z)\rangle$, $z \in U \cap V$, $x \in F_c(U)$, $y \in F_c^d(V)$, is uniformly bounded and consists of holomorphic functions; therefore it is equicontinuous.

(ii) Since $z \to T_z$ is analytic, for $f \in H(U, X)$ the function $z \to T_z f(z)$ belongs to $H(U, X)$ (by Def. 1.12) and by Cor. 2.6 the function $t(f, g)(z) = \langle T_z f(z), g(z)\rangle$ is analytic. We have thus defined a bilinear operator $t: H^\infty(U, X) \times H^\infty(U, X^*) \to H(U)$, (where $H^\infty(\cdot)$ consists of all uniformly bounded sections in $H(\cdot)$). Considering evaluations at $z \in U$ we check easily that t has closed graph. Since $H^\infty(U, X)$ and $H^\infty(U, X^*)$ are complete (by Cor. 2.2), t is bounded by the closed graph theorem. Thus for any $C > 0$ the set of functions $z \to \langle T_z f(z), g(z)\rangle$, $z \in U$, $\|f\|_U \leqslant C$, $\|g\|_U \leqslant C$ is bounded in $H(U)$ and so it is equicontinuous. Therefore $\{T_z\}$ is continuous. Q.E.D.

PROPOSITION 3.4. If $\{T_z\}_{z \in M}$ is a continuous family of operators in a continuous family of Banach spaces $\{X_z\}_{z \in M}$, then the set-valued function $z \to \sigma(T_z): M \to 2^{\mathbf{C}}$ is upper semicontinuous.

PROOF. For sake of simplicity we will consider only the case of metrizable (and locally compact) M. It suffices to prove the following two assertions.

Assertion 1. The function $z \to \|T_z\|$ is locally bounded on M.

Assertion 2. The multifunction $z \to \sigma_\pi(T_z)$ has closed graph. The same holds for the multifunction $z \to \sigma_\pi(T_z^*)$.

Concerning Assertion 1, consider U and $C > 0$ as in Def. 3.2 (ii). Since the set $\{z \to |\langle T_z x(z), y(z)\rangle|: x \in F(U),\ y \in F^d(U),\ \|x\|_U \leqslant C, \|y\|_U \leqslant C\}$ is equicontinuous and pointwise bounded, its supremum, say $\varphi(z)$, is continuous. By Def. 3.2 (ii) $\|T_z\| \leqslant \varphi(z)$ for $z \in U$. The assertion follows.

We will check only the first statement of Assertion 2. (The other one can be confirmed in a similar way.) Let $U \in \mathbf{U}$, $z(n) \in U$, $n \geqslant 0$,

$w_n \in \sigma(T_{z(n)})$, $n \geq 1$ and $z(n) \to z(0)$, $w_n \to w_0$. We have to show that $w_0 \in \sigma_\pi(T_{z(0)})$. By definition of σ_π and condition (ii) of Def. 3.2, there are $x_n \in F(U)$, such that $\|x_n\|_U \leq C$ (a constant chosen as in Def. 3.2) and

$$(3.1) \qquad \|(T_{z(n)} - w_n)x_n(z(n))\|_{z(n)} \leq 1/n ,$$

while $\|x_n(z(n))\|_{z(n)} = 1$. Observe first that

$$(3.2) \qquad \inf_n \|x_n(z(0))\|_{z(0)} > 0 .$$

Suppose to the contrary that $x_n(z(0)) \to 0$. Consider for each n the set of functions $|\langle x_n(z), y(z)\rangle|$, $\|y\|_U \leq C$, $y \in F^d(U)$, and denote by $\varphi_n(z)$ its supremum. Since the union of these sets is equicontinuous and bounded, the sequence $\{\varphi_n\}$ is equicontinuous. On the oher hand $\varphi_n(z(n)) \geq \|x_n(z(n))\| \geq 1$ and $\varphi_n(z(0)) \to 0$. Since this contradicts equicontinuity, (3.2) holds.

We show now that $\|(T_{z(0)} - w_0)x_n(z(0))\|_{z(0)} \to 0$. Consider for each n the set of functions $|\langle (T_z - w_n)x_n(z), y(z)\rangle|$, $z \in U$, $y \in F^d(U)$, $\|y\|_U \leq C$, and denote its supremum by $\varphi_n(z)$. The union of these sets is equicontinuous and pointwise bounded, therefore $\{\varphi_n(z)\}$ form an equicontinuous sequence. By (3.1), $\varphi_n(z(n)) \leq C/n$. Since

$$z(n) \to z(0), \quad \varphi_n(z(0)) \to 0, \qquad i.e. \qquad \|(T_{z(0)} - w_n)x_n(z(0))\|_{z(0)} \to 0,$$

and since $w_n \to w_0$, $\|(T_{z(0)} - w)x_n(z(0))\|_{z(0)} \to 0$. This and (3.2) settles Assertion 2. Q.E.D.

COROLLARY 3.5. Let $\{T_z\}_{z \in M}$ be a continuous family of operators in a continuous family of Banach spaces. Assume that all T_z are invertible. Then $z \to \|T_z^{-1}\|_z$ is locally bounded.

PROOF (*Sketch*). If not, then, similarly as in the proof of Assertion 2 above there are U, C, $z(n) \to z(0) \in U$, $\|x_n\|_U \leq C$, $x_n \in F(U)$, such that (3.1) holds with $w_n = 0$. In the same way as above one obtains that $\|T_{z(0)}x_n(z(0))\| \underset{z(0)}{\to} 0$ while (3.2) holds, which contradicts the invertibility of $T_{z(0)}$. Q.E.D.

PROOF OF TH. 3.1. Part (i) of the Theorem follows from Propositions 3.3 (ii) and 3.4; moreover, $\|T_z\|$ is locally bounded. For arbitrary

$a \in U$ choose V, a neighbourhood of a, with compact closure in U and such that $C(V) < \infty$. Let $L > \|T_a^{-1}\| \, C(V)$ and let G be the set of all functions $T_z^* f(z)$, $z \in V$, where $f \in H(U, X^*)$ and $\|f\|_V \leqslant L$. Then G is uniformly bounded on V and $\{g(a) : g \in G\}$ contains the unit sphere of X_a^*. Thus Th. 2.5 can be applied. If $h \in H(V, X^*)$, there is a disc $D = D(a, r)$, functions $g_n \in G$ and scalars $\alpha_n \in \mathbf{C}$ such that

$$(3.3) \qquad h(z) = \sum_{n=0}^{\infty} \alpha_n g_n(z)(z-a)^n, \quad \text{and} \quad \sum_{n=0}^{\infty} |\alpha_n| r^n < \infty .$$

Since $g_n(z) = T_z f_n(z)$, where $\|f_n\|_V \leqslant L$, the series $\sum_{n=0}^{\infty} \alpha_n f_n(z)(z-a)^n$ converges absolutely in D to some $f \in H(V, X^*)$. By (3.3) $h(z) = T_z^* f(z)$ for $z \in D$, *i.e.* the function $z \to (T_z^*)^{-1} h(z)$ is analytic in D, and so in V (a is arbitrary). By Corollary 2.6, for every $k \in H(V, X)$ the function $\langle k(z), f(z) \rangle = \langle k(z), (T_z^*)^{-1} h(z) \rangle = \langle T_z^{-1} k(z), h(z) \rangle$ is analytic. Since h was arbitrary in $H(V, X^*)$, the section $z \to T_z^{-1} k(z)$ is analytic. Thus family $\{T_z^{-1}\}$ is analytic in U. Q.E.D.

REMARK. It can be shown, using results of Rochberg et al. [6, Def. 2.4 and Th. 3.1] that the interpolation spaces of the type $B\{z\}$ considered by the named authors admit natural structure of an analytic family of Banach spaces, with constants $C(U) = 1$. (The details will be discussed elsewhere.) Thus the results of Ransford [5], who studied spectra of interpolated operators, seem to be related to Theorems 1 and 3 of this paper.

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