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Discrete Valuation Domains and Ranks of Their Maximal Extensions.

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The problem of measuring the size of a valuation domain R inside its maximal immediate extension S led L. Salce and the second author to the definition of two functions, the *completion defect* c_R and the *total defect* d_R, from the set of the ideals of R into the class of cardinal numbers [8].

In this paper we prove some formulae that connect the two functions c_R and d_R. When we restrict our attention to the discrete valuation domains with the ascending chain condition (a.c.c.) on prime ideals, these formulae allow us to compute d_R as a function of c_R. Moreover we are able to determine all functions c_R and d_R that can arise as R ranges in the class of the discrete valuation domains of prime characteristic p with the a.c.c. on prime ideals. This involves the construction of rather complicated but interesting examples of rings (Theorem 8).

There are two main differences of notation between this paper and [8]. Firstly, we just consider the two functions c_R and d_R as defined on the set of *prime* ideals, and not the set of all ideals; secondly, the functions c_R and d_R will take only natural numbers and the symbol ∞ as their values, not arbitrary infinite cardinal numbers. The reason for these choices is twofold: on the one hand, it becomes easier to define, understand and employ the functions c_R and d_R; on the other hand,

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our formulae would not hold anymore if they were interpreted as formulae among infinite cardinal numbers.

Recall that an ordered abelian group is *discrete* if the quotient groups of successive convex subgroups are each isomorphic to the additive group $\mathbb{Z}$ of integers. A valuation domain is *discrete* if its value group is discrete, and is a *DVR* if its value group is isomorphic to $\mathbb{Z}$. If K is a field and G is an ordered abelian group, we may consider the set K^G of all mappings $G \to K$. If we define the support of f, $\mathrm{Supp}(f) = \{x \in G: f(x) \neq 0\}$ for any $f \in K^G$, then the set $K((G)) = \{f \in K^G: \mathrm{Supp}(f)$ is well-ordered by the order of $G\}$ is a field under the pointwise addition and the convolution product. The field $K((G))$ is called *Hahn's field* of G over K, and its subring $K[[G]] = \{f \in K((G)): x \geq 0$ for all $x \in \mathrm{Supp}(f)\}$ is a valuation domain, called the *long power series ring* of G over K. The field of Laurent power series and its subring of formal power series are obtained as particular cases of this construction.

We use R, $R^\wedge$ and S to denote a valuation domain, its completion and a maximal immediate extension of R respectively. Let $\mathrm{Spec}(R)$ denote the set of all prime ideals of R, totally ordered by inclusion. If $P \in \mathrm{Spec}(R)$, we define the *completion defect* at P, $c_R(P)$, and the *total defect* at P, $d_R(P)$, as the rank of the torsion free R/P-module $(R/P)^\wedge$ (the completion of the valuation domain R/P) and the rank of the torsion-free R/P-module S/PS, respectively. Equivalently $c_R(P)$ is the degree of the field of fractions of $(R/P)^\wedge$ over the field of fractions of R/P, and $d_R(P)$ is the degree of the field of fractions of S/PS over the field of fractions of R/P. In particular, if $P \leqslant Q$ are two prime ideals, then $c_R(Q) = c_{R/P}(Q/P)$; similarly for d_R. Note that for P prime and in the finite case these definitions coincide with the technical ones of [8].

We shall view c_R and d_R as functions $\mathrm{Spec}(R) \to \mathbb{N} \cup \{\infty\}$. Then d_R is a decreasing function, i.e., $P \leqslant Q$ implies $d_R(P) \geqslant d_R(Q)$. Moreover, $c_R(M) = d_R(M) = 1$ at the maximal ideal M of R, and $c_R(P) \leqslant d_R(P)$ for all $P \in \mathrm{Spec}(R)$.

Finally, in this paper $E_R(R/P)$ denotes the injective envelope of the R-module R/P, and an ordinal number λ is the set of the ordinal numbers less than λ.

1. Computation of defects.

In our first proposition we determine the total defect of the prime ideals of R that have not an immediate successor in the ordering of

Spec(R), i.e., the prime ideals that are equal to the intersection of the primes properly containing them.

PROPOSITION 1. *Let R be a valuation domain and let P be a prime ideal such that $P = \cap \{(Q: Q > P, Q$ a prime ideal in $R\}$. Then $d(P) = c(P)\cdot \sup\{d(Q): Q > P, Q$ a prime ideal in $R\}$.*

PROOF. The equality holds trivially if one of the factors on the right hand side of the equation is ∞, because $c \leqslant d$ and d is a decreasing function. Therefore we may suppose that $c(P) < \infty$ and the set $\{d(Q): Q > P, Q \in \operatorname{Spec}(R)\}$ has a largest element n which is a natural number; moreover, factoring out the prime ideal P, we may suppose $P = 0$. If $R^\wedge$ and S denote the completion and a maximal immediate extension of R, and K, $K(R^\wedge)$, $K(S)$ denote the fields of fractions of R, $R^\wedge$ and S respectively, then the equation $[K(S):K] = [K(S): K(R^\wedge)]\cdot[K(R^\wedge):K]$ may be written as $d_R(0) = c_R(0)\cdot \operatorname{rank}_{R^\wedge} S$. Therefore we must show that $\operatorname{rank}_{R^\wedge} S = \sup\{d_R(Q): Q \neq 0, Q \in \operatorname{Spec}(R)\}$, i.e., that $\operatorname{rank}_{R^\wedge} S = \sup\{\operatorname{rank}_{R/Q}(S/QS): Q \neq 0, Q \in \operatorname{Spec}(R)\}$. Since $R/Q \simeq R^\wedge/QR^\wedge$ for any $Q \neq 0$, we may suppose R complete.

Now n is the largest rank, i.e., there exists a prime ideal $L \neq 0$ such that $\operatorname{rank}_{R/Q}(S/QS) = n$ for all prime ideals Q, $0 \neq Q \leqslant L$. We must prove that $\operatorname{rank}_R S = n$. Let $s_1, \ldots, s_n$ be n elements of S such that their images $s_i + LS$ in S/LS are linearly independent over R/L. Then it is easy to prove that their images $s_i + QS$ in S/QS are linearly independent over R/Q whenever $Q \leqslant L$ (this is also true if $Q = 0$). Let us prove that the linearly independent subset $\{s_1, \ldots, s_n\}$ of S is maximal. If s is any element of S, the set $\{s + QS, s_1 + QS, \ldots, s_n + QS\} \subseteq S/QS$ is linearly dependent over R/Q whenever $Q \neq 0$ and $Q \leqslant L$, and therefore $as \in \sum_i Rs_i + QS \subseteq S$ for some $a \in R$, $a \notin Q$. Hence $s \in \sum_i R_Q s_i + QS \subseteq \sum_i Ks_i + QS$ for all $Q \neq 0$. Since $\{QS: Q \in \operatorname{Spec}(R), Q \neq 0\}$ is a basis of neighbourhoods of 0 for the valuation topology on S, it follows that in the topological vector space $K(S)$ over the topological field K, the set S is contained in the closure of the vector subspace $\sum_i Ks_i$. But the closure of a subspace is a subspace, and therefore $\sum_i Ks_i$ is dense in $K(S)$. Since $\sum_i Ks_i$ has dimension n and every finite dimensional subspace of a topological vector space over a complete topological field is a closed set [1, § 5, n. 2, Cor. of Prop. 4], the vector space $K(S) = \sum_i Ks_i$ has dimension n. Hence $\operatorname{rank}_R S = n$. □

When the prime ideal P has an immediate successor Q in Spec (R), i.e., when the set of all the prime ideals of R properly containing P has a least element Q, we are able to prove a similar formula when the residue ring R_Q/P is a DVR. As we prove in the next lemma, the exact formula is $d(P) = c(P)\cdot d(Q)$.

This equality does not hold in general if the valuation domain R_Q/P is not a DVR, not even if R has rank 1, i.e., the prime ideals of R are only 0 and the maximal ideal M. To see this, take for R any complete valuation domain of rank 1 which is not maximal. Then $d(0) > 1$ and $c(0) = d(M) = 1$.

In the proof of the next lemma we shall need the following remark: if R is any valuation domain, I an ideal in R, S a maximal immediate extension of R and K, $K(S)$ denote the fields of fractions of R, S, respectively, then $K(S)$ and $\mathrm{Hom}_R(K, K(S)/IS)$ are isomorphic R-modules. To see this, apply the functor $\mathrm{Hom}_R(K, -)$ to the exact sequence

$$0 \to IS \to K(S) \to K(S)/IS \to 0$$

and obtain the exact sequence

$$\mathrm{Hom}_R(K, IS) \to \mathrm{Hom}_R(K, K(S)) \to \mathrm{Hom}_R(K, K(S)/IS) \to \mathrm{Ext}^1_R(K, IS)\,.$$

In this sequence $\mathrm{Hom}_R(K, IS) = 0$ because IS has no nonzero divisible submodules, $\mathrm{Hom}_R(K, K(S)) \cong K(S)$, and $\mathrm{Ext}^1_R(K, IS) \cong \mathrm{Ext}^1_S(K \otimes S, IS) = 0$, because S is a flat R-algebra and S is a maximal valuation domain [4, Th. A3 and Th. 51].

LEMMA 2. *Let R be a valuation domain and let $P < Q$ be prime ideals in R. If R_Q/P is a DVR, then $d(P) = c(P)\cdot d(Q)$.*

PROOF. If one of the factors on the right hand side of the equation is ∞, the equality holds trivially. Therefore we may suppose $c(P) < \infty$ and $d(Q) < \infty$ and, factoring out the prime ideal P, we can suppose $P = 0$, so that R_Q is a DVR. The remark preceding the statement of the lemma, applied to the rings R and R_Q and their common ideal Q, gives $K(S) \cong \mathrm{Hom}_R(K, K(S)/QS)$ and $K(S') \cong \mathrm{Hom}_{R_Q}(K, K(S')/QS')$, where S, S' are maximal immediate extensions of R, R_Q respectively. Moreover $K(S)/QS = E_S(S/QS)$ [3] and $E_S(S/QS) \cong E_R(S/QS)$, because S is a flat R-algebra (so that every injective S-module is also injective as an R-module) and every element of S is the product of an element of R and a unit of S. Hence $K(S)/QS \simeq E_R(S/QS)$. But S/QS has Goldie dimension $d(Q) < \infty$ [8, Cor. 3.4] and all nonzero cyclic R-submodules of S/QS are isomorphic to R/Q. It follows that $K(S)/QS \simeq \simeq E_R(R/Q)^{d(Q)}$. Similarly $K(S')/QS' \simeq E_{R_Q}(R_Q/Q)$, because $d_{R_Q}(Q) = 1$. Therefore $K(S) \simeq \mathrm{Hom}_R(K, E_R(R/Q))^{d(Q)}$ and $K(S') \simeq \mathrm{Hom}_{R_Q}(K, E_{R_Q}(R_Q/Q))$. But $E_R(R/Q) \simeq E_{R_Q}(R_Q/Q)$ [9, Prop. 5.6], and $\mathrm{Hom}_R(K, E_R(R/Q)) \simeq \mathrm{Hom}_{R_Q}(K, E_{R_Q}(R_Q/Q))$. We conclude that the R-modules $K(S)$ and $K(S')^{d(Q)}$ are isomorphic. But $\mathrm{rank}_R K(S) = \mathrm{rank}_R S = d(0)$, and $\mathrm{rank}_R K(S')^{d(Q)} = d(Q)\cdot c(0)$ because S' is a maximal immediate extension of R_Q and R_Q is a DVR, so that S' is the completion of both R_Q and R. Hence $d(0) = d(Q)\cdot c(0)$. □

When the valuation domain R is discrete, Lemma 2 applies to any prime ideal P with an immediate successor Q, so that $d(Q) = \sup\{d(L): L > P,\ L \text{ a prime ideal in } R\}$. Thus combining Proposition 1 and Lemma 2 we obtain

THEOREM 3. *Let R be a discrete valuation domain and let P be a prime nonmaximal ideal of R. Then $d(P) = c(P)\cdot \sup\{d(Q): Q > P,\ Q$ a prime ideal in $R\}$.* □

Theorem 3 enables us to compute d as a function of c for the discrete valuation domains with the a.c.c. on prime ideals. These domains were studied in [10], under the name of *totally branched valuation domains*, in relation to the structure of their modules.

COROLLARY 4. *Let R be a discrete valuation domain with the a.c.c. on prime ideals. Then $d(P) = \prod\{c(Q): Q \geqslant P,\ Q \in \mathrm{Spec}(R)\}$ for all prime ideals P in R; in particular the mapping c determines the mapping d.*

PROOF. Since R has the a.c.c. on prime ideals, the set $\mathrm{Spec}(R)$ is well-ordered under reverse inclusion, i.e., $\mathrm{Spec}(R) = \{P_\lambda: \lambda \leqslant \alpha\}$ for some ordinal α where $P_\lambda > P_\mu$ if $\lambda < \mu \leqslant \alpha$. We must prove that $d(P_\mu) = \prod_{\lambda\leqslant\mu} c(P_\lambda)$. Induction on μ. If $\mu = 0$, P_μ is the maximal ideal of R, and $d(P_0) = c(P_0) = 1$. If $\mu = \mu' + 1$ the conclusion follows from Lemma 2. If μ is a limit ordinal, either $\sup\left\{\prod_{\lambda\leqslant\mu'} c(P_\lambda): \mu' < \mu\right\} = \sup\{d(P_{\mu'}): \mu' < \mu\} = \infty$, in which case the result holds trivially, or $\left\{\prod_{\lambda\leqslant\mu'} c(P_\lambda): \mu' < \mu\right\} = \{d(P_{\mu'}): \mu' < \mu\}$ is bounded, in which case $\prod_{\lambda\leqslant\mu} c(P_\lambda) = c(P_\mu)\cdot\prod_{\lambda<\mu} c(P_\lambda) = c(P_\mu)\cdot\sup\left\{\prod_{\lambda\leqslant\mu'} c(P_\lambda): \mu' < \mu\right\}$ and the result follows by the inductive hypothesis and Proposition 1. □

The formula of Corollary 4 may not hold for an arbitrary discrete valuation domain, not even if it has the d.c.c. on prime ideals. This is shown by the following example, where we construct a discrete valuation domain with $\mathrm{Spec}(R)$ order isomorphic to $\Omega + 1$, $c(P) = 1$ for all $P \in \mathrm{Spec}(R)$, and $d(0) \neq 1$.

EXAMPLE 5. Let Ω be the first uncountable ordinal and let $\mathbb{Z}^\Omega$ be the lexicographic product of Ω copies of $\mathbb{Z}$, that is, $\mathbb{Z}^\Omega$ is the direct product of $|\Omega|$ copies of $\mathbb{Z}$ ordered lexicographically, which means that if one views the elements of $\mathbb{Z}^\Omega$ as functions $\Omega \to \mathbb{Z}$, then for f, $g \in \mathbb{Z}^\Omega$, $f < g$ if and only if $f(\alpha) < g(\alpha)$ where $\alpha = \min\{\beta : \beta < \Omega,\ f(\beta) \neq g(\beta)\}$. Then there exists a valuation domain R with value group $\mathbb{Z}^\Omega$ (and therefore with $\mathrm{Spec}(R)$ order isomorphic to $\Omega + 1$), which is not maximal, but such that R/P is a complete valuation domain for all $P \in \mathrm{Spec}(R)$.

To see this, let K be a field and let R be the subring of $K[\![\mathbb{Z}^\Omega]\!]$ consisting of all the long power series of $K[\![\mathbb{Z}^\Omega]\!]$ with countable (well-ordered) support. It is obvious that R is a valuation domain and that $K[\![\mathbb{Z}^\Omega]\!]$ is a proper immediate extension of R. This implies that $\mathbb{Z}^\Omega$ is the value group of R and that R is not maximal. Moreover for every ordinal $\xi < \Omega$ the canonical isomorphism of ordered groups $\mathbb{Z}^\xi \oplus \mathbb{Z}^\Omega \cong \mathbb{Z}^\Omega \cong \mathbb{Z}^{\xi+\Omega}$ implies that $K[\![\mathbb{Z}^\Omega]\!]/P^*$ is canonically isomorphic to $K[\![\mathbb{Z}^\Omega]\!]$ for every prime non maximal ideal P^* of $K[\![\mathbb{Z}^\Omega]\!]$, and this ring isomorphism induces a ring isomorphism $R/P \cong R$ for every prime non maximal ideal P of R. Therefore in order to prove that R/P is complete for all $P \in \mathrm{Spec}(R)$ it is sufficient to prove that R is complete. This is an easy exercise. $\square$

In the next section we will consider and completely solve the problem of determining the functions c_R that can arise from discrete valuation domains R of nonzero characteristic with the a.c.c. on prime ideals. For these rings Corollary 4 yields the function d.

Since the construction in the next section is rather complicated, we now give an example of a discrete valuation domain of nonzero characteristic with a finite number of prime ideals and (almost) arbitrary completion defects on these primes. This example is based on an idea of Nagata's [6]. For an example of a DVR of characteristic zero with $c(0) = 2$ see the example of Terjanian in [7].

EXAMPLE 6. Let $p^{l(0)} = 1$, $p^{l(1)}$, ..., $p^{l(n)}$ be a finite sequence of powers of a prime number p. We construct a discrete valuation domain of dimension n, with prime ideals $P_0 > P_1 > \ldots > P_n = 0$ such that $c(P_i) = p^{l(i)}$ for $i = 0, \ldots, n$.

We first recall Nagata's example [6, Example E 3.3, p. 207]: if K is a field of characteristic p, x is an indeterminate over K, $[K:K^p] = \infty$ and $a \geqslant 0$, then there exists a field K' such that $K \cdot K^p((x)) \leqslant K' \leqslant K((x))$ and $[K((x)):K'] = p^a$. Here $K \cdot K^p((x))$ is the compositum of the subfields K and $K^p((x))$ of $K((x))$. (Nagata's example is only for $a = 1$, but a slight modification of his arguments yields our version).

Now fix a field K_0 of characteristic p such that $[K_0:K_0^p] = \infty$, and let $x_1, \ldots, x_n$ be algebraically independent indeterminates over K_0.

By induction on i it is now immediate to construct fields $K_1, \ldots, K_n$ such that $K_{i-1} \cdot K_{i-1}^p((x_i)) \leqslant K_i \leqslant K_{i-1}((x_i))$ and $[K_{i-1}((x_i)):K_i] = p^{l(i)}$. Remark that if $[K_{i-1}:K_{i-1}^p] = \infty$ then $[K_{i-1}((x_i)):K_{i-1}^p((x_i))] = \infty$, and since $[K_{i-1}((x_i)):K_i] < \infty$, it follows that $[K_i: K_{i-1}^p((x_i))] = \infty$; but $K_i^p \leqslant K_{i-1}^p((x_i))$, so that in particular $[K_i:K_i^p] = \infty$. Hence the inductive step is given by Nagata's example.

Set $V_i = K_i \cap K_{i-1}[[x_i]]$, so that V_i is the valuation ring of the valuation induced on K_i by the usual rank one valuation of $K_{i-1}[[x_i]]$. Since $K_{i-1}[x_i] \leqslant V_i$, the completion of V_i is $K_{i-1}[[x_i]]$. Now set

$$R = K_0 + x_1 V_1 + \ldots + x_n V_n;$$

the ring R is an iterated « $D + M$ construction » (see [2]). Since the V_i's are DVR, R is a discrete valuation domain of rank n with prime ideals $P_0 > P_1 > \ldots > P_n = 0$, where $P_i = x_{i+1} V_{i+1} + x_{i+2} V_{i+2} + \ldots + x_n V_n$ for $0 \leqslant i \leqslant n - 1$. Moreover R/P_i is naturally isomorphic to $K_0 + x_1 V_1 + \ldots + x_i V_i$, so that in particular K_i is the field of fractions of R/P_i and the set $\{x_i^m V_i : m \geqslant 0\}$ is a basis of neighbourhoods of zero in the valuation topology of R/P_i. But then the completion of R/P_i is isomorphic to the completion of V_i, which is $K_{i-1}[[x_i]]$. Hence $c(P_i) = \operatorname{rank}_{V_i} K_{i-1}[[x_i]] = [K_{i-1}((x_i)):K_i] = p^{l(i)}$, as we wanted. $\square$

2. A theorem of realization.

In this section we shall construct a discrete valuation domain R with the a.c.c. on prime ideals (and thus with Spec(R) well-ordered by reverse inclusion) with arbitrary order type for Spec(R) and arbitrarily fixed function c_R. We make our construction in Theorem 8. First we need a remark.

REMARK 7. If the valuation domain R has characteristic $p \neq 0$, then $d(P)$ is either ∞ or a power of p for every prime ideal P of R.

This follows from some exercises of § 8, Ch. 5 of Bourbaki's book [1]: factoring out the prime ideal P of R we may suppose $P = 0$, and we must prove that if $[K(S):K]$ is finite, then it is a power of p. Here, as usual, $K(S)$ and K are the field of fractions of a maximal immediate extension S of R and the field of fractions of R, respectively. By Bourbaki's exercise 6 *c*), S is an Henselian ring. By [11, Cor. 1 of Th. 9] the restriction of the valuation of $K(S)$ to every subfield of $K(S)$ of finite codimension is Henselian, because S has prime characteristic. Therefore R is Henselian, and by Bourbaki's exercise 9 *a*) (Ostrowski's Theorem) $[K(S):K]$ is a power of p, because S is an immediate extension of R and therefore the ramification index and the residual degree are both equal to one.

We are ready to prove our theorem. For the terminology about p-basis and p-independence we refer the reader to Matsumura's book [5].

THEOREM 8. *Let α be an ordinal number, $l\colon \alpha + 1 \to \mathbf{N} \cup \{\infty\}$ a mapping with $l(0) = 0$ and p a prime number. Then there exist a discreet valuation domain R and an order antiisomorphism $\alpha + 1 \to \operatorname{Spec}(R)$, $\lambda \mapsto P_\lambda$, such that $c(P_\lambda) = p^{l(\lambda)}$ for every $\lambda \leqslant \alpha$.*

PROOF. Let G_α be the free abelian group with basis $\{g_\lambda\colon \lambda < \alpha\}$. Regard the elements of G_α as functions $\alpha \to \mathbb{Z}$ that vanish almost everywhere. If $f, g \in G_\alpha$, set $f < g$ if $f(\lambda) < g(\lambda)$ where $\lambda = \max\{\mu\colon \mu < \alpha,\ f(\mu) \neq g(\mu)\}$. Then G_α is a totally ordered group and its convex subgroups are, for all $\lambda \leqslant \alpha$, $G_\lambda = \{f \in G_\alpha\colon f(\mu) = 0$ for every $\mu \geqslant \lambda,\ \mu < \alpha\}$.

Fix a field K of characteristic p such that the dimension $[K:K^p]$ of K as a vector space over K^p is greater or equal to $\max\{|\alpha|, \aleph_0\}$, where $|\alpha|$ is the cardinality of α. Let B be a p-basis of K over K^p (see [5]); B has cardinality $\geqslant \max\{|\alpha|, \aleph_0\}$.

Let $K((G_\lambda))$ denote Hahn's field. Recall that the elements of $K((G_\lambda))$ may be viewed as formal series of type $\sum c_g X^g$, where $c_g \in K$, X^g is a symbol for each $g \in G_\lambda$, and $c_g = 0$ for all g but a well-ordered subset of G_λ. We may suppose that $K = K((G_0)) < K((G_1)) < \ldots < K((G_\lambda)) < < \ldots < K((G_\alpha))$.

If $\lambda < \mu \leqslant \alpha$ there is a canonical ring homomorphism $\pi_{\mu\lambda}\colon K[[G_\mu]] \to K[[G_\lambda]]$ defined by $\sum_{g \in G_\mu} c_g X^g \mapsto \sum_{g \in G_\lambda} c_g X^g$ for every $\sum_{g \in G_\mu} c_g X^g \in K[[G_\mu]]$.

Since $|B| \geqslant \max\{|\alpha|, \aleph_0\}$, the set B contains distinct elements

$b(\lambda, \mu, n, m)$, indexed by λ, μ, n, m, where λ, μ are ordinal numbers $\leqslant \alpha$, and $n, m \in \mathbf{N}$.

Fix $\nu \leqslant \alpha$; since the set $\{mg_\mu : m \in \mathbf{N}, \mu < \nu\}$ is a well ordered subset of G_ν, the field $K((G_\nu))$ contains the elements

$$f(\lambda, n, \nu) = \sum_{m \in \mathbf{N}, \mu < \nu} b(\lambda, \mu, n, m) X^{mg_\mu}$$

indexed by λ, n, ν, with $\lambda, \nu \leqslant \alpha$ and $n \in \mathbf{N}$.

CLAIM 1. Fix $\nu \leqslant \alpha$. The set $A_\nu = \{f(\lambda, n, \nu) : \nu \leqslant \lambda \leqslant \alpha,\ n \in \mathbf{N}\} \cup \cup \{b(\lambda, \mu, n, m) : \nu \leqslant \lambda \leqslant \alpha,\ \nu \leqslant \mu \leqslant \alpha,\ n, m \in \mathbf{N}\}$ is a subset of $K((G_\nu))$ which is p-independent over $K^p((G_\nu))$.

PROOF. It is sufficient to prove that every element of A_ν does not belong to the subfield of $K((G_\nu))$ generated by $K^p((G_\nu))$ and all the other elements of A_ν. Fix λ_0 with $\nu \leqslant \lambda_0 \leqslant \alpha$, and $n_0 \in \mathbf{N}$; we will prove that $f(\lambda_0, n_0, \nu)$ does not belong to the subfield F of $K((G_\nu))$ generated by $K^p((G_\nu))$ and $A_\nu \setminus \{f(\lambda_0, n_0, \nu)\}$. The elements of F are long power series with coefficients in $K^p(\{b(\lambda, \mu, n, m) : \nu \leqslant \lambda \leqslant \alpha,\ \mu < \nu,\ n, m \in \mathbf{N}\} \setminus \setminus \{b(\lambda_0, \mu, n_0, m) : \mu < \nu,\ m \in \mathbf{N}\})$ $(\{b(\lambda, \mu, n, m) : \nu \leqslant \lambda \leqslant \alpha,\ \nu \leqslant \mu \leqslant \alpha, n, m \in \mathbf{N}\})$, and the coefficients of $f(\lambda_0, n_0, \nu)$ are $\{b(\lambda_0, \mu, n_0, m) : \mu < \nu, m \in \mathbf{N}\}$. Since the $b(\lambda, \mu, n, m)$'s are in B, which is p-independent over K^p, the coefficients of $f(\lambda_0, n_0, \nu)$ do not belong to the field generated by the coefficients of the elements of F. Therefore $f(\lambda_0, n_0, \nu) \notin F$.

Similarly $b(\lambda_0, \mu_0, n_0, m_0) \notin K^p((G_\nu))$ $(A_\nu \setminus \{b(\lambda_0, \mu_0, n_0, m_0)\})$. This proves Claim 1. □

Let C_ν be a subset of $K((G_\nu))$ such that $A_\nu \cap C_\nu = \emptyset$ and $B_\nu = A_\nu \cup C_\nu$ is a p-basis of $K((G_\nu))$ over $K^p((G_\nu))$.

CLAIM 2. Fix ν, ξ with $\xi < \nu \leqslant \alpha$. The set $D_\nu = \{f(\lambda, n, \nu) : \nu \leqslant \lambda \leqslant \alpha, n \in \mathbf{N}\}$ is a subset of $K((G_\nu))$ p-independent over $K((G_\xi)) \cdot K^p((G_\nu))$, the compositum of the fields $K((G_\xi))$ and $K^p((G_\nu))$.

PROOF. Fix $\lambda_0 \geqslant \nu$ and $n_0 \in \mathbf{N}$. It is sufficient to prove that the element $f(\lambda_0, n_0, \nu) \in D_\nu$ does not belong to the subfield of $K((G_\nu))$ generated by $K((G_\xi)) \cdot K^p((G_\nu))$ and $D_\nu \setminus \{f(\lambda_0, n_0, \nu)\}$.

By way of contradiction, suppose that $f(\lambda_0, n_0, \nu) \in K((G_\xi)) \cdot K^p((G_\nu))$ $(D_\nu \setminus \{f(\lambda_0, n_0, \nu)\})$. Then there is a finite subset S of $K((G_\xi))$ such that $f(\lambda_0, n_0, \nu) \in K^p((G_\nu))(S)$ $(D_\nu \setminus \{f(\lambda_0, n_0, \nu)\})$. By Claim 1 B_ξ is a p-basis of $K((G_\xi))$ over $K^p((G_\xi))$. Therefore there is a finite subset B'_ξ

of B_ξ such that $S \subseteq K^p((G_\xi))(B'_\xi)$. In particular, $f(\lambda_0, n_0, \nu) \in K^p((G_\nu))$ $(B'_\xi)\,(D_\nu \setminus \{f(\lambda_0, n_0, \nu)\})$.

Now $B'_\xi \subseteq (A_\xi \cap B'_\xi) \cup C_\xi$, and thus $f(\lambda_0, n_0, \nu) \in F$, where $F = K^p((G_\nu))(C_\xi)\,(A_\xi \cap B'_\xi)\,(D_\nu \setminus \{f(\lambda_0, n_0, \nu)\}) \subseteq K((G_\nu))$.

But the canonical isomorphism of ordered groups $G_\nu \simeq G_\xi \oplus \mathbf{Z}^{\nu-\xi}$ allows us to identify $K((G_\nu))$ and $K((G_\xi))((\mathbf{Z}^{\nu-\xi}))$, i.e., we may view the elements of $K((G_\nu))$ as long power series with coefficients in $K((G_\xi))$. The coefficients of $f(\lambda_0, n_0, \nu)$ in $K((G_\xi))$ are $f(\lambda_0, n_0, \xi)$ and the $b(\lambda_0, \mu, n_0, m)$'s with $\xi \leqslant \mu < \nu$ and $m \in \mathbf{N}$.

The coefficients in $K((G_\xi))$ of the elements of F are in the field $F' = K^p((G_\xi))\ (C_\xi)\ (A_\xi \cap B'_\xi)\ (\{f(\lambda, n, \xi) : \lambda \geqslant \nu,\ n \in \mathbf{N}\} \setminus \{f(\lambda_0, n_0, \xi)\})$ $(\{b(\lambda, \mu, n, m) : \lambda \geqslant \nu,\ \xi \leqslant \mu < \nu,\ n,\ m \in \mathbf{N}\} \setminus \{b(\lambda_0, \mu, n_0, m) : \xi \leqslant \mu < \nu,$ $m \in \mathbf{N}\})$.

Now the set $A_\xi \cap B'_\xi$ is finite, and the set $\{b(\lambda_0, \mu, n_0, m) : \xi \leqslant \mu < \nu,$ $m \in \mathbf{N}\}$ is infinite. Therefore $b(\lambda_0, \mu_0, n_0, m_0) \notin A_\xi \cap B'_\xi$ for suitable μ_0 with $\xi \leqslant \mu_0 < \nu$ and $m_0 \in \mathbf{N}$. Since $B_\xi = A_\xi \cup C_\xi$ is a p-basis of $K((G_\xi))$ over $K^p((G_\xi))$, it follows that $b(\lambda_0, \mu_0, n_0, m_0) \notin F'$. Therefore $f(\lambda_0, n_0, \nu) \notin F$, contradiction. This proves Claim 2. □

We now construct a family of fields $\{K_\nu : \nu \leqslant \alpha\}$ satisfying the following conditions:

a) $K^p((G_\nu)) \leqslant K_\nu \leqslant K((G_\nu))$ for all $\nu \leqslant \alpha$;

b) if $\mu < \nu \leqslant \alpha$, then $K \leqslant K_\mu \leqslant K_\nu$ and $\pi_{\nu\mu}(K_\nu \cap K[\![G_\nu]\!]) = K_\mu \cap K[\![G_\mu]\!]$ (the homomorphisms $\pi_{\nu\mu}$ have been defined in the fourth paragraph of the proof of this theorem);

c) if $n \in \mathbf{N}$ and $\nu < \lambda \leqslant \alpha$, then $f(\lambda, n, \nu) \in K_\nu$;

d) if $K_\nu^\wedge$ is the completion of K_ν in the topology induced by the valuation topology of $K((G_\nu))$, then $[K_\nu^\wedge : K_\nu] = p^{l(\nu)}$.

The construction is by transfinite induction on $\nu \leqslant \alpha$. For $\nu = 0$, we set $K_\nu = K$. Then the four conditions *a*)-*d*) are trivially satisfied (note that $l(0) = 0$).

Case of a non-limit ordinal. Suppose $\nu + 1 \leqslant \alpha$ and suppose that $\{K_\lambda : \lambda \leqslant \nu\}$ has been defined and satisfies properties *a*)-*d*). Consider the field $L = K_\nu \cdot K^p((G_{\nu+1})) \subseteq K((G_{\nu+1}))$, compositum of fields. The elements $f(\lambda, n, \nu+1) \in K((G_{\nu+1}))$, $n \in \mathbf{N}$, $\nu + 1 \leqslant \lambda \leqslant \alpha$, are p-independent over L, because they are p-independent over $K((G_\nu)) \cdot K^p((G_{\nu+1})) \geqslant L$ (Claim 2). Moreover the elements $f(\lambda, n, \nu+1)$ belong to the closure of L in the topological field $K((G_{\nu+1}))$ (i.e., they belong to the completion $L^\wedge$ of L), because $f(\lambda, n, \nu+1) = f(\lambda, n, \nu) + \sum_{m\in\mathbf{N}} b(\lambda, \nu, n, m) X^{mg_\nu}$ is the limit of the sequence

$$\left\{f(\lambda, n, \nu) + \sum_{m\leqslant i} b(\lambda, \nu, n, m) X^{mg_\nu} \colon i \in \mathbf{N}\right\},$$

which is contained in $K_\nu \cdot K \cdot K^p((G_{\nu+1})) = L$, because $f(\lambda, n, \nu) \in K_\nu$ by the inductive hypothesis, $b(\lambda, \nu, n, m) \in K$ and $X^{mg_\nu} \in K^p((G_{\nu+1}))$.

Therefore there exists a p-basis $D_{\nu+1}$ of $L^\wedge$ over L with $f(\lambda, n, \nu+1) \in D_{\nu+1}$ for all $n \in \mathbf{N}$, $\nu+1 \leqslant \lambda \leqslant \alpha$. Set

$$K_{\nu+1} = L\big(D_{\nu+1} \setminus \{f(\nu+1, i, \nu+1) \colon 1 \leqslant i \leqslant l(\nu+1)\}\big).$$

Let us show that $K_{\nu+1}$ satisfies the required properties:

a) and *c*) are trivial.

b) Obviously $K_\nu \leqslant K_{\nu+1}$. Since $\pi_{\nu+1,\mu} = \pi_{\nu\mu} \circ \pi_{\nu+1,\nu}$ for every $\mu \leqslant \nu$, it is sufficient to prove that

$$\pi_{\nu+1,\nu}(K_{\nu+1} \cap K[\![G_{\nu+1}]\!]) = K_\nu \cap K[\![G_\nu]\!].$$

Now $L \leqslant K_{\nu+1} \leqslant L^\wedge$, so that $L \cap K[\![G_{\nu+1}]\!] \leqslant K_{\nu+1} \cap K[\![G_{\nu+1}]\!] \leqslant L^\wedge \cap \cap K[\![G_{\nu+1}]\!]$.

On the other hand, since $L^\wedge \cap K[\![G_{\nu+1}]\!]$ is the completion of $L \cap K[\![G_{\nu+1}]\!]$, their proper homomorphic images are canonically isomorphic, and, in particular, $\pi_{\nu+1,\nu}(L \cap K[\![G_{\nu+1}]\!]) = \pi_{\nu+1,\nu}(L^\wedge \cap \cap K[\![G_{\nu+1}]\!])$. From this we obtain that $\pi_{\nu+1,\nu}(L \cap K[\![G_{\nu+1}]\!]) = = \pi_{\nu+1,\nu}(K_{\nu+1} \cap K[\![G_{\nu+1}]\!])$. Thus it remains to prove that $\pi_{\nu+1,\nu}(L \cap \cap K[\![G_{\nu+1}]\!]) = \pi_{\nu+1,\nu}(K_\nu \cdot K^p((G_{\nu+1})) \cap K[\![G_{\nu+1}]\!])$ is equal to $K_\nu \cap K[\![G_\nu]\!]$, and for this it is sufficient to show that $K_\nu \cap K[\![G_\nu]\!]$ contains the image of $K_\nu \cdot K^p((G_{\nu+1})) \cap K[\![G_{\nu+1}]\!]$, the other inclusion being trivial. Now $K_\nu \cdot K^p((G_{\nu+1})) \leqslant K((G_{\nu+1}))$, and $K((G_{\nu+1}))$ is canonically isomorphic to $K((G_\nu))((X))$, X an indeterminate. If we indentify $K((G_{\nu+1}))$ and $K((G_\nu))((X))$ via this canonical isomorphism, then $K_\nu \cdot K^p((G_{\nu+1})) \leqslant K_\nu((X))$ by *a*). Therefore

$$\pi_{\nu+1,\nu}(K_\nu \cdot K^p((G_{\nu+1})) \cap K[\![G_{\nu+1}]\!]) \leqslant \pi_{\nu+1,\nu}(K_\nu((X)) \cap K[\![G_{\nu+1}]\!]) \leqslant K_\nu.$$

This proves Property *b*).

d) Since $D_{\nu+1}$ is a p-basis of $L^\wedge$ over L, it is obvious that $[L^\wedge : K_{\nu+1}] = p^{l(\nu+1)}$. Moreover $L \leqslant K_{\nu+1} \leqslant L^\wedge$ so that $K_{\nu+1}$ is dense in $L^\wedge$; since $L^\wedge$ is complete, we conclude that $K^\wedge_{\nu+1} = L^\wedge$.

Case of a limit ordinal. Suppose $\nu \leqslant \alpha$ is a limit ordinal and suppose that $\{K_\lambda : \lambda < \nu\}$ has been defined and satisfies properties *a*)-*d*).

Consider the fields $K'_\nu = \bigcup_{\lambda<\nu} K_\lambda \leqslant K((G_\nu))$ and $K'^\wedge_\nu$, the closure of K'_ν in $K((G_\nu))$. Since $K^p((G_\lambda)) \leqslant K_\lambda$ by Property *a*), it follows that $K'_\nu \geqslant \bigcup_{\lambda<\nu} K^p((G_\lambda))$, so that $K'^\wedge_\nu \geqslant \left(\bigcup_{\lambda<\nu} K^p((G_\lambda))\right)^\wedge = K^p((G_\nu))$.

Set $L = K'_\nu \cdot K^p((G_\nu)) \leqslant K'^\wedge_\nu \leqslant K((G_\nu))$; since $K'_\nu \leqslant L \leqslant K'^\wedge_\nu$, the closure $L^\wedge$ of L in $K((G_\nu))$ is $K'^\wedge_\nu$.

If $\lambda \geqslant \nu$ and $n \in \mathbf{N}$, the series $f(\lambda, n, \nu)$ is the limit of $\{f(\lambda, n, \mu): \mu < \nu\}$ in the topological field $K((G_\nu))$. But $f(\lambda, n, \mu) \in K_\mu$ by Property *c*), and thus it follows that $f(\lambda, n, \mu) \in K'_\nu$ for all $\mu < \nu$, so that $f(\lambda, n, \nu) \in L^\wedge$. Therefore $D_\nu = \{f(\lambda, n, \nu): \lambda \geqslant \nu,\ n \in \mathbf{N}\} \subseteq L^\wedge$. Let us check that D_ν is p-independent over L: otherwise, a finite subset F_0 of D_ν would be p-dependent over $L = K'_\nu \cdot K^p((G_\nu)) = \bigcup_{\lambda<\nu} \left(K_\lambda \cdot K^p((G_\nu))\right)$. Since F_0 is finite, there would exist $\lambda_0 < \nu$ such that F_0 is p-dependent over $K_{\lambda_0} \cdot K^p((G_\nu))$. Therefore D_ν would be p-dependent over $K_{\lambda_0} \cdot K^p((G_\nu))$, and this contradicts Claim 2.

Since D_ν is a subset of $L^\wedge$ p-independent over L, there exists a p-basis E_ν of $L^\wedge$ over L containing D_ν. Set

$$K_\nu = L(E_\nu \setminus \{f(\nu, i, \nu): 1 \leqslant i \leqslant l(\nu)\}),$$

so that K_ν is a subfield of $L^\wedge$.

Let us show that K_ν satisfies properties *a*)-*d*).

a) is obvious, because $K^p((G_\nu)) \leqslant L \leqslant K_\nu \leqslant L^\wedge \leqslant K((G_\nu))$.

b) It is clear that $K_\mu \leqslant K'_\nu \leqslant L \leqslant K_\nu$ for all $\mu < \nu$. Since $K_\mu \leqslant K_\nu$, $\pi_{\nu\mu}(K_\nu \cap K[\![G_\nu]\!])$ contains $K_\mu \cap K[\![G_\mu]\!]$. Now, $K'_\nu \leqslant K_\nu \leqslant L^\wedge = K'^\wedge_\nu$ so that $K'_\nu \cap K[\![G_\nu]\!] \leqslant K_\nu \cap K[\![G_\nu]\!] \leqslant K'^\wedge_\nu \cap K[\![G_\nu]\!] = (K'_\nu \cap K[\![G_\nu]\!])^\wedge$, and since

$$\pi_{\nu\mu}(K'_\nu \cap K[\![G_\nu]\!]) = \pi_{\nu\mu}\left((K'_\nu \cap K[\![G_\nu]\!])^\wedge\right)$$

we obtain

$$\pi_{\nu\mu}(K_\nu \cap K[\![G_\nu]\!]) = \pi_{\nu\mu}(K'_\nu \cap K[\![G_\nu]\!]) = \pi_{\nu\mu}\Big(\bigcup_{\lambda<\nu} K_\lambda \cap K[\![G_\lambda]\!]\Big) = \pi_{\nu\mu}\Big(\bigcup_{\lambda<\nu,\,\lambda>\mu} (K_\lambda \cap K[\![G_\lambda]\!])\Big).$$

By the inductive hypothesis, for every λ such that $\mu<\lambda<\nu$, we have

$$\pi_{\nu\mu}(K_\lambda \cap K[\![G_\lambda]\!]) = \pi_{\lambda\mu}\circ\pi_{\nu\lambda}(K_\lambda \cap K[\![G_\lambda]\!]) = \pi_{\lambda\mu}(K_\lambda \cap K[\![G_\lambda]\!]) = K_\mu \cap K[\![G_\mu]\!].$$

This proves b).

c) If $n\in\mathbf{N}$ and $\nu<\lambda\leqslant\alpha$, then $f(\lambda, n, \nu)\in D_\nu\subseteq E_\nu$, so that $f(\lambda, n, \nu)\in K_\nu$.

d) We have remarked that $K_\nu^\wedge = L^\wedge$. Since E_ν is a p-basis of $L^\wedge$ over L, it is clear that $[L^\wedge : K_\nu] = p^{l(\nu)}$. This proves d).

Thus the construction of the family $\{K_\lambda : \lambda\leqslant\alpha\}$ is complete.

Set $R = K_\alpha \cap K[\![G_\alpha]\!]$, so that R is a valuation domain. We prove that R is the valuation domain required in the statement of the theorem.

Since $K\leqslant R$ and $K^p[\![G_\alpha]\!]\leqslant R$ by properties a) and b), the extension of valuation domains $R\leqslant K[\![G_\alpha]\!]$ is immediate, so that R is discrete with value group G_α. The prime ideals of R

$$P_0 > P_1 > \ldots > P_\lambda > \ldots > P_\alpha = 0$$

are the kernels of the restrictions to R of the canonical homomorphisms $\pi_{\alpha\lambda}: K[\![G_\alpha]\!] \to K[\![G_\lambda]\!]$; therefore $R/P_\lambda \cong \pi_{\alpha\lambda}(R) = K_\lambda \cap K[\![G_\lambda]\!]$ by Property b). This gives $c(P_\lambda) = p^{l(\lambda)}$ for all $\lambda\leqslant\alpha$ by Property d).

The desired conclusion follows. □

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