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On the Automorphism Group of Planes of Figueroa Type.

U. DEMPWOLFF (*)

1. Figueroa constructed in [2] a new class of finite projective planes. This construction was generalized by Hering and Schaeffer in [3] and in [1] it was pointed out, that this construction yields infinite planes too. While Hering and Schaeffer determined the automorphism group in the finite case we like to do this for the infinite planes.

2. Let K be a cyclic Galois extension of degree 3 of the field F. Represent the points $\mathfrak{R}$ of $\mathfrak{T} = PG(2, K)$ by $P = \langle(x_1, x_2, x_3)\rangle$, the lines $\mathfrak{L}$ by $g = \langle(y_1, y_2, y_3)^t\rangle$ $(x_i, y_i \in K)$, and $P \in g$ iff $\sum x_i y_i = 0$. Define partitions $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2 \cup \mathfrak{R}_3$ and $\mathfrak{L} = \mathfrak{L}_1 \cup \mathfrak{L}_2 \cup \mathfrak{L}_3$ where $\langle(x_1, x_2, x_3)\rangle \in \mathfrak{R}_i$ (respectively $\langle(x_1, x_2, x_3)^t\rangle \in \mathfrak{L}_i$) iff $\dim_F\langle x_1, x_2, x_3\rangle = i$. Then $\mathfrak{T}_1 = (\mathfrak{R}_1, \mathfrak{L}_1)$ is a subplane isomorphic to $PG(2, F)$ and $B = PGL(3, F)$ induces on $\mathfrak{T}$ respectively on $\mathfrak{T}_1$ collineations by $P\gamma = \langle v\gamma\rangle$, $g\gamma = \langle\gamma^{-1}u^t\rangle$ for $P = \langle v\rangle \in \mathfrak{R}$, $g = \langle u^t\rangle \in \mathfrak{L}$, and $\gamma \in B$. If $S \in \mathfrak{R}_3$ there is precisely one line $s \in \mathfrak{L}_3$ such that $S \notin s$ and $B_S = B_s$ (and vice versa). This defines a map μ which interchanges $\mathfrak{R}_3$ and $\mathfrak{L}_3$. Moreover there are precisely two points S_1, $S_2 \in s \cap \mathfrak{R}_3$ with $B_S = B_{S_1} = B_{S_2}$. For $s \in \mathfrak{R}_3$ define

$$s^* = (s \cap \mathfrak{R}_2) \cup \{S_1, S_2\} \cup \{t\mu : s\mu \in t \in \mathfrak{L}_3\}, \qquad \mathfrak{L}_3^* = \{s^* : s \in \mathfrak{L}_3\},$$

and $\mathfrak{L}^* = \mathfrak{L}_1 \cup \mathfrak{L}_2 \cup \mathfrak{L}_3^*$. Then by [1] $\mathfrak{T}^* = (\mathfrak{R}, \mathfrak{L}^*)$ is a nondesargue-

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sian projective plane for $|F| > 2$. Set $G = \mathrm{Aut}(\mathfrak{T}^*)$. We like to verify:

PROPOSITION. *G fixes the subplane $\mathfrak{T}_1$ and $G \simeq Z \times A$, where $A \simeq \simeq P\Gamma L(3, F)$ and $Z \simeq Z_3$.*

REMARK. G has index 2 in the group of correlations (see [3]).

PROOF. By [2, 3] we may assume that $|F| = \infty$.
Let H be the subgroup of G which fixes $\mathfrak{T}_1$:

(1) *H contains a subgroup $A \times Z$, $A \simeq P\Gamma L(3, F)$, $Z \simeq Z_3$.*

This follows precisely as in [3].

(2) $H = A \times Z$.

Let M be the normal subgroup of H which fixes $\mathfrak{T}_1$ pointwise. Take $(P, g) \in \mathfrak{K}_1 \times \mathfrak{L}_1$ and pick a perspectivity $\tau \in B(P, g)$. For $\vartheta \in M$ we have $\tau^{-1}\vartheta^{-1}\tau\vartheta \in M \cup B(P, g) = 1$. Thus M centralizes B. Suppose $g \in \mathfrak{L}^* - \mathfrak{L}_1$ is a line fixed by $\vartheta \in M$. Then all points of $g \cap \mathfrak{K}_2$ are fixed by ϑ as they all lie on exactly one line of $\mathfrak{L}_1$: Thus $\mathfrak{K}_2$ is pointwise fixed and we have $\vartheta = 1$.

Thus M acts fixed-point-free on $\mathfrak{K}_2 \cup \mathfrak{K}_3$. In particular for $S \in \mathfrak{K}_3$ M acts faithful and semiregular on the three fixed points S, S_1, S_2 of B_S. Thus $M = Z$.

(3) *Let G_1 be the subgroup of G which fixes $\mathfrak{K}_2$ and $\mathfrak{L}_2$. Then $G_1 = H$.*

For convenience we use for the moment a coordinate transformation and identify points in $\mathfrak{K}_i$ (lines in $\mathfrak{L}_i$) with $\langle(x_1, x_2, x_3)\rangle$ (respectively $\langle(x_1, x_2, x_3)^t\rangle$ iff)

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & x_3 \\ \bar{x}_2 & \bar{x}_3 & \bar{x}_1 \\ \bar{\bar{x}}_3 & \bar{\bar{x}}_1 & \bar{\bar{x}}_2 \end{pmatrix} = i$$

where $K \ni x \to \bar{x} \in K$ denotes a Galois automorphism of order 3 (see also [1, 2, 3]).

Suppose $H \subset G_1$. Let $\gamma \in G_1 - H$ and set $\mathfrak{K}' = \mathfrak{K}_1\gamma$, $\mathfrak{L}' = \mathfrak{L}_1\gamma$, and $\mathfrak{T}' = (\mathfrak{K}', \mathfrak{L}')$. By our assumption $\mathfrak{K}' \cap \mathfrak{K}_1 = \emptyset$, $\mathfrak{L}' \cap \mathfrak{L}_1 = \emptyset$. We may assume $s^* \in \mathfrak{L}'$ for $s = \langle(1, 0, 0)^t\rangle$ and pick $P = \langle(1, a, b)\rangle \in \mathfrak{K}_3 - \mathfrak{K}'$. As H_s is transitive on $s \cap \mathfrak{K}_2$ and H_g is transitive on $g \cap \mathfrak{K}_1$, $g \cap \mathfrak{K}_2$ for $g \in \mathfrak{L}_1$, we have that H'_{s^*} is transitive on $s^* \cap \mathfrak{K}_3$ where $H' =$

$\gamma^{-1}H\gamma$. Thus $V = \langle(0, 1, 0)\rangle$ and $U = \langle(0, 0, 1)\rangle$ lie in $\mathfrak{K}'$. Then $P\cdot V$, $P\cdot U$ lie in $\mathfrak{L}_2$ forcing norm(a) = norm$(b) = 1$. Thus $\mathfrak{K}'$ contains all points $\langle(1, a, a)\rangle$ with $0 \neq a \in F$, $a^3 \neq 1$, and $0 \neq 1 + 2a^3 - 3a^2$. Then $g = \langle(0, -1, 1)^t\rangle \in \mathfrak{L}_2$ contains more then one point of $\mathfrak{K}'$, a contradiction.

From now on we make the assumption $H \subset G$.

(4) *G is flag-transitive on $\mathfrak{T}^*$.*

By (3) we may assume that G is transitive on the lines of $\mathfrak{L}^*$. Pick $m \in \mathfrak{L}_2$ and a nontrivial elation τ in G of the form $\tau \in G(P, m)$.

Case 1. $P \in \mathfrak{K}_1$. Suppose for all $r \in \mathfrak{L}_1$, $P \in r$ we have $(r \cap \mathfrak{K}_1)\tau = r \cap \mathfrak{K}_1$. Then $\tau \in H$, a contradiction. Thus there is an $r \in \mathfrak{L}_1$, $P \in r$, and a $T \in r \cap \mathfrak{K}_1$ with $T\tau \in \mathfrak{K}_2$. So $\langle H_r, \tau\rangle$ is transitive on r and we are done.

Case 2. $P \in \mathfrak{K}_2 \cup \mathfrak{K}_3$. Set $\{C\} = m \cap \mathfrak{K}_1$. Choose $T \in \mathfrak{K}_1 - \{C\}$, such that $P\cdot T \in \mathfrak{L}_2$. Suppose $C\cdot(T\tau) \notin \mathfrak{L}_1$. Then take an elation $1 \neq \sigma \in G(C, C\cdot T)$. With $\tau^{-1}\sigma\tau$ we are in case 1. Suppose next $C\cdot(T\tau) \in \mathfrak{L}_1$. Pick an elation $1 \neq \sigma \in G(T, C\cdot T)$. Then $\tau^{-1}\sigma\tau$ has axis in $\mathfrak{L}_1$ and center in $\mathfrak{K}_2$. The dual argument of case 1 proves the assertion.

(5) *$G - H$ contains an elation with axis and center in $\mathfrak{T}_1$.*

By (4) it is obvious that we can find a homology $1 \neq \tau \in G(M, g)$ with $g \in \mathfrak{L}_1$ and $M \in \mathfrak{K}_2$. Pick $C \in g \cap \mathfrak{K}_1$ such that $g' = M\cdot C \in \mathfrak{L}_1$: Then τ normalizes $G(C, g)$. If $H(C, g)$ is not τ-invariant, we are done. Suppose τ normalizes $H(C, g)$. Then τ acts fixed point free by conjugation on $H(C, g)$. Take $N \in g' \cap \mathfrak{K}_1$, $N \neq C$. Now $H(N, g)$ acts transitive by conjugation on $H(C, g)$. Thus we find $\tau' \in H(N, g)$ such that $\varrho = \tau'\tau$ centralizes $1 \neq \sigma \in H(C, g)$. ϱ is not a homology as otherwise its center would be σ-invariant. As ϱ fixes g' we have $\varrho \in G(C, g)$. Take $Q \in \mathfrak{K}_1 - (g \cup g')$. Then $Q\tau' \in \mathfrak{K}_1 - (g \cup g')$ and $Q\varrho$ lies on $(Q\tau')\cdot M$. Now $(Q\tau')\cdot M \in \mathfrak{L}_2$ and g' is the line of $\mathfrak{L}_1$ containing M. Thus $Q\varrho \in \mathfrak{K}_2$ and we are done.

(6) *Pick σ as in* (5). *We may assume that σ moves the points of $\mathfrak{K}$ as*

$$\sigma(\theta) = \begin{pmatrix} 1 & \theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \textit{where } \theta \in K - F.$$

We may assume that $a = \langle(1, 0, 0)^t\rangle$ is the axis and $C = \langle(0, 1, 0)\rangle$ is the center of σ and that $X = \langle(1, 0, 0)\rangle$ is moved onto $X' = \langle(1, \theta, 0)\rangle$ for some $\theta \in K - F$. Pick $g \in \mathfrak{L}_1$, $C \in g$, $g \neq X \cdot C$. For $Y \in g$, $Y \neq C$, we have $Y\sigma = X' \cdot R \cap g$, where $R = X \cdot Y \cap a$. Clearly $Y\sigma$, R, and Y lie in $\mathfrak{K}_1 \cup \mathfrak{K}_2$. Thus $Y\sigma = Y\sigma(\theta)$ by the definition of $\mathfrak{T}^*$. In particular σ acts like $\sigma(\theta)$ on the lines of $\mathfrak{L}_1$ through C. Suppose $b \in \mathfrak{L}_2$, $C \in b$. Take $Y \in b$, $T \in a \cap \mathfrak{L}_1$ and $g \in \mathfrak{L}_1$, $C \in g$. So $T \cdot Y \in \mathfrak{L}_1 \cup \mathfrak{L}_2$ and $Y\sigma$ is determined by the image of $T \cdot Y \cap g$ under σ. Hence $Y\sigma = Y\sigma(\theta)$, too.

Take now an arbitrary $m \in \mathfrak{L}_1 \cup \mathfrak{L}_2$. Then

$$m\sigma = \{P\sigma : P \in m\} = \{P\sigma(\theta) : P \in m\} \in \mathfrak{L}.$$

As $\mathfrak{L}_3^* \cap \mathfrak{L}_3 = \emptyset$, σ leaves $\mathfrak{L}_3^*$ invariant. Thus $\mathfrak{T}_1$ is σ-invariant and $\sigma \in H$, the final contradiction.

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