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MARTIN HUBER

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A Simple Proof for a Theorem of Chase.

MARTIN HUBER (*)

Dedicated to Professor Laszlo Fuchs
on the occasion of his sixtieth birthday.

In this note all groups are abelian. A group G is called a *W-group* (Whitehead group) if $\operatorname{Ext}(G, \mathbb{Z}) = 0$. Whitehead's problem asks whether every *W-group* is free. This problem has been solved by Shelah [S1, S2] in the sense that the answer depends on the underlying set theory. The deepest result on Whitehead's problem prior to Shelah's work is Chase's theorem, saying that under $2^{\aleph_0} < 2^{\aleph_1}$ every *W-group* G is *strongly* $\aleph_1$*-free* i.e., G is $\aleph_1$-free and every countable subset of G is contained in a countable $\aleph_1$-pure subgroup of G. In fact, Chase proved the following somewhat stronger result.

THEOREM [C; Thm. 2.3]. Assume $2^{\aleph_0} < 2^{\aleph_1}$. If G is a torsion-free group such that $\operatorname{Ext}(G, \mathbb{Z})$ is torsion, then G is strongly $\aleph_1$-free.

Among other nontrivial tools, Chase's proof uses the derived functors of the inverse limit. An entirely different proof has been given in [EH]; in there the above theorem is a corollary to a more general result involving the set-theoretic principle « weak diamond ». That in the given situation weak diamond is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$ is a nontrivial result of set theory due to Devlin and Shelah [DS].

In this note an alternative proof is given for the above theorem which the author has found a few years ago (cf. p. 167, Remark 2 in [EH]). Since the present proof is both shorter and more elementary

(*) Indirizzo dell'A.: Mathematisches Institut der Universität, Albertstrasse 23 b, D-7800 Freiburg i. Br., Rep. Fed. Tedesca.

than those in [C] or [EH], I think it is well worth being published.

We shall make use of the description of Ext in terms of factor sets. Recall [F; § 49] that $\mathrm{Ext}(G, \mathbb{Z})$ can be identified with $\mathrm{Fact}(G, \mathbb{Z})/\mathrm{Trans}(G, \mathbb{Z})$ where $\mathrm{Fact}(G, \mathbb{Z})$ is the group of factor sets on G to $\mathbb{Z}$ and $\mathrm{Trans}(G, \mathbb{Z})$ is the subgroup of transformation sets. A transformation set is a factor set δg, given by

$$\delta g(x, y) = g(x) - g(x + y) + g(y) ,$$

where $g: G \to \mathbb{Z}$ is a map with $g(0) = 0$.

Our proof is divided into three steps, the first of which consists in establishing a lemma.

Lemma. Let K be a subgroup of H such that $\mathrm{Ext}(H/K, \mathbb{Z})$ contains an element of infinite order. Then there is an element $\tilde{f} \in \mathrm{Fact}(H, \mathbb{Z})$ extending $0 \in \mathrm{Fact}(K, \mathbb{Z})$ such that for all $n > 0$ there is no map $g: H \to \mathbb{Z}$ for which $g \restriction K = 0$ and $n\tilde{f} = \delta g$.

Proof. Let $h \in \mathrm{Fact}(H/K, \mathbb{Z})$ represent an infinite order element of $\mathrm{Ext}(H/K, \mathbb{Z})$. Let $\pi: H \to H/K$ denote the natural map and define $\tilde{f} \in \mathrm{Fact}(H, \mathbb{Z})$ by

$$\tilde{f}(x, y) = h(\pi x, \pi y) .$$

We claim that $\tilde{f}$ satisfies the required conditions. Clearly $\tilde{f}$ extends $0 \in \mathrm{Fact}(K, \mathbb{Z})$. Suppose that there is an $n > 0$ and a function $g: H \to \mathbb{Z}$ such that $g \restriction K = 0$ and $n\tilde{f} = \delta g$. Then for each pair $(x, y) \in H \times K$ we have

$$g(x) - g(x + y) = \delta g(x, y) = n\tilde{f}(x, y) = nh(\pi x, 0) = 0 .$$

Therefore g is actually a function on H/K i.e., there is a $\bar{g}: H/K \to \mathbb{Z}$ with $\bar{g}\pi = g$. It follows readily that $nh = \delta\bar{g}$, contradicting the choice of h. □

The second and main task is to prove the following proposition. This is in fact contained in [C; Thm. 1.8]; we notice that the use of factor sets simplifies its proof considerably. By $r_0(A)$ we denote the torsion-free rank of the group A.

Proposition. Let G be a nonzero torsion-free group that does not contain any subgroup H of smaller cardinality such that G/H is $\aleph_1$-free. Then $r_0(\mathrm{Ext}(G, \mathbb{Z})) = 2^{|G|}$.

Proof. Let $\varkappa$ denote the cardinality of G. Suppose first that $\varkappa = \aleph_0$. By hypothesis G is not free; thus, as is well known, $r_0(\operatorname{Ext}(G, \mathbb{Z})) = 2^{\aleph_0}$ (cf. [C; L. 1.4]). Now suppose that $\varkappa$ is uncountable. The hypothesis enables us to define inductively an increasing chain $\{G_\alpha : \alpha < \varkappa\}$ of pure subgroups of G with the following properties:

(o) $G_0 = 0$;

(i) $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ if α is a limit ordinal;

(ii) for all $\alpha < \varkappa$, $G_{\alpha+1}/G_\alpha$ is nonzero and countable;

(iii) for all $\alpha < \varkappa$, $\operatorname{Hom}(G_\alpha, \mathbb{Z}) = 0$.

We have to choose $G_{\alpha+1}$ as a pure subgroup of G which contains G_α such that $G_{\alpha+1}/G_\alpha$ is nonzero, countable, and satisfies $\operatorname{Hom}(G_{\alpha+1}/G_\alpha, \mathbb{Z}) = 0$. That this is possible follows from the facts that G/G_α contains a nonfree countable subgroup and that every countable group is the direct sum of a free group and a group C satisfying $\operatorname{Hom}(C, \mathbb{Z}) = 0$. (For the latter see [F; Cor. 19.3]). For all $\alpha < \varkappa$, since G_α is pure in G, it follows that $\operatorname{Ext}(G_{\alpha+1}/G_\alpha, \mathbb{Z})$ contains an element of infinite order.

Let $G' = \bigcup_{\alpha<\varkappa} G_\alpha$. We will prove that $r_0(\operatorname{Ext}(G', \mathbb{Z})) = 2^\varkappa$. To this end we assign to each sequence $\eta \in 2^\alpha$, $\alpha < \varkappa$, a factor set f_η on G_α such that

(a) if ξ is an initial segment of η, then f_η extends f_ξ;

(b) if $\xi \neq \eta$ have the same domain α, then $f_\xi - f_\eta$ represents an element of $\operatorname{Ext}(G_\alpha, \mathbb{Z})$ of infinite order.

The definition is by induction on the domain of η. Suppose f_η has been defined, where $\eta \in 2^\alpha$. Let $\eta^\frown\langle 0\rangle$, $\eta^\frown\langle 1\rangle$ denote the two extensions of η in $2^{\alpha+1}$. Extend f_η to $f_{\eta^\frown\langle 0\rangle}$ arbitrarily (possible, cf. [HHS; Lemma 1]) and then define $f_{\eta^\frown\langle 1\rangle}$ according to the lemma above, so that there is no $g\colon G_{\alpha+1} \to \mathbb{Z}$ for which $g\restriction G_\alpha = 0$ and $n(f_{\eta^\frown\langle 1\rangle} - f_{\eta^\frown\langle 0\rangle}) = \delta g$ for any $n > 0$. We claim that $f_{\eta^\frown\langle 1\rangle} - f_{\eta^\frown\langle 0\rangle}$ represents an element of $\operatorname{Ext}(G_{\alpha+1}, \mathbb{Z})$ of infinite order. Suppose to the contrary that there is a map $h\colon G_{\alpha+1} \to \mathbb{Z}$ such that for some $n > 0$, $n(f_{\eta^\frown\langle 1\rangle} - f_{\eta^\frown\langle 0\rangle}) = \delta h$. Then of course $(\delta h)\restriction G_\alpha = 0$; hence $h\restriction G_\alpha$ is a homomorphism. But by definition of G_α, $\operatorname{Hom}(G_\alpha, \mathbb{Z}) = 0$, whence we obtain a contradiction.

It follows that the factor sets f_η, $\eta \in \bigcup_{\alpha<\varkappa} 2^\alpha$, satisfy (a) and (b), and in the limit we obtain $2^\varkappa$ factor sets on G' to $\mathbb{Z}$ which represent pairwise different elements of $\operatorname{Ext}(G', \mathbb{Z})$ modulo torsion. We conclude that

$$2^{|G|} \leqslant r_0(\operatorname{Ext}(G, \mathbb{Z})) \leqslant 2^{|G|},$$

the latter inequality being obvious. $\square$

The final step is to deduce the Theorem from the Proposition: Assume that $2^{\aleph_0} < 2^{\aleph_1}$ and let G be an uncountable torsion-free group which is not strongly $\aleph_1$-free. Thus G contains a countable subgroup H such that for every countable subgroup K containing H, G/K is not $\aleph_1$-free. Therefore by the proposition we have $r_0(\operatorname{Ext}(G/H, \mathbb{Z})) \geqslant 2^{\aleph_1}$. Considering exactness of the sequence

$$\operatorname{Hom}(H, \mathbb{Z}) \to \operatorname{Ext}(G/H, \mathbb{Z}) \to \operatorname{Ext}(G, \mathbb{Z})$$

we infer that $r_0(\operatorname{Ext}(G, \mathbb{Z})) \geqslant 2^{\aleph_1}$, since $|\operatorname{Hom}(H, \mathbb{Z})| = 2^{\aleph_0}$ and $2^{\aleph_0} < 2^{\aleph_1}$. Hence, in particular, $\operatorname{Ext}(G, \mathbb{Z})$ is not a torsion group. $\square$

Finally we wish to mention that the Theorem is not true without the hypothesis of $2^{\aleph_0} < 2^{\aleph_1}$. It follows from results in [S1] and [S3] that in a model of Martin's Axiom plus the denial of the Continuum Hypothesis there exist *W-groups* which are not strongly $\aleph_1$-free.

REFERENCES

[C] S. Chase, *On group extensions and a problem of J. H. C. Whitehead*, in *Topics in Abelian Groups*, pp. 173-197, Chicago: Scott, Foresman and Co., 1963.

[DS] K. Devlin - S. Shelah, *A weak version of $\diamondsuit$ which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Israel J. Math., **29** (1978), pp. 239-247.

[EH] P. Eklof - M. Huber, *On the rank of Ext*, Math. Z., **174** (1980), pp. 159-185.

[F] L. Fuchs, *Infinite Abelian Groups*, Vol. I, London, New York, Academic Press, 1970.

[HHS] H. Hiller - M. Huber - S. Shelah, *The structure of* $\operatorname{Ext}(A, \mathbb{Z})$ *and* $V = L$, Math. Z., **162** (1978), pp. 39-50.

[S1] S. Shelah, *Infinite abelian groups. Whitehead problem and some constructions*, Israel J. Math., **18** (1974), pp. 243-256.

[S2] S. Shelah, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Israel J. Math., **21** (1975), pp. 319-349.

[S3] S. Shelah, *On uncountable abelian groups*, Israel J. Math., **32** (1979), pp. 311-330.

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