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Some Cardinal Invariants for Valuation Domains.

LUIGI SALCE - PAOLO ZANARDO

Introduction.

The condition on a valuation domain F of being maximal, which goes back to Krull [8], and the very close condition of being almost maximal, due to Kaplansky [7], were extensively investigated by many authors, on account of their importance for the consequences deriving for the ring structure of R and for many classes of R-modules.

On the contrary, the problem of measuring in some way the failure of the maximality did not receive too much attention up to now; the only contributions in this direction known by the authors are by Brandal [1], and by Facchini and Vamos [2].

As any valuation domain R is a subring of a maximal valuation domain S, which is an immediate extension of it, it is natural to try to measure the failure of the maximality for R by looking for cardinal invariants which measure, roughly speaking, how large is S over R.

In this paper, given any ideal I of R, we will introduce two cardinal invariants associated with I: the *completion defect at* I, denoted by $c_R(I)$, and the *total defect at* I, denoted by $d_R(I)$; their definition, which seems very technical, raised up naturally in the investigation of indecomposable finitely generated modules in [13].

The completion defect $c_R(I)$ measures how large is $(R/I)^\wedge$, the com-

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pletion of R/I in the ideal topology, and the total defect $d_R(I)$ measures how large is S/IS over R/I; recall that both R/I and $(R/I)^\wedge$ are contained in S/IS up to canonical isomorphisms.

It is noteworthy that the invariants that we are introducing do not depend on the ring structure of S, which is not unique up to isomorphism, but only on the R-module structure of S, which is a pure-injective hull of R (see [7] and [12]).

In the first section we introduce the *breadth ideal* (*of non maximality*) of the valuation domain R, a concept originally due to Brandal [1], and the *breadth ideal o a unit* of S, a concept defined in a slightly different way by Ostrowsky [11], Kaplansky [7] and Nishi [10].

The breadth ideals of the units of S are used in the second section to define the completion defect and the total defect at an ideal I of R. The main result in this section is an inequality which relates the total defect at an ideal I with the completion defects at the ideals containing I. This inequality however is in general strict, as is shown, for a special class of discrete valuation domains, by Facchini and the second author in [3].

In section 3 we compare the total defect at an ideal I with the Goldie dimension $g_R(I)$ of S/IS as an R/I-module; they turn out to be equal if I is a prime ideal, while in the non-prime case the total defect becomes generally larger.

We remark that the invariants of the valuation domain R that we investigate here play a relevant role in the study of many classes of R-modules: besides finitely generated modules (see [13]), the R-modules JS/IS, where $I > J$ are fractional ideals of R(see [4]); indecomposable injective R-modules (see [2]) and torsion-free R-modules of finire rank (see [5]).

1. The breadth.

R will always denote a valuation domain, P its maximal ideal and Q its field of quotients. Recall that R is maximal if it is linearly compact (in the discrete topology); R is almost maximal if every proper factor of it is linearly compact.

A valuation domain S containing R as a subring is an immediate extension of R if

(i) every ideal of S is of the form IS, where I is an ideal of R, and $IS \cap R = I$;

(ii) S/PS is naturally isomorphic to R/P or, equivalently, $S = PS + R$.

An immediate extension S of R is maximal if, given any valuation domain S' containing S as a proper subring, either (i) or (ii) fails for S'.

It is well known (see [7] or [12]) that every valuation domain R is contained in a maximal immediate extension S, which is a maximal valuation domain. However S is uniquely determined up to R-isomorphism only, and not as a ring, unless R is almost maximal, in which case S is the completion of R (in the valuation tolopogy). R coincides with S if and only if it is maximal.

Brandal considered in [1] the family of ideals of R

$$\mathcal{F} = \{I \leqslant R : R/I \text{ is not linearly compact}\}$$

and he showed that either $\mathcal{F} = \emptyset$, or there exists a prime ideal L of R such that

$$\mathcal{F} = \{I : I \leqslant L\} \quad \text{or} \quad \mathcal{F} = \{I : I < L\}.$$

Fixed a maximal immediate extension S of R, we reformulate this result by introducing the following subset of R, called the *breadth of* R (with a more meaningful term we could call it the *breadth of non maximality of* R):

$$B(R) = \{a \in R : S > R + aS\}.$$

Notice that $B(R) = \emptyset$ whenever $S = R + aS$ for all $a \in R$; this happens exactly if $S = R$, i.e. if R is maximal; thus from now on we shall assume that R is a valuation domain not maximal, so $B(R)$ is an ideal of R.

PROPOSITION 1.1. Let R be a valuation domain not maximal. Then its breadth $B(R)$ is a prime ideal of R such that:

$$B(R) = \cap \{I : R/I \text{ is linearly compact}\}$$
$$= \cup \{I : R/I \text{ is not linearly compact}\}.$$

PROOF. Assume that $a, b \in R \setminus B(R)$. Then $S = R + aS = R + bS$, and $bS = b(R + aS)$ implies $S = R + b(R + aS) = R + abS$. Therefore $ab \notin B(R)$, so that $B(R)$ is prime.

If R/I is linearly compact, then $R/I \simeq S/IS$ in a natural way, therefore $S = R + IS$; thus $I \geqslant B(R)$. It follows that $B(R)$ is contained in $\cap \{I: R/I$ is linearly compact$\}$. Conversely, if $I > B(R)$, then $S = R + IS$, thus $R/I \simeq S/IS$ is linearly compact. Being $B(R)$ prime, either $B(R) = P$, in which case the first equality is trivial, or $B(R)$ is the intersection of the ideals properly containing it, thus the first equality is obvious. The second equality can be proved in a similar way. ///

From Proposition 1.1 it follows that $B(R)$ does not depend on the choice of S, and that it coincides with the ideal L quoted in the Brandal's result. Notice that R is almost maximal (and not maximal) if and only if $B(R) = 0$.

The valuation domain $R/B(R)$ is always almost maximal; Brandal gives examples in [1] showing that $R/B(R)$ can be maximal or not.

Let us denote by $U(S)$ and (UR) respectively the multiplicative groups of the units of S and R. Every $0 \neq x \in S$ can be written in a unique way, up to units of E, in the form $x = \varepsilon r$, with $\varepsilon \in U(S)$ and $r \in R$; by this reason we will confine ourselves to consider units of S in the following discussion.

Given any $\varepsilon \in U(S) \setminus R$, consider the ideal of R, called the *breadth of* ε

$$B(\varepsilon) = \{a \in R: \varepsilon \notin R + aS\}.$$

Remark. Our definition of breadth of a unit of S is essentially the same as the one given by Nishi [10], which is a slight modification of the original definition of breadth of a pseudoconvergent set of elements of R given by Kaplansky [7], and originally due to Ostrowsky [11]. The definition of breadth (of non maximality) of R is originated by the two above definitions.

From the definitions of $B(R)$ and $B(\varepsilon)$ it trivially follows that $B(\varepsilon) \leqslant B(R)$. Conversely, let $a \in B(R)$; then $S > R + aS$, thus there exists $\varepsilon \in U(S)$ such that $\varepsilon \notin R + aS$, therefore $a \in B(\varepsilon)$; we have proved

Proposition 1.2. Let R be a valuation domain not maximal. Then $B(R) = \cup \{B(\varepsilon): \varepsilon \in U(S) \setminus R\}$.

The following result will be useful in the next section; it is similar to [10, Prop. 6].

LEMMA 1.3. Let R be a valuation domain not maximal and $\varepsilon \in U(S)\setminus R$. If $u \in U(R)$ and $0 \neq r \in P$, then $B(u + r\varepsilon) = rB(\varepsilon)$.

PROOF. $\varepsilon \notin R + aS (a \in R)$ if and only if $r\varepsilon \notin R + raS$, and this obviously is equivalent to $u + r\varepsilon \notin R + raS$. ///

We introduce the following notation: given $I \leqslant R$, let $f_I\colon S \to S/IS$ be the canonical surjection; then the image $f_I R$ of R is a subring of S/IS isomorphic to R/I; its completion, whenever R/I is Hausdorff, is denoted by $(f_I R)^\wedge$. Notice that, being $f_I R$ pure in S/IS and S/IS complete, we have the following inclusions:

$$f_I R \leqslant (f_I R)^\wedge \leqslant S/IS\,. \tag{1}$$

The topology considered above, as in the following proposition, on the factor ring R/I is the «ideal topology», which has as a basis of neighborhoods of 0 the ideals $(aR)/I$, $a \in R\setminus I$.

PROPOSITION 1.4. Let R be a valuation domain, and $I \leqslant R$. Then R/I is Hausdorff and non complete if and only if $I = B(\varepsilon)$ for some $\varepsilon \in U(S)\setminus R$.

PROOF. In order to show that $R/B(\varepsilon)$ is Hausdorff, it is enough to prove that $a \notin B(\varepsilon)$ implies $pa \notin B(\varepsilon)$ for some $p \in P$. So let $\varepsilon \in R + aS$; then $\varepsilon = r + as (r \in R,\ s \in S)$. But $S = R + PS$ implies that $s = t + ps'$, for some $t \in R$, $p \in P$ and $s' \in S$; therefore we get: $\varepsilon = r + at + aps' \in R + paS$, as we want. Clearly $\varepsilon + B(\varepsilon)S \notin f_{B(\varepsilon)}R$, but it is the limit of a Cauchy net of elements of $f_{B(\varepsilon)}R$: for, given $r \notin B(\varepsilon)$, $\varepsilon \in R + rS$ implies that there exists $u_r \in U(R)$ such that $\varepsilon - u_r \in rS$; thus $\varepsilon + B(\varepsilon)S$ is the limit of the Cauchy net $\{u_r + B(\varepsilon)S : r \notin B(\varepsilon)\}$. So we have proved that $R/B(\varepsilon) \cong f_{B(\varepsilon)}R$ is not complete.

Conversely, assuming that R/I is Hausdorff and not complete, from the inclusions (1) we get an element $\varepsilon \in U(S)\setminus R$ such that $\varepsilon + IS$ is the limit of a Cauchy net $\{u_r + IS : r \notin I\}$ in $f_I R$. So $\varepsilon \in R + rS$ if and only if $r \notin I$, therefore $I = B(\varepsilon)$. ///

From the proof of the preceding proposition we deduce the following

COROLLARY 1.5. Let $\varepsilon \in U(S)\setminus R$, and $I \leqslant R$. Then $\varepsilon + IS \in f_I R$ if and only if $B(\varepsilon) < I$; $\varepsilon + IS \in (f_I R)^\wedge \setminus (f_I R)$ if and only if $B(\varepsilon) = I$. ///

A particular case is when $I = 0$; then the elements of the completions $\hat{R}$ of R are exactly those $x = r\varepsilon \in S(0 \neq r \in R,\ \varepsilon \in U(S))$ such that $B(\varepsilon) = 0$. It was shown by Nishi [10] that $\hat{R}$ is the center $Z(A)$ of the ring $A = \operatorname{End}_R E(R/P)$, which is isomorphic to $\operatorname{End}_R(S)$; so we have the inclusions:

$$R \leqslant \hat{R} = Z(A) \leqslant S \leqslant A = \operatorname{End}_R E(R/P) \cong \operatorname{End}_R S .$$

2. The completion defect and the total defect.

We introduce now a new concept, which first appeared in [13]. Let R be a valuation domain not maximal, and S a maximal immediate extension of R; let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in U(S)$ and $I \leqslant P$; we say that the ε_i's are *u-independent over I* if

$$a_0 + \sum_1^n a_i \varepsilon_i \in IS \qquad (a_i \in R,\ 0 \leqslant i \leqslant n) \tag{2}$$

implies $a_i \in P$ for all i. Conversely, if (2) holds for some $a_i \in U(R)$ the ε_i's are said *u-dependent over I.*

LEMMA 2.1. (i) If $\varepsilon \in U(S) \setminus R$, then ε is u-independent over $B(\varepsilon)$. (ii) If $\varepsilon_1, \ldots, \varepsilon_n \in U(S)$ are u-independent over $I \leqslant P$, then

$$\varepsilon_i \notin R \text{ and } B(\varepsilon_i) \geqslant I \text{ for all } i .$$

PROOF. (i) If $a_0 + a_1 \varepsilon \in B(\varepsilon) S$, then $a_0 \notin P$ if and only if $a_1 \notin P$ and, in this case, $\varepsilon \in B(\varepsilon) S + R$, which is absurd.

(ii) If, for some j, $\varepsilon_j \in R$, then (2) holds with $a_0 = \varepsilon_j$, $a_j = -1$ and $a_i = 0$ for $0 \neq i \neq j$. Assume now that $B(\varepsilon_j) < I$ for some j. Then $\varepsilon_j + IS = u + IS$ for some $u \in U(R)$, so (2) holds with $a_0 = u$, $a_j = -1$ and $a_i = 0$ for $0 \neq i \neq j$. ///

We say that a family $\{\varepsilon_\lambda : \lambda \in \Lambda\}$ of units of S not in R is u-independent over an ideal $I \leqslant P$, if any finite subset of it is u-indipendent over I; so the u-independence is a property of finite character, and maximal families of units with this property do exist.

Having fixed the ideal $I \leqslant P$, we consider all the families $\{\varepsilon_\lambda\}_{\lambda \in \Lambda}$ of units of S which are u-independent over I, such that $B(\varepsilon_\lambda) = I$ for all $\lambda \in \Lambda$. Let $c_R(I)$ be the minimal cardinal such that $c_R(I) \geqslant |\Lambda| + I$ for all these families.

We call $c_R(I)$ the *completion defect of R at I*; clearly $c_R(I)$ is an invariant of R not depending on the choice of S, being the u-independence defined by linearity.

Obviously R is almost maximal if and only if $c_R(I) = 1$ for all nonzero ideals I.

The following result compares the completion defects at isomorphic ideals.

PROPOSITION 2.2. Let $I \cong J$ be isomorphic ideals of R contained in P. Then $c_R(I) = c_R(J)$.

PROOF. It is enough to show that, given a family $\{\varepsilon_\lambda : \lambda \in \Lambda\} \subseteq S$ which is u-independent over I, where $I = B(\varepsilon_\lambda)$ for all $\lambda \in \Lambda$, there exists a family $\{\eta_\lambda : \lambda \in \Lambda\}$ which is u-independent over J, where $J = B(\eta_\lambda)$ for all $\lambda \in \Lambda$. Being $I \cong J$, there exists $a \in R$ such that either $J = aI$ or $aJ = I$; we can assume $a \in P$, otherwise $J = I$ and the claim is trivial. If $J = aI$, let $\eta_\lambda = 1 + a\varepsilon_\lambda$ for all $\lambda \in \Lambda$. Then $B(\eta_\lambda) = J$ for all $\lambda \in \Lambda$ follows from Lemma 1.3. Assume now that

$$a_0 + \sum_1^n a_i \eta_{\lambda_i} \in JS \ (a_i \in R,\ 0 \leqslant i \leqslant n);$$

then $a_0 + \sum_1^n a_i(1 + a\varepsilon_{\lambda_i}) \in aIS$ implies that

$$a^{-1}\left(a_0 + \sum_1^n a_i\right) + \sum_1^n a_i \varepsilon_{\lambda_i} \in IS$$

thus, by the u-independence of the ε_λ's, we deduce that $a_1, \ldots, a_n \in P$, and $a^{-1}\left(a_0 + \sum_1^n a_i\right) \in P$; it follows that $a_0 \in P$ too.

Conversely, assume that $aJ = I (a \in P)$. Notice that $a \notin I$, because $J \leqslant P$. Being $B(\varepsilon_\lambda) = I$, there exists an $u_\lambda^a \in U(R)$ such that $\varepsilon_\lambda = u_\lambda^a + a\eta_\lambda$ for some $\eta_\lambda \in S$. Without loss of generality, we can assume that $\eta_\lambda \in U(S)$: for, if $\eta_\lambda \in PS$, substitute u_λ^a and η_λ respectively by $u_\lambda^a - a \in U(R)$ and $1 + \eta_\lambda \in U(S)$. From $aJ = I = B(\varepsilon_\lambda)$ and from Lemma 1.3, we deduce that $aJ = aB(\eta_\lambda)$, so $B(\eta_\lambda) = J$. Assume now that

$$a0 + \sum_1^n a_i \eta_{\lambda_i} \in JS \qquad (a_i \in R,\ 0 \leqslant i \leqslant n)\,.$$

Then

$$\left(aa_0 - \sum_1^n a_i u^a_{\lambda_i}\right) + \sum_1^n a_i(u_{\lambda_i} + a\eta_{\lambda_i}) \in aJS = IS;$$

recalling that $u^a_{\lambda_i} + a\eta_{\lambda_i} = \varepsilon_{\lambda_i}$ $(1 \leqslant i \leqslant n)$, and that the ε_{λ_i}'s are u-independent over I, it follows that $a_1, \ldots, a_n \in P$; then $a_0 + \sum_1^n a_i \eta_{\lambda_i} \in JS$ implies $a_0 \in P$ too. ///

Given an ideal $I \leqslant P$, we introduce now another invariant; we consider all the families $\{\varepsilon_\lambda : \lambda \in \Lambda\}$ of units of S as in the definition of $c_R(I)$, but we assume only that the ε_λ's are u-independent over I, without assuming that $B(\varepsilon_\lambda) = I$ for all $\lambda \in \Lambda$; thus, by Lemma 2.1, we only know that $B(\varepsilon_\lambda) \geqslant I$ for all $\lambda \in \Lambda$. Let $d_R(I)$ be the minimal cardinal such that $d_R(I) \geqslant |\Lambda| + 1$ for all these families. We call $d_R(I)$ the *total defect of R at I*; here too we notice that $d_R(I)$ is an inavriant of R not depending on the choice of S.

The following result is an immediate consequence of the definition.

LEMMA 2.3. (i) If $I \leqslant J \leqslant P$, then $d_R(I) \geqslant d_R(J)$.

(ii) $d_R(I) = 1$ if and only if $I > B(R)$ or $I = B(R)$ and $R/B(R)$ is complete.

(iii) R is almost maximal if and only if $d_R(I) = 1$ for every $I \neq 0$. ///

Given an R-module M and an ideal $I \leqslant P$, we say that the elements $x_1, \ldots, x_n \in M$ are *linearly independent over I* if $x_1 + IM, \ldots, x_n + IM$ are linearly independent elements of the R/I-module M/IM, i.e. if $\sum_1^n a_i x_i \in IM (a_i \in R)$ implies that $a_i \in I$ for all i. Obviously one can extend this definition to a family of elements of M.

Recall that, if M is torsion-free, then the rank $rk_R\, M$ of M is the dimension of the Q-vector space $M \otimes_R Q$, where Q is the field of quotients of R, or, equivalently, the cardinality of a maximal system of linearly independent elements of M.

PROPOSITION 2.4. Let R be a valuation domain and I a prime ideal of R. Then $c_R(I) = rk_{R/I}(R/I)^\wedge$ and $d_R(I) = rk_{R/I} S/IS$.

PROOF. Given a family of elements $\{x_\lambda\colon \lambda \in \Lambda\}$ of $(R/I)^\wedge$ which are linearly independent over I, one can assume, without loss of generality, that $x_\lambda \in U(S)$ for all $\lambda \in \Lambda$, and that one of them, say $x_{\bar\lambda}$, is 1. It follows trivially from the definition that $\{x_\lambda\colon \lambda \neq \bar\lambda\}$ is a family of elements u-independent over I, and $B(x_\lambda) = I$ by Corollary 1.5; therefore $rk_{R/I}(R/I)^\wedge \leqslant c_R(I)$. In a similar way one can see that $rk_{R/I} S/IS \leqslant d_R(I)$.

Conversely, to prove that $c_R(I) \leqslant rk_{R/I}(R/I)^\wedge$ (respectively $d_R(I) \leqslant rk_{R/I} S/IS$) it is enough to show that, given a family $\{\varepsilon_\lambda\colon \lambda \in \Lambda\}$ of units of S with $B(\varepsilon_\lambda) = I$ for all $\lambda \in \Lambda$ (resp. with $B(\varepsilon_\lambda) \geqslant I$), which are u-independent over I, then $\{1, \varepsilon_\lambda\colon \lambda \in \Lambda\}$ are linearly independent over I. Assume that

$$a_0 + \sum_1^n a_i \varepsilon_{\lambda_i} \in IS \qquad (a_1 \in R);$$

if some $a_i \notin I$, let a_j be one of the coefficients not in I with minimal value. By multiplying by a_j^{-1}, we get

$$a_j^{-1}\Big(a_0 + \sum_1^n a_i \varepsilon_{\lambda_i}\Big) \in a_j^{-1} IS = IS\ ,$$

because $a_j I = I$; the last relation is absurd, because the coefficient of ε_{λ_i} is equal to 1, which contradicts the u-independence of $\{\varepsilon_\lambda\colon \lambda \in \Lambda\}$ over I. ///

We wish to compare now the total defect $d_R(I)$ at the ideal I with the completion defects $c_R(J)$ at the ideals $J \geqslant I$.

LEMMA 2.5. For every $i = 1, \ldots, n$, let E_i be a family of units of S u-independent over the ideals J_i, such that $J_i = B(\varepsilon)$ for all $\varepsilon \in E_i$. If $J_1 > J_2 > \ldots > J_n$, and the J_i's are pairwise non isomorphic, then $\cup\{E_i\colon 1 \leqslant i \leqslant n\}$ is u-independent over J_n.

PROOF. We induct on n, the claim being trivial for $n = 1$. So, assume that $n > 1$ and that $\cup\{E_i\colon 1 \leqslant i \leqslant t\}$ is u-independent over J_t, for $1 \leqslant t \leqslant n-1$. Let

$$a_0 + \sum_1^k a_j \varepsilon_j + \sum_1^m b_h \eta_h \in J_n S \tag{3}$$

where a_j, $b_h \in R\setminus\{0\}$ $(0 \leqslant j \leqslant k)$, $1 \leqslant h \leqslant m$; $\varepsilon_j \in \bigcup\{E_i : 1 \leqslant i \leqslant n-1\}$ for $1 \leqslant j \leqslant k$ and $\eta_h \in E_n$ for $1 \leqslant h \leqslant m$. First, notice that $a_1, \ldots, a_k \in P$: for, let $r \in J_n \setminus J_{n-1}$; then, for all $h \leqslant m$ there exists $v_h^r \in U(R)$ such that $\eta_h - v_h^r \in rS$. It follows that

$$a_0 + \sum_1^k a_j \varepsilon_j + \sum_1^m b_h v_h^r \in J_{n-1} S$$

and the u-independence of the ε_j's implies that $a_1, \ldots, a_k \in P$. Recall now that $B(\varepsilon_j)$ is one of the J_i's, for $1 \leqslant i \leqslant n-1$, for all j; we will show that

$$a_j B(\varepsilon_j) = B(1 + a_j \varepsilon_j) < J_n \text{ for all } j .$$

Being J_n not isomorphic to $J_1, \ldots, J_{n-1}$, $a_j B(\varepsilon_j) \neq J_n$ for all j; let

(4) $$A_1 = \{j : a_j B(\varepsilon_j) < J_n\}; \qquad A_2 = \{j : a_j B(\varepsilon_j) > J_n\} .$$

Let $t \in J_n \setminus \bigcup\{a_j B(\varepsilon_j) : j \in A_1\}$; for all $j \in A_1$ we can choose an element $w_j^t \in U(R)$ such that

$$(1 + a_j \varepsilon_j) - w_j^t \in tS < J_n S ;$$

substituting in (3), we get:

(5) $$a_0 - k1 + \sum_{j \in A_1} w_j^t + \sum_{j \in A_2} (1 + a_j \varepsilon_j) + \sum_1^m b_h \eta_h \in J_n S$$

where the sums with indexes in A_1, A_2 ate intended to be 0 whenever A_1 or A_2 is void.

Assume now that $A_2 \neq \emptyset$. Let $c \in \bigcap_{j \in A_2} a_j B(\varepsilon_j) \setminus J_n$, and choose for all $h \leqslant m$, $v_h^c \in U(R)$ such that $\eta_h - v_h^c \in cS$; substituting in (5) we get:

(6) $$a_0 - k1 + \sum_{j \in A_1} w_j^t + \sum_{j \in A_2} (1 + a_j \varepsilon_j) + \sum_1^m b_h v_h^c \in \bigcap_{j \in A_2} a_j B(\varepsilon_j) S ;$$

we will show that (6) is absurd. Let $j_0 \in A_2$ be such that a_{j_0} has minimal value among the a_j's with $j \in A_2$; then

$$(7) \qquad a_{j_0}^{-1}(a_0 - k1 + \sum_{j\in A_1} w_j^t + \sum_1^m b_h v_h^c + |A_2|\cdot 1) + \\ + \sum_{j\in A_2} a_{j_0}^{-1} a_j \varepsilon_j \in \bigcap_{j\in A_2} a_{j_0}^{-1} a_j B(\varepsilon_j) S\,;$$

notice that in (7) $a_{j_0}^{-1}a_j \in R$ for all $j \in A_2$, and that

$$B(\varepsilon_{j_0}) \geqslant a_{j_0}^{-1} \bigcap_{j\in A_2} a_j B(\varepsilon_j)$$

therefore also the first summand in (7) is in R. But then (7) is absurd, because the coefficient of ε_{j_0} is 1 and by the inductive hypothesis. Thus we have proved that $A_2 = \emptyset$, therefore (5) becomes simply:

$$(8) \qquad (a_0 - k1 + \sum_{j\in A_1} w_j^t) + \sum_1^m b_h \eta_h \in J_n S\,;$$

by the u-independence of the η_h's over J_n, (8) gives that $b_1, \ldots, b_m \in P$; being $a_1, \ldots, a_k \in P$, from (3) it follows that $a_0 \in P$. ///

Given an ideal $J \leqslant R$, let $[J]$ denote the isomorphism class of J; if $I \leqslant R$ is another ideal, then $[J] \geqslant I$ means that there exists $J' \in [J]$ such that $J' \geqslant I$. By Proposition 2.2, we can define $c_R[J]$ as the common value $c_R(J')$, where J' ranges over $[J]$. We can easily obtain from the preceding Lemma 2.5 the following

PROPOSITION 2.6. $d_R(I) \geqslant \sum \{c_R[J] - 1 : [J] \geqslant I\} + 1$. ///

The inequality in Proposition 2.6 is in general strict, as is shown by Facchini and the second author in [3]; actually, they prove a multiplicative formula relating $d_R(I)$ and the $c_R[J]$'s, $[J] \geqslant I$, for I a prime ideal of a discrete valuation domain R with $\operatorname{Spec} R$ well ordered by the opposite inclusion; they also give a realization theorem for these domains with preassigned completion defects, using an idea of Nagata [9].

3. – Total defect and Goldie dimension.

Let I be an ideal of the valuation domain R, and let $g_R(I)$ denote the Goldie dimension of S/IS as an R/I-module. If I is a prime ideal, then $g_R(I) = rk_{R/I}S/IS$. It follows from the definitions that, for an arbitrary ideal I, $g_R(I) \leqslant d_R(I)$, and Proposition 2.4 shows that this inequality becomes an equality if I is a prime ideal.

Recall that, if $0 \neq I \leqslant P$, then the subset of R

$$I^{\#} = \{r \in R : rI < I\}$$

is a prime ideal, which is the union of all the ideals $< R$ isomorphic to I (see [10] and [4]). If $I = 0$, we set $I^{\#} = 0$.

LEMMA 3.1. Given any $I < R$, $g_R(I) = g_R(I^{\#})$.

PROOF. It is enough to show that, given $\varepsilon_1, \ldots, \varepsilon_n \in U(S)$, they are linearly independent over $I^{\#}$ if and only if they are linearly independent over I. So, assume that they are linearly independent over $I^{\#}$ and let $\sum_1^n a_i\varepsilon_i \in IS$. If some $a_i \notin I$, let $a \in R$ be such that $v(a) = \min\{v(a_i) : 1 \leqslant i \leqslant n\}$. Then

$$a^{-1} \sum_1^n a_i \varepsilon_i \in a^{-1}IS \leqslant I^{\#}S$$

because $a^{-1}I$ is an ideal isomorphic to I, hence $a^{-1}I \leqslant I^{\#}$. But this is a contradiction, because some $a^{-1}a_i$ is a unit.

Conversely, let $\varepsilon_1, \ldots, \varepsilon_n$ be linearly independent over I and let $x = \sum_1^n a_i\varepsilon_i \in I^{\#}S$. If $x \in IS$, then $a_i \in I$ for all i, hence $a_i \in I^{\#}$ for all i. If $x \notin IS$, then $x = r\eta$, where $\eta \in U(S)$ and $r \in I^{\#} \setminus I$. Being $I^{\#}$ the union of the ideals isomorphic to I, there exists an ideal $J \leqslant I^{\#}$ such that $tJ = I$ for some $t \in P$ and $r \in J$. Then $tr \in I$, therefore $t\sum_1^n a_i\varepsilon_i \in IS$; the independence of the ε_i's over I implies that $ta_i \in I = tJ$, hence $a_i \in J \leqslant I^{\#}$ for all i. ///

Recall that an ideal $I < R$ is archimedean if $I^{\#} = P$. As an immediate consequence of the preceding lemma we get

COROLLARY 3.2. Given two ideals $I \simeq J$, then $g_R(I) = g_R(J)$; moreover $g_R(I) = 1$ if I is archimedean.

PROOF. The first claim follows from the equality $I^{\#} = J^{\#}$; the second equality follows from the isomorphism $S/PS \simeq R/P$. ///

Lemma 3.1 and Proposition 2.4 give the following

COROLLARY 3.4. Given any ideal $I < R$, $g_R(I) = d_R(I^{\#})$. ///

COROLLARY 3.5. Given any ideal $I < R$, then $g_R(I) = 1$ if and only if either $I^{\#} > B(R)$, or $I^{\#} = B(R)$ and $R/B(R)$ is complete. ///

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