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Rendiconti del Seminario Matematico della Università di Padova, tome 74 (1985), p. 129-137

<http://www.numdam.org/item?id=RSMUP_1985__74__129_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 74 (1985)

Group Graded Rings and Smash Products.

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SUNTO - Mediante una nuova caratterizzazione del Smash prodotto si dimostrano direttamente e in forma un po' più generale i teoremi di dualità di Cohen e Montgomery.

Introduction.

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a k-algebra graded by a finite group G, with R_e the component corresponding to the identity element e of G (k is a commutative ring). In the paper [1] M. Cohen and S. Montgomery define the ring $R \# k[G]^*$, called the « Smash product » associated to the graded ring R. This ring may be used to obtain many properties of the graded ring R. The main tools are provided by the two Duality Theorems: Duality Theorem for Action and Duality Theorem for Coaction (see Theorem 3.2 and 3.5 of [1]). In this paper we give a new characterization of the Smash Product $R \# k[G]^*$ (Theorems 1.2 and 1.3) and we deduce directly from it and in a little more general form Cohen and Montgomery Duality Theorems (Theorems 2.2 and 2.3).

1. The rings $\mathrm{End}_{R\text{-gr}}(U)$, $\mathrm{End}_R(U)$ and their structure.

If $R = \bigoplus_{\sigma \in G} R_\sigma$ is a k-algebra graded by a finite group G, we denote by R-gr the category of all unital right graded R-modules. If $M = \bigoplus_{\sigma \in G} M_\sigma$, $N = \bigoplus_{\sigma \in G} N_\sigma$ are two objects of R-gr, then the morphisms in R-gr are R-homomorphisms $f\colon M \to N$ such that $f(M_\sigma) \subset f(N_\sigma)$ for all $\sigma \in G$. It is well known that R-gr is a Grothendieck category (see [2]).

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If $M = \bigoplus_{\lambda \in G} M_\lambda$ is a graded R-module and $\sigma \in G$, then $M(\sigma)$ is the graded module obtained from M by putting $M(\sigma)_\lambda = M_{\lambda\sigma}$; the graded module $M(\sigma)$ is called the *σ-suspension* of M [2]. It is well-known that the mapping $M \to M(\sigma)$ defines a functor from R-gr to R-gr which is an equivalence of categories for all $\sigma \in G$. $M \in R$-gr is said to be *G-invariant* [2] if for all $\sigma \in G$, $M \simeq M(\sigma)$ in R-gr. Consider now the graded modules M and N. A R-linear mapping $f\colon M \to N$ is said to be a graded morphism of degree τ, $\tau \in G$, if $f(M_\sigma) \subset N_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree τ build up an additive subgroup $\mathrm{HOM}_R(M, N)_\tau$ of $\mathrm{Hom}_R(M, N)$. It is clear that $\mathrm{HOM}_R(M, N) = \bigoplus_{\tau \in G} \mathrm{HOM}_R(M, N)_\tau$ is a graded abelian group of type G and $\mathrm{HOM}_R(M, N)_e = \mathrm{Hom}_{R\text{-gr}}(M, N)$. In particular, if $M = N$, then $\mathrm{HOM}_R(M, N) = \mathrm{END}_R(M)$ is a graded ring of type G. In the sequel we will denote by $U = \bigoplus_{\sigma \in G} R(\sigma)$. Since $\{R(\sigma)\}_{\sigma \in G}$ is a family of generators for R-gr [2], it follows that U is a generator for R-gr. When G is a finite group, we also remark that $\mathrm{END}_R(U) = \mathrm{End}_R(U)$ (see [2]).

PROPOSITION 1.1. [2] *If $G = \{g_1 = e, g_2, \ldots, g_n\}$ is a finite group, then the ring* $\mathrm{End}_R(U)$ *is isomorphic to the matrix ring $M_n(R)$ equipped with the grading*

$$M_n(R) = \bigoplus_{\lambda \in G} M_n(R)_\lambda,$$

where

$$M_n(R)_\lambda = \begin{pmatrix} R_{g_1\lambda g_1^{-1}} & R_{g_1\lambda g_2^{-1}} \cdots & R_{g_1\lambda g_n^{-1}} \\ R_{g_2\lambda g_1^{-1}} & R_{g_2\lambda g_2^{-1}} \cdots & R_{g_2\lambda g_n^{-1}} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ R_{g_n\lambda g_1^{-1}} & R_{g_n\lambda g_2^{-1}} \cdots & R_{g_n\lambda g_n^{-1}} \end{pmatrix}.$$

In particular, the ring $\mathrm{End}_{R\text{-gr}}(U)$ *is isomorphic to the matrix ring*

$$\begin{pmatrix} R_e & R_{g_1 g_2^{-1}} \cdots & R_{g_1 g_n^{-1}} \\ R_{g_2 g_1^{-1}} & R_e \quad \cdots & R_{g_2 g_n^{-1}} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ R_{g_n g_1^{-1}} & R_{g_n g_2^{-1}} \cdots & R_e \end{pmatrix}.$$

PROOF. See Corollary I.5.3 of [2].

By an *action* of a group G on a k-algebra A we mean a group morphism $\alpha: G \to \mathrm{Aut}_k(A)$; let α_g denote the image of $g \in G$ in $\mathrm{Aut}_k(A)$. We may define the *skew group ring* (or *trivial crossed product*) denoted by $A * G$, as being the free right and left A-module with basis $\{g: g \in G\}$ and with moltiplication given by $(ag)\cdot(bh) = a\alpha_g(b)gh$, where $a, b \in A$, $g, h \in G$. The ring $A * G$ is a graded ring: $A * G = \bigoplus_{\sigma\in G} (A * G)_\sigma$, where $(A * G)_\sigma = A\sigma = \{a\sigma: a \in A\}$.

THEOREM 1.2. *If* $G = \{g_1 = e, g_2, \ldots, g_n\}$ *is a finite group, then the ring* $\mathrm{End}_R(U)$ *is isomorphic to the skew group ring* $\mathrm{End}_{R\text{-gr}}(U) * G$.

PROOF. By Proposition 1.1 we have that $\mathrm{End}_R(U)$ is isomorphic to the matrix ring $M_n(R)$. We consider the set

$$U_\lambda = \begin{pmatrix} 0\cdots 0 & R_e & 0\cdots 0 \\ 0\cdots R_e & 0 & 0\cdots 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0\cdots\cdots & \cdots & R_e\cdots 0 \end{pmatrix}$$

where on the first row R_e is on the k_1-th position, k_1 being such that $g_{k_1} = g_1\lambda$; on the second row R_e is on the k_2-th position, where k_2 is such that $g_{k_2} = g_2\lambda$; ...; on the n-th row R_n is on the k_n-th position, where $g_{k_n} = g_n\lambda$. Since G is a group it is easy to see that $\{1, 2, \ldots, n\} = (k_1, k_2, \ldots, k_n\}$. Moreover one may see that $U_\lambda \subset M_n(R)_\lambda$. Let $u_\lambda \in U_\lambda$,

$$u_\lambda = \begin{pmatrix} 0\cdots 0 & 1 & 0\cdots 0 \\ 0\cdots 1 & 0 & \cdots\cdots 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0\cdots\cdots & 1 & \cdots\cdots 0 \end{pmatrix}.$$

From the definition of U_λ it follows that u_λ has a 1 on each column and all the other entries are 0. We will show now that the system $\{u_\lambda\}_{\lambda\in G}$ has the property that $u_\lambda u_\mu = u_{\lambda\mu}$, for all $\lambda, \mu \in G$. Let

$$u_\mu = \begin{pmatrix} 0\cdots & 1 & \cdots\cdots 0 \\ 0\cdots & 1 & 0\cdots 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0\cdots\cdots & \cdots & 1\ \ 0 \end{pmatrix}$$

where 1 in the first row is on the l_1-th column, where $g_{l_1} = g_1\mu$; 1 in the second row is on the l_2-th column, where $g_{l_2} = g_2\mu$; ...; 1 in the n-th row is on the l_n-th column, where $g_{l_n} = g_n\mu$.

There exists a unique column, say the s-th, such as its intersection with the k_1-th row has a 1 and the rest of its entries are zero. Thus we have that $g_s = g_{k_1}\mu$. Since $g_{k_1} = g_1\lambda$, then $g_s = g_1\lambda\mu$ and so in the matrix $u_\lambda u_\mu$ we have 1 on the first row in the s-th position, all the other entries of the first row being zero. Hence the first row of the matrix $u_\lambda u_\mu$ is the same as the first row of the matrix $u_{\lambda\mu}$. Using the same argument for the other rows, we deduce that $u_\lambda u_\mu = u_{\lambda\mu}$. In particular, since u_e is equal to the unit matrix, we obtain that $u_\lambda^{-1} = u_{\lambda^{-1}}$, for each $\lambda \in G$. Using now Theorem 5.3.23 of [2], we obtain that $\mathrm{End}_R(U)$ is isomorphic to the skew group ring $\mathrm{End}_{R\text{-gr}}(U) * G$. Q.E.D.

Let now $A = \bigoplus_{g\in G} A_g$ be a graded k-algebra, where k is a commutative ring and G is a finite group. By [1], the construction of the smash product $A \# k[G]^*$ is the following: $A \# k[G]^*$ is the free left and right A-module with basis $\{p_g : g \in G\}$, a set of orthogonal idempotents whose sum is 1 and with the moltiplication given by the rule: $(ap_g)\cdot(bp_h) = ab_{gh^{-1}}p_h$, $a, b \in A$.

THEOREM 1.3. *Let* $R = \bigoplus_{\sigma\in G} R_\sigma$ *be a graded k-algebra, where* $G = \{g_1 = e, g_2, \ldots, g_n\}$ *is a finite group. Then the ring* $\mathrm{End}_{R\text{-gr}}(U)$ *is isomorphic to the smash product*

$$R \# k[G]^* .$$

PROOF. We have seen that the ring $\mathrm{End}_{R\text{-gr}}(U)$ is isomorphic to the matrix ring

$$T = \begin{pmatrix} R_e & R_{g_1g_2^{-1}} \cdots R_{g_1g_n^{-1}} \\ R_{g_2g_1^{-1}} & R_e \quad \cdots R_{g_2g_n^{-1}} \\ \cdots & \cdots\cdots\cdots \\ R_{g_ng_1^{-1}} & R_{g_ng_1^{-1}} \cdots R_e \end{pmatrix}.$$

Define $\varphi : R \to T$ as follows: if $a = \sum_{g\in G} a_g$, $a_g \in R_g$ for all $g \in G$, then put

$$\varphi(a) = \begin{pmatrix} a_e & a_{g_1g_2^{-1}} \cdots a_{g_1g_n^{-1}} \\ a_{g_2g_1^{-1}} & a_e \quad \cdots a_{g_2g_n^{-1}} \\ \cdots\cdots & \cdots\cdots\cdots \\ a_{g_ng_1^{-1}} & a_{g_ng_2^{-1}} \cdots a_e \end{pmatrix}.$$

It is clear that φ is additive and injective. It is straightforward to check that if $b \in R$, then we have $\varphi(ab) = \varphi(a)\varphi(b)$. Consequently, φ is a ring morphism. Let $S = \varphi(R)$. Hence S is a subring of T. Consider the elements:

$$p_{g_k} = \begin{pmatrix} 0 \cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots \\ 0 \cdots 0 \; 1 \cdots 0 \\ 0 \cdots\cdots\cdots 0 \end{pmatrix} \leftarrow k\text{-th row}$$

$\uparrow$
k-th column

i.e. p_{g_k} is a matrix with 1 at the intersection of the k-th row with the k-th column, all its other entries being zero. Then the system of elements $\{p_{g_k}\}_{1 \le k \le n}$ is a system of orthogonal idempotents whose sum is 1. It is clear that

$$T = Sp_{g_1} + Sp_{g_2} + \ldots + Sp_{g_n} = p_{g_1}S + p_{g_2}S + \ldots + p_{g_n}S$$

Let us prove that $\{p_{g_k}\}$, $1 \le k \le n$, is a linear independent system over the ring S. Let $\sum_{k=1}^{n} s_k p_{g_k} = 0$, where $s_k \in S$. Hence

$$s_k = \begin{pmatrix} a_e^k & a^k_{g_1g_2^{-1}} \cdots a^k_{g_1g_n^{-1}} \\ a^k_{g_2g_1^{-1}} & a_e^k \quad \cdots a^k_{g_2g_n^{-1}} \\ \cdots\cdots & \cdots\cdots\cdots \\ a^k_{g_ng_1^{-1}} & a^k_{g_ng_2^{-1}} \quad a_e^k \end{pmatrix}.$$

Then

$$\sum_{k=1}^{n} s_k p_{g_k} = \begin{pmatrix} a_e^1 & 0\cdots 0 \\ a^1_{g_2g_1^{-1}} & 0\cdots 0 \\ \cdots & \cdots \\ a_{g_ng_1^{-1}}^1 & 0\cdots 0 \end{pmatrix} + \begin{pmatrix} 0 & a^2_{g_1g_2^{-1}} & 0\cdots 0 \\ 0 & a_e^2 & 0\cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & a^2_{g_ng_2^{-1}} & 0\cdots 0 \end{pmatrix} + \cdots +$$

$$+ \begin{pmatrix} 0\cdots 0 & a^n_{g_1g_n^{-1}} \\ 0\cdots 0 & a^n_{g_2g_n^{-1}} \\ \cdots & \cdots \\ 0\cdots 0 & a_e^n \end{pmatrix}$$

and hence $a_e^k = a^k_{g_1g_2^{-1}} = \ldots = a^k_{g_1g_n^{-1}} = 0$, for all k, $1 \leqq k \leqq n$. Thus $s_1 = s_2 = \ldots = s_n = 0$. The ring S is a graded ring of type G with the grading $\{S_g : g \in G\}$, where

$$S_g = \begin{pmatrix} 0\cdots & R_g & 0\cdots 0 \\ 0 & R_g & \cdots & 0\cdots 0 \\ \cdots & \cdots & \cdots \\ 0\cdots\cdots\cdots & R_g\cdots \end{pmatrix}$$

and R_g is on the first row on the k_1-th position, k_1 being such that $g = g_1 g_{k_1}^{-1}$, on the second row on the k_2-th position such that $g = g_2 g_{k_2}^{-1}$, ..., on the n-th row on the k_n-th position such that $g = g_n g_{k_n}^{-1}$. Thus it is clear that $\varphi: R \to S$ becomes an isomorphism of graded rings. To finish the proof we need to show that:

$$(sp_g)(tp_h) = st_{gh^{-1}} p_h, \text{ for all } s, t \in S .$$

To see this let $g = g_m$, $h = g_\varrho$ and

$$t = \begin{pmatrix} b_e & b_{g_1g_2^{-1}} \cdots b_{g_1g_n^{-1}} \\ b_{g_2g_1^{-1}} & b_e \quad \cdots b_{g_2g_n^{-1}} \\ \cdots & \cdots \\ b_{g_ng_1^{-1}} & b_{g_n} \quad \cdots b_e \end{pmatrix}.$$

We have that

$$p_{g_m}t = \begin{pmatrix} 0 & 0 \cdots 0 & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots 0 & \cdots 0 \\ b_{g_m g_1^{-1}} & b_{g_m g_2^{-1}} \cdots & b_{g_m g_n^{-1}} \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots 0 & \cdots 0 \end{pmatrix} \leftarrow m\text{-th row}$$

and thus

$$p_{g_m}(tp_{g_l}) = \begin{pmatrix} 0 \cdots 0 \cdots & 0 \\ \cdots & \cdots \\ 0 \cdots b_{g_m g_l^{-1}} \cdots & 0 \\ \cdots & \cdots \\ 0 \cdots 0 \cdots\cdots & 0 \end{pmatrix} \leftarrow m\text{-th row}\,.$$

$\uparrow$ l-th column

On the other hand, since $t_{g_m g_l^{-1}}$ is a matrix which has on the first row a single non zero element, $b_{g_m g_l^{-1}}$, on the k_1-th position, for which $g_m g_l^{-1} = g_1 g_{k_1}^{-1}$, on the second row a single non zero element, $b_{g_m g_l^{-1}}$, on the k_2-th position, where $g_m g_l^{-1} = g_2 g_{k_2}^{-1}$, ..., on the n-th row a single non zero element, $b_{g_m g_l^{-1}}$, on the k_n-th position such that $g_m g_l^{-1} = g_n g_{k_n}^{-1}$, it is clear that $k_m = 1$ and $k_s \neq 1$ for $s \neq m$, $1 \leqq s \leqq n$. We deduce that

$$t_{g_m g_l^{-1}} p_{g_l} = \begin{pmatrix} 0 \cdots\cdots\cdots & 0 \\ \cdots & \cdots \\ 0 \cdots b_{g_m g_l^{-1}} \cdots & 0 \\ \cdots & \cdots \\ 0 \cdots 0 \cdots\cdots & 0 \end{pmatrix} \leftarrow m\text{-th row}$$

$\uparrow$ l-th column

and thus $p_g(tp_h) = t_{gh^{-1}} p_h$, so clearly $(sp_g)(tp_h) = st_{gh^{-1}} p_h$, for all s, $t \in S$ and g, $h \in G$.

2. Cohen-Montgomery duality theorems.

The notation in this section will be the same as the one in section 1.

THEOREM 2.1. *Let* $R = \bigoplus_{\sigma \in G} R_\sigma$ *be a graded ring of type* G, *where* G *is a finite group. If we denote* $U = \bigoplus_{\sigma \in G} R(\sigma)$, *then the functor* $M \to \mathrm{Hom}_{R\text{-gr}}(U, M)$ *is an equivalence from the category* R-gr *to the category* $\mathrm{End}_{R\text{-gr}}(U)$-mod.

PROOF. Since $\{R(\sigma)\}_{\sigma \in G}$ is a set of generators for R-gr [2], then U is a generator for R-gr. On the other hand, U is a finitely generated projective R-module. Hence U is a small projective generator for R-gr and therefore, after a classical result of B. Mitchel (see [3]), the functor $M \to \mathrm{Hom}_{R\text{-gr}}(U, M)$ is an equivalence between the categories R-gr and $\mathrm{End}_{R\text{-gr}}(U)$-mod. Q.E.D.

REMARKS:

1) Bearing in mind Theorem 1.2, Theorem 2.1 is nothing else that Theorem 2.2 of [1].

2) If $M = \bigoplus_{\sigma \in G} M_\sigma$, then $\mathrm{Hom}_{R\text{-gr}}(U, M) \simeq \bigoplus_{\sigma \in G} \mathrm{Hom}_{R\text{-gr}}(R(\sigma), M)$. Since $\mathrm{Hom}_{R\text{-gr}}(R(\sigma), M) = M_\sigma$, then $\mathrm{Hom}_{R\text{-gr}}(U, M) \simeq M$.

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring. R is called a *crossed product* if for every $\sigma \in G$ there exists a homogeneous invertible element $u_\sigma \in R_\sigma$. The structure of crossed products is given in Theorem I.3.23 of [2].

THEOREM 2.2. *Let* $R = \bigoplus_{\sigma \in G} R_\sigma$ *be a graded k-algebra which is a crossed product, where* G *is a finite group with* $n = |G|$. *Then*

$$R \# k[G]^* \simeq M_n(R_e) .$$

PROOF. Since R is a crossed product, then $R \simeq R(\sigma)$ in the category R-gr for any $\sigma \in G$. Therefore $U = \bigoplus_{\sigma \in G} R(\sigma) \simeq R^{(n)}$.

LEMMA. *Let* $\mathcal{C}$ *be an abelian category and* M *an object of* $\mathcal{C}$. *Then, for* $n > 0$

$$\mathrm{End}_{\mathcal{C}}(M^{(n)}) \simeq M_n(\mathrm{End}_{\mathcal{C}}(M)) .$$

PROOF. Straightforward (see [3]).

We may use the Lemma to obtain that $\mathrm{End}_{R\text{-gr}}(U) \simeq M_n(\mathrm{End}_{R\text{-gr}}(R))$. But $\mathrm{End}_{R\text{-gr}}(U) \simeq R_e$ and therefore $\mathrm{End}_{R\text{-gr}}(U) \simeq M_n(R_e)$. We apply now Theorem 1.3 and obtain that

$$R \# k[G]^* \simeq M_n(R_e). \qquad \text{Q.E.D.}$$

REMARK. This result is a slight extension of Theorem 3.2 (Duality Theorem for Action) of Cohen and Montgomery [1], which is given in the case $R = S * G$ is a skew group ring.

THEOREM 2.3. (Duality for Coactions) (see [1]). *Let* $R = \bigoplus_{\sigma\in G} R_\sigma$ *be a graded k-algebra, where G is a finite group with* $n = |G|$. *Then*

$$(R \# k[G]^*) * G \simeq M_n(R).$$

PROOF. By Theorem 1.2 and Theorem 1.3 we have that $\mathrm{End}_R(U) \simeq$ $\simeq (R \# k[G]^*) * G$. Since $U = \bigoplus_{\sigma\in G} R(\sigma)$, then in the category R-mod it is $U \simeq R^{(n)}$. Therefore

$$\mathrm{End}_R(U) \simeq \mathrm{End}_R(R^{(n)}) \simeq M_n(R). \qquad \text{Q.E.D.}$$

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Manoscritto pervenuto in redazione il 18 luglio 1984.