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About Abhyankar's Conjectures on space lines.

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SUNTO - Il principale risultato di questo lavoro è che la ben nota quintica di Abhyankar $\mathcal{C}_5: (t + t^5, t^4, t^3)$ è una linea rettificabile, cioè esiste un automorfismo $\Phi: \mathbb{A}^3 \to \mathbb{A}^3$ che trasforma $\mathcal{C}_5$ in una retta.

Introduction.

The considerations of this paper will be done over an algebraically closed field k of characteristic zero. In $\mathbb{A}_k^3$ we say that two curves $\mathcal{C}$, $\mathcal{C}'$ or two surfaces $\mathcal{F}$, $\mathcal{F}'$ are *equivalent* if there exists an automorphism $\Phi: \mathbb{A}_k^3 \to \mathbb{A}_k^3$ such that $\Phi(\mathcal{C}) = \mathcal{C}'$ or $\Phi(\mathcal{F}) = \mathcal{F}'$ respectively. A curve which is isomorphic to a straight line, will be called shortly a line; a line which is equivalent to a straight line will be called a *rectifiable* line. An automorphism of $\mathbb{A}_k^3$ which is the product of linear and triangular automorphisms will be called *elementary*.

By an *elementarily rectifiable* line we mean a line which is rectified by an elementary automorphism.

In Chapter 11) of Prof. Abhyankar's paper « On the semigroup of a meromorphic curve » [1], there are open problems about lines in A_k^n ($n \geqslant 3$) and some Conjectures. Among them there are:

Conjecture 2) (p. 413) which says, in particular, that a line $\mathcal{C}$: $(x(t), y(t), z(t))$ of $\mathbb{A}_k^3$, such that no one of deg $x(t)$, deg $y(t)$, deg $z(t)$ belongs to the semigroup generated by the others two is non rectifiable; and, as part of Conjecture 2), is separately formulated

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Conjecture 3) (p. 414) which says, in particular, that the following curves

$$\mathcal{C}_n : (t + t^n, t^{n-1}, t^{n-2}) \qquad (n \geqslant 5)$$

are all non rectifiable.

In a previous paper [3], among other things, we disproved Conjecture 2), providing examples of lines, satisfying the above condition on degrees, that are actually rectifiable. We did not then tackle the problem of proving or disproving Conjecture 3), thinking that probably Conjecture 2) had been stated as a generalization of Conjecture 3), with may be not many attempts to disprove it; whereas it seems to us that this was not so for Conjective 3), and in particular with $\mathcal{C}_5 : (t + t^5, t^4, t^3)$, the well known Abhyankar's quintic curve (see [2]) p. 98-99. Now we disprove Conjecture 3) just for the curve $\mathcal{C}_5$, showing it to be rectifiable. The compexity of the automorphism that rectifies it (see end of § 2) shows however that Conjecture 3) was reasonably formulated: anyway the $\mathcal{C}_n$, with $n \geqslant 6$, are still there supporting Conjecture 1) ([1], p. 413) which, in particular, asserts the existence of non rectifiable lines in $\mathbb{A}^3_k$.

§ 1. We shall state some general facts for lines in $\mathbb{A}^3_k$. In what follows, given a line $\mathcal{C}$: $(x(t), y(t), z(t))$, we shall always suppose t to be a global regular parameter.

PROPOSITION 1. *If* $\mathcal{C}$: $(x(t), y(t), z(t))$ *is a line that meets transversally in a single point a plane,* $\mathcal{C}$ *is elementarily rectifiable.*

PROOF. No restriction in supposing the plane to be $\{X = 0\}$. Then $x(t)$ must be linear in t, or $\mathcal{C}$ would meet $\{X = 0\}$ in more then one point, or in one point but not transversally, in case $x(t)$ should be a power of a linear polynomial. Suppose $x(t) = a_0 + a_1 t$, $a_1 \neq 0$, $y(t) = b_0 + \ldots + b_m t^m$ and $z(t) = c_0 + \ldots + c_n t^n$, and consider the elementary automorphism $\Psi \circ \Phi$, where

$$\Phi = \begin{pmatrix} \frac{1}{a_1} X - \frac{a_0}{a_1} \\ Y \\ Z \end{pmatrix}, \qquad \Psi = \begin{pmatrix} X \\ Y - b_0 - \ldots - b_m X^m \\ Z - c_0 - \ldots - c_n X^n \end{pmatrix}$$

we have $\Psi \circ \Phi(\mathcal{C}) : (t, 0, 0)$.

PROPOSITION 2. *A necessary and sufficient condition for a line* $\mathcal{C}$: $(x(t), y(t), z(t))$ *to be rectifiable is that it meets transversally in a single point a surface* $\mathcal{F}$ *which is equivalent to a plane.*

PROOF. Let Φ' an automorphism transorming $\mathcal{F}$ into a plane α: $\Phi'(\mathcal{F}) = \alpha$. Then, since Φ' preserves intersection multiplicities, $\Phi'(\mathcal{C})$ is a line that meets transversally the plane $\Phi'(\mathcal{F}) = \alpha$ in a single point. By Prop. 1), there exists an elementary automorphism Φ'' such that $\Phi''(\Phi'(\mathcal{C}))$ is a straight line l: so $\Phi'' \circ \Phi'(\mathcal{C}) = l$, and $\mathcal{C}$ is rectifiable. Conversely, let Φ an automorphism such that $\Phi(\mathcal{C})$ is a straight line l, and let α be a plane transversal to l: then $\Phi = \Phi^{-1}(\alpha)$ is a surface equivalent to a plane that meets transversally $\mathcal{C} = \Phi^{-1}(l)$ in a single point.

PROPOSITION 3). *Let* $\mathcal{C}$: $(x(t), y(t), z(t))$ *be a line and suppose* $F(X, Y, Z)$ *is a polynomial such that* $t = F(x(t), y(t), z(t))$. *Then a necessary and sufficient condition for* $\mathcal{C}$ *to be rectifiable is that among all the polynomials of the kind* $F(X, Y, Z) + H(X, Y, Z)$, *with* $H(X, Y, Z) \in \mathfrak{I}(\mathcal{C})$ *there is one* $P(X, Y, Z) = F(X, Y, Z) + H(X, Y, Z)$, *such that* $\mathcal{F} = \{P(X, Y, Z) = 0\}$ *is a surface equivalent to a plane*

PROOF. Let us remember that, given an automorphism $\Phi: \mathbb{A}^3 \to \mathbb{A}^3$, with

$$\Phi = \begin{pmatrix} P(X, Y, Z) \\ Q(X, Y, Z) \\ R(X, Y, Z) \end{pmatrix}, \qquad \Phi^{-1} = \begin{pmatrix} P'(X, Y, Z) \\ Q'(X, Y, Z) \\ R'(X, Y, Z) \end{pmatrix}$$

and given a surface $\mathcal{G} = \{G(X, Y, Z) = 0\}$, one has

$$\Phi(\mathcal{G}) = \{G(P'(X, Y, Z), Q'(X, Y, Z), R'(X, Y, Z)) = 0\}$$

and

$$\Phi^{-1}(\mathcal{G}) = \{G(P(X, Y, Z), Q(X, Y, Z), R(X, Y, Z)) = 0\}.$$

Now suppose $\mathcal{C}$ rectifiable. Then we can find an automorphism

$$\Phi = \begin{pmatrix} P(X, Y, Z) \\ Q(X, Y, Z) \\ R(X, Y, Z) \end{pmatrix}$$

such that

$$\Phi(\mathcal{C}) = \begin{pmatrix} P(x(t), y(t), z(t)) \\ Q(x(t), y(t), z(t)) \\ R(x(t), y(t), z(t)) \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}.$$

We have $\Phi^{-1}(\{X = 0\}) = \{P(X, Y, Z) = 0\}$, so that $\{P(X, Y, Z) = 0\}$ is a surface equivalent to a plane with $P(x(t), y(t), z(t)) = t$; beeing

$$P(x(t), y(t), z(t)) - F(x(t), y(t), z(t)) = 0 ,$$

this means $PX, Y, Z) - F(X, Y, Z) = H(X, Y, Z) \in \mathfrak{J}(\mathcal{C})$, so that $P(X, Y, Z) = F(X, Y, Z) + H(X, Y, Z)$ is the required polynomial.

Conversely if $\exists P(X, Y, Z) = F(X, Y, Z) + H(X, Y, Z)$, with $H(X, Y, Z) \in \mathfrak{J}(\mathcal{C})$, and $\{P(X, Y, Z) = 0\}$ is equivalent to a plane, this means that $\mathcal{C}$ meets $\{P(X, Y, Z) = 0\}$ only in the point $(x(0), y(0), z(0))$ and transverasally: by Prop. 2) $\mathcal{C}$ is then a rectifiable line.

§ 2. Let us come now to our quintic $\mathcal{C}_5$: $(t + t^5, t^4, t^3)$. We find easily

$$t = (t + t^5) - (t + t^5)t^4 + (t^3)^3 = x(t) - x(t)y(t) + z(t)^3 ,$$

so that we can say $F(X, Y, Z) = X - XY + Z^3$ to be a polynomial such that $F(x(t), y(t), z(t)) = t$. Then we look for an $H(X, Y, Z) \in \mathfrak{J}(\mathcal{C}_5)$ such that $\{F(X, Y, Z) + H(X, Y, Z) = 0\}$ is a surface equivalent to a plane. Take

$$H(X, Y, Z) = XY + Z^3 - X^2Z + Z^3Y ;$$

it belongs to $\mathfrak{J}(\mathcal{C}_5)$: indeed

$$(t + t^5)t^4 + (t^3)^3 - (t + t^5)^2t^3 + (t^3)^3t^4 =$$
$$= t^5 + t^9 + t^9 - t^5 - 2t^9 - t^{13} + t^{13} = 0 .$$

Now consider

$$F(X, Y, Z) + H(X, Y, Z) = X - ZX^2 + 2Z^3 + Z^3Y :$$

the surface $\mathcal{F}_4 = \{X - ZX^2 + 2Z^3 + Z^3Y = 0\}$ is actually isomorphic to a plane, having the following parametrisation with its inverse

$$\begin{cases} x = -u^3v \\ y = v + u^4v^2 - 2\,, \\ z = u \end{cases} \qquad \begin{cases} u = z \\ v = y + 2 - \left(x^2 - (y+2)z^2\right)^2 \end{cases}.$$

Beeing the equation of $\mathcal{F}_4$ linear in Y, by a well known result of A. Sathaye [4], $\mathcal{F}$ is equivalent to a plane, so by Prop. 3), we can say that $\mathcal{C}_5$ is a rectifiable line.

In the next page we give the equations of the automorphism Φ that straightens $\mathcal{C}_5$: $(t + t^5, t^4, t^3)$ to $\mathcal{C}$: $(t, 0, 0)$ toghether with those of its inverse Φ^{-1}.

§ 3. In this paragraph we should like to make some comments about the interesting questions on m-flats in $\mathbf{A}_k^n$ raised by Prof. Abhyankar in [1], Chapter 11) and some remarks on the above results.

REMARK 1). Prop. 3) could provide a useful tool for proving or disproving that a line $\mathcal{C}$ in $\mathbf{A}_k^3$ is rectifiable.

Of course a deeper investigation on the set $F(X, Y, Z) + \mathfrak{J}(\mathcal{C})$ is called for. Let us remember that $\mathfrak{J}(\mathcal{C})$ is a prime ideal with two generators, because $\mathcal{C}$ is complete intersection of two surfaces.

REMARK 2). Prop. 1), 2) and 3) hold also in $\mathbf{A}_k^n$ with the obvious slight changes in the proofs.

REMARK 3) We think that the notion of «rigidity» for a line given in [1], p. 413, should be reconsidered, in order not to render even trivially false Conjectures 2) and 3). Indeed every line $\mathcal{C}$ in $\mathbf{A}_k^n$ which lies in a hyperplane $\mathcal{H}$ is elementarily rectifiable. No restriction in supposing $\mathcal{H} = \{X_n = 0\}$, so that it is $\mathcal{C}$: $(P_1(t), \ldots, P_{n-1}(t), 0)$. Then we can find a polynomial $F(X_1, \ldots, X_{n-1})$ such that

$$F\left(P_1(t), \ldots, P_{n-1}(t)\right) = t$$

Consider then the two elementary automorphisms of $\mathbf{A}_k^n$

$$\Phi = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \\ X_n + F(X_1, \ldots, X_{n-1}) \end{pmatrix}, \qquad \Psi = \begin{pmatrix} X_1 - P_1(X_n) \\ \vdots \\ X_{n-1} - P_{n-1}(X_n) \\ X_n \end{pmatrix}$$

$$\Phi = \begin{pmatrix} Z^3Y + 2Z^3 + X - ZX^2 \\ -2 - 5X^4 + 6X^5Z - 2X^6Z^2 + [1 + 2XZ + 4X^2Z^2 - 12X^3Z^3 + 6X^4Z^4](Y+2) + [Z^4 + 6XZ^5 - 6X^2Z^6](Y+2)^2 + 2(Y+2)^3Z^6 \\ Z - (Z^3Y + 2Z^3 + X - ZX^2)^3 \end{pmatrix},$$

$$\Phi^{-1} = \begin{pmatrix} -(Z+X^3)^3(Y+2) + X + (Z+X^3)X^2 + 2X^3(Z+X^3)^2 \\ [1 - 2X(Z+X^3) - 2X^2(Z+X^3)^2 - 4X^3(Z+X^3)^3](Y+2) + (Z+X^3)^4(Y+2)^2 + 5X^4 + 4X^5(Z+X^3) + 4X^6(Z+X^3)^2 - 2 \\ Z + X^3 \end{pmatrix}.$$

we find

$$\Psi\circ\Phi(\mathcal{C}) = \Psi(\Phi(\mathcal{C})) = \Psi\begin{pmatrix} P_1(t) \\ \vdots \\ P_{n-1}(t) \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t \end{pmatrix}.$$

The problem is that, for $n \geqslant 4$, a line can be « rigid », but lie in a hyperplane $\mathcal{H}$ of the kind $\mathcal{H} = \{X_i = 0\}$: for example $\mathcal{C}$: $(t + t^6, t^5, t^4, 0)$.

REMARK 4). It could be useful to point out another necessary and sufficient condition for a line $\mathcal{C}$ in A_k^3 to be rectifiable: it is that $\mathcal{C}$ is contained in a surface $\mathcal{F}$ equivalent to a plane. Indeed, if $\mathcal{C} \subset \mathcal{F}$ and Φ is an automorphism such that $\Phi(\mathcal{F})$ is a plane α, then $\Phi(\mathcal{C})$ is a line of α and, as it is well known (see [1], Ch. 10), there is an automorphism ψ of α such that $\psi(\Phi(\mathcal{C}))$ is a line l. If Ψ is an automorphism of $\mathbf{A}_k^3$ such that $\Psi|_\alpha = \psi$, we have

$$\Psi\circ\Phi(\mathcal{C}) = \Psi(\Phi(\mathcal{C})) = \psi(\Phi(\mathcal{C})) = l$$

so that $\mathcal{C}$ is rectifiable. The converse is obvious.

REMARK 5). We think that one can answer in the positive to Question 3 in [1], p. 414, which asks if every line in $\mathbf{A}_k^n$ is equivalent to a rigid line. Indeed given a line $\mathcal{C}$: $(x_1(t), \dots, x_n(t))$ or $\deg x_i(t) > 0$, $\forall i$, or (see Remark 3)) $\mathcal{C}$ is already equivalent to a straight line (which can be taken rigid). In the first case, if $\mathcal{C}$ is not rigid, then by an elementary automorphism Φ one can transform $\mathcal{C}$ in a line $\Phi(\mathcal{C})$: $(x_1'(t), \dots, x_n'(t))$ which, if $\deg x_i'(t) > 0$, $\forall i$, has however the total degree $\sum_{i=1}^{n} \deg x_i'(t) < \sum_{i=1}^{n} \deg x_i(t)$; in the other case $\Phi(\mathcal{C})$ would be already equivalent to a rigid straight line. If $\Phi(\mathcal{C})$ is not rigid the process can continue stopping necessarily, by the continuous decreasing of the total degree.

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