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Note on the Regular Digraphs.

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ABSTRACT - In this paper, we give two sufficient conditions, in which, the conjecture of [1] is true.

Introduction.

Our graph-theoretic terminology is standard and is based essentially on the book of Harary [3]. For a vertex v of a digraph D , we denote by $id(v)$ and $od(v)$ the indegree and outdegree, respectively, of v . If $id(v) = od(v) = r$, then we speak of the *degree* of v . If every vertex of D has degree r , then D is said to be *regular of degree r* or simply *r -regular*. The *girth* of a digraph D containing directed cycles (circuits) is the length of the smallest cycle in D . For $n \geq 2$ and $r \geq 1$, $g(r, n)$ is the minimum number of vertices in an r -regular digraph having girth n . From [1], we have

$$(1) \quad g(r, n) \leq r(n-1) + 1,$$

and the following

CONJECTURE. For all $r \geq 1$, $n \geq 2$, $g(r, n) = r(n-1) + 1$.

In this paper, we give two sufficient conditions, in which, this conjecture is true.

The main results. Throughout, for any real number x , we use $[x]$ to denote the integer less than or equal to x , and all the digraphs

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considered are finite, without loops or multiple edges. If $D = (V, E)$ is a digraph with V as the vertex set and E as the edge set, then, for every $W \subseteq V$, we denote:

$$V^+(W) = \{v \in V : (w, v) \in E, w \in W\}$$

and

$$V^-(W) = \{v \in V : (v, w) \in E, w \in W\}.$$

THEOREM 1. *Let $D = (V, E)$ be an r -regular digraph of girth n with $r \geq 1$ and $n \geq 2$. If there exist a vertex $v \in V$ and the sets $V_i, W_i \subseteq V$, $i = 1, 2, \dots, \lfloor n/2 \rfloor$, such that*

- (a) $V_1 = V^-(\{v\}), W_1 = V^+(\{v\})$,
- (b) $|V_i| = |W_i| = r, i = 1, 2, \dots, \lfloor n/2 \rfloor$,
- (c) $V_i \subseteq [V^-(\bigcup_{j < i} V_j) - \bigcup_{j < i} V_j], i = 2, 3, \dots, \lfloor n/2 \rfloor$,
- (d) $W_i \subseteq [V^+(\bigcup_{j < i} W_j) - \bigcup_{j < i} W_j], i = 2, 3, \dots, \lfloor n/2 \rfloor$,

then $g(r, n) = r(n - 1) + 1$.

PROOF. Obviously, from (c) and (d), we have

$$(2) \quad V_i \cap V_j = \emptyset, \quad i, j = 1, 2, \dots, \lfloor n/2 \rfloor,$$

$$(3) \quad W_i \cap W_j = \emptyset, \quad i, j = 1, 2, \dots, \lfloor n/2 \rfloor.$$

Now, we prove, by induction on i , that for every $1 \leq i \leq \lfloor n/2 \rfloor$ and $x \in V_i$, there exists a directed path from x to v of length $\leq i$. Obviously, by (a), that is true for $i = 1$, and consider it to be true for $j < i - 1$. Since $V_i \subseteq V^-(\bigcup_{j < i} V_j)$, there exists $z \in \bigcup_{j < i} V_j$, such that $(x, z) \in E$. By induction hypothesis, the path of length $\leq i - 1$, from z to v , joined with (x, z) , is a path from x to v of length $\leq i$. Similarly, for every $1 \leq i \leq \lfloor n/2 \rfloor$ and $y \in W_i$, there exists a path from v to y of length $\leq i$.

Suppose that, for some $1 \leq i \leq \lfloor n/2 \rfloor$, $v \in V_i$. Thus, there exists a cycle (from v to v) of length $\leq i \leq \lfloor n/2 \rfloor < n$, contradicting the definition of n . Hence,

$$(4) \quad v \notin V_i, \quad i = 1, 2, \dots, \lfloor n/2 \rfloor.$$

Similarly, we have

$$(5) \quad v \notin W_i, \quad i = 1, 2, \dots, \lfloor n/2 \rfloor.$$

Suppose that, for some $1 \leq i \leq \lfloor n/2 \rfloor$ and $1 \leq j \leq \lfloor (n-1)/2 \rfloor$, $V_i \cap W_j \neq \emptyset$. Let then $t \in V_i \cap W_j$. Thus the union of the path from t to v (of length $\leq i$) with the path from v to t (of length $\leq j$) is a cycle of length $\leq i + j \leq \lfloor n/2 \rfloor + \lfloor (n-1)/2 \rfloor = n-1 < n$, contradicting the definition of n . Hence,

$$(6) \quad V_i \cap W_j = \emptyset, \quad i = 1, 2, \dots, \lfloor n/2 \rfloor, \quad j = 1, 2, \dots, \lfloor (n-1)/2 \rfloor.$$

Thus, from (b) and (2)-(6), we have $g(r, n) \geq r \lfloor n/2 \rfloor + r \lfloor (n-1)/2 \rfloor = r[\lfloor n/2 \rfloor + \lfloor (n-1)/2 \rfloor] = r(n-1)$, and, by (1), the theorem is proved (Q.E.D.).

THEOREM 2. *Let $D = (V, E)$ be an r -regular digraph of girth n with $r \geq 1$ and $n \geq 2$. If $r(\lfloor n/2 \rfloor - 1) < n$, then $g(r, n) = r(n-1) + 1$.*

PROOF. Let $v \in V$ arbitrary chosen, and $V_1 = V^-(\{v\})$, $W_1 = V^+(\{v\})$. Suppose that, for every $j < i \leq \lfloor n/2 \rfloor$, we have constructed the sets V_i, W_i satisfying (b)-(d).

Thus, because V_i 's are pairwise disjoint and of cardinality r , we have

$$\left| \bigcup_{j < i} V_j \right| = \sum_{j=1}^{i-1} |V_j| = r(i-1), \text{ i.e., } \left| \bigcup_{j < i} V_j \right| \leq r(\lfloor n/2 \rfloor - 1) < n.$$

Hence, the subdigraph of D , induced by $\bigcup_{j < i} V_j$ (the subdigraph of D whose vertex set is $\bigcup_{j < i} V_j$ and whose edge set is the set of those edges of D that have both ends in $\bigcup_{j < i} V_j$), does not contain cycles. But, according to [3], if a digraph has no cycles and which fails to consist only of isolated vertices, then it contains a transmitter (a vertex with positive outdegree and zero indegree) and a receiver (a vertex with positive indegree and zero outdegree). Hence, there exists $x \in \bigcup_{j < i} V_j$, such that $V^-(\{x\}) \cap \left(\bigcup_{j < i} V_j \right) = \emptyset$. Obviously, $V_i := V^-(\{x\})$ satisfies (b) and (c). Similarly, we can construct W_i . In this way, we come in the hypothesis of the theorem 1, and, therefore, we have $g(r, n) = r(n-1) + 1$ (Q.E.D.).

COROLLARY 1. [1]. *For the following values of r and n , the conjecture is true:*

$$r = 1 \text{ and } n \text{ arbitrary } (n \geq 2),$$

$$r \text{ arbitrary } (r \geq 1) \text{ and } n = 2, 3.$$

PROOF. It follows from the theorem 2 (Q.E.D.).

COROLLARY 2. [2]. *The conjecture is true for $r = 2$ and arbitrary $n(n \geq 2)$.*

PROOF. It follows from the theorem 2 (Q.E.D.).

COROLLARY 3. *The conjecture is true for the following values of r and n :*

$$r = 3, \quad n = 4, 5, 7.$$

$$r = 4, \quad n = 5.$$

PROOF. It follows from the theorem 2 (Q.E.D.).

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