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An Existence Theorem for Bounded Solutions of Differential Equations in Banach Spaces.

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SUMMARY - In this note we shall give sufficient conditions for the existence of bounded solutions of the differential equation $y' = f(t, y)$, $y(0) = x_0$ on the half-line $t \geqslant 0$. Here f is a function with values in a Banach space satisfying some regularity Ambrosetti type condition expressed in terms of the « measure of noncompactness α ».

Let $J = [0, \infty)$, and let $(E, \|\cdot\|)$ be a Banach space. Assume that $f: J \times E \to E$ is a function which satisfies the following conditions: (1) for each fixed $x \in E$ the mapping $t \mapsto f(t, x)$ is measurable; (2) for each fixed $t \in J$ the mapping $x \mapsto f(t, x)$ is continuous; and (3) $\|f(t, x)\| \leqslant G(t, \|x\|)$ for $(t, x) \in J \times E$, where the function G is monotonically nondecreasing in the second variable such that $t \mapsto G(t, u)$ is locally bounded for any fixed $u \in J$ and $t \mapsto G(t, y(t))$ is measurable for each continuous bounded function y of J into itself.

Let $x_0 \in E$. By (PC) we shall denote the problem of finding a solution of the differential equation

$$y' = f(t, y)$$

satisfying the initial condition $y(0) = x_0$.

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We deal with the problem (PC) using a method developed by Ambrosetti [1]. This method is based on the properties of the measure of noncompactness α. The proof of our theorem is sugested by a paper of Stokes [7] concerning finite-dimensional vector differential equations.

The measure of noncompactness $\alpha(X)$ of a nonempty bounded subset X of E is defined as the infimum of all $\varepsilon > 0$ such that there esists a finite convering of X by sets of diameter $\leqslant \varepsilon$. For properties of α the reader is referred to [2], [3], [5].

Denote by $C(J)$ the set of all continuous functions from J to E. The set $C(J)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of J.

Let us put

$$X(t) = \{x(t) : x \in X\}, \qquad X_t = \bigcup \{X(s) : 0 \leqslant s \leqslant t\},$$

and

$$\int_0^t f(s, X(s))\, ds = \left\{ \int_0^t f(s, x(s))\, ds : x \in X \right\}$$

for $t \in J$ and $X \subset C(J)$. Moreover, we use the standard notations. The closure of a subset A of a topological vector space, its convex hull and its closed convex hull be denoted, respectively, by $\overline{A}$, $\operatorname{conv} A$ and $\overline{\operatorname{conv}}\, A$. For a mapping F defined on A we denote by $F[A]$ the image of A under F.

The Ascoli theorem we state as follows: $X \subset C(J)$ is conditionally compact if and only if X is almost equicontinuous and $\overline{X(t)}$ is compact for every $t \in J$. We shall use also the following result due to Ambrosetti [1]: If I is a compact subset of J and Y is a bounded equicontinuous subset of the usual Banach space of continuous E-valued functions on I, then

$$\alpha\left(\bigcup \{Y(t) : t \in I\}\right) = \sup \left\{\alpha\left(Y(t)\right) : t \in I\right\}.$$

Our result be proved by the following fixed point theorem of Furi and Vignoli type (see e.g. [6], Theorem 2):

Let $\mathfrak{X}$ be a nonempty closed convex subset of $C(J)$. Let $\Phi : 2^{\mathfrak{X}} \to [0, \infty)$ be a function such that $\Phi(X \cup \{x\}) = \Phi(X)$, $\Phi(\overline{\operatorname{conv}}\, X) = \Phi(X)$, $\Phi(X_1) \leqslant \Phi(X_2)$ whenever $X_1 \subset X_2$, and if $\Phi(X) = 0$ then $\overline{X}$ is compact, for every $x \in \mathfrak{X}$ and every subsets X, X_1, X_2 of $\mathfrak{X}$. Suppose that T is a continuous mapping of $\mathfrak{X}$ into itself and $\Phi(T[X]) < \Phi(X)$ for arbitrary subset X of $\mathfrak{X}$ with $\Phi(X) > 0$. Under the hypotheses, T has a fixed point in $\mathfrak{X}$.

THEOREM. Let h, L be functions of J into itself such that h is nondecreasing with $h(0) = 0$ and $h(t) < t$ for $t > 0$, and L is measurable and integrable on compact subsets of J with $\sup\left\{\int_0^t L(s)\,ds : t \in J\right\} \leqslant 1$. Suppose that the scalar inequality

$$g(t) \geqslant \|x_0\| + \int_0^t G(t, g(s))\,ds$$

has a bounded solution g existing on J; denote by Z_0 the set of all $x \in E$ with $\|x\| \leqslant r_0 = \sup\{g(t) : t \in J\}$. Assume, moreover, that for any $t > 0$, $\varepsilon > 0$ and $X \subset Z_0$ there exists a closed subset Q of $[0, t]$ such that mes $([0, t] \setminus Q) < \varepsilon$ and

$$\alpha(f[I \times X]) \leqslant \sup\{L(s) : s \in I\} \cdot h(\alpha(X))$$

for each closed subset I of Q.

Then (PC) has at least one solution y defined on J and $\|y(t)\| \leqslant g(t)$ for $t \in J$.

PROOF. Denote by $\mathfrak{X}$ the set of all $x \in C(J)$ such that $\|x(t)\| \leqslant g(t)$ on J and

$$\|x(\sigma) - x(\tau)\| \leqslant \left|\int_\sigma^\tau G(s, r_0)\,ds\right|$$

for σ, τ in J. The set $\mathfrak{X}$ is a closed convex bounded and almost equi continuous subset of $C(J)$.

Let us put

$\Phi(X) = \sup\{\alpha(X(t)) : t \in J\}$ for a subset X of $\mathfrak{X}$. Obviously $\Phi(X) < \infty$, $\Phi(X_1) \leqslant \Phi(X_2)$ for $X_1 \subset X_2$, and $\Phi(X \cup \{x\}) = \Phi(X)$ for $x \in \mathfrak{X}$. Since

$$(\overline{\text{conv}}\, X)(t) = (\overline{\text{conv}\, X})(t) \subset \overline{(\text{conv}\, X)(t)} \subset \overline{\text{conv}\,(X(t))}\ ,$$

so

$$\alpha\big((\overline{\text{conv}}\, X)(t)\big) \leqslant \alpha\big(\overline{\text{conv}(X(t))}\big) = \alpha(X(t))$$

for $t \in J$. Hence

$$\sup\left\{\alpha\left((\overline{\operatorname{conv}} X)(t)\right) : t \in J\right\} \leqslant \sup\left\{\alpha(X(t)) : t \in J\right\} ,$$

and consequently, $\Phi(\overline{\operatorname{conv}} X) = \Phi(X)$. If $\Phi(X) = 0$ then $\overline{X(t)}$ is compact for every $t \in J$; therefore Ascoli's theorem proves that $\overline{X}$ is compact in $C(J)$.

To apply our fixed point theorem we define the continuous mapping T as follows: for $y \in C(J)$,

$$(Ty)(t) = x_0 + \int_0^t f\big(s, y(s)\big)\, ds .$$

Modyfying the reasoning from [7] we infer that $T[\mathfrak{X}] \subset \mathfrak{X}$.

Let X be a subset of $\mathfrak{X}$ such that $\Phi(X) > 0$. To prove the theorem it remains to be show that $\Phi\big(T[X]\big) \leqslant h\big(\Phi(X)\big)$.

To this end, fix t in J. Let $\varepsilon > 0$, and let $\delta = \delta(\varepsilon) > 0$ be a number such that

$$\int_A G(s, r_0)\, ds < \varepsilon/2$$

for each measurable $A \subset [0, t]$ with mes $(A) < \delta$. By the Luzin theorem there exists a closed subset B_1 of $[0, t]$ with mes $\big([0, t] \setminus B_1\big) < \delta/2$ and the function L is continuous on B_1. Furthermore, by our comparison condition, there exists a closed subset B_2 of $[0, t]$ such that mes $\big([0, t] \setminus B_2\big) < \delta/2$ and

$$\alpha\big(f[I \times X_t]\big) \leqslant \sup\{L(s) : s \in I\} \cdot h\big(\alpha(X_t)\big)$$

for each closed subset I of B_2.

Define:

$$A = A_1 \cup A_2, \qquad B = [0, t] \setminus A ,$$

where $A_i = [0, t] \setminus B_i$ for $i = 1, 2$. Since L is uniformly continuous on B, for any given $\varepsilon' > 0$ there exists $\eta > 0$ such that $t', t'' \in B$ and $|t' - t''| < \eta$ implies $\alpha(X_t)|L(t') - L(t'')| < \varepsilon'$. For a positive integer $m > t/\eta$, let $t_0 = 0 < t_1 < \ldots < t_m = t$ be the partion of the interval $[0, t]$ with $t_j = m^{-1}t + t_{j-1}$ $(j = 1, 2, \ldots, m)$. Moreover, let $I_j = [t_{j-1}, t_j] \setminus A$ and let s_j be a point in I_j such that $L(s_j) = \sup\{L(s) : s \in I_j\}$.

By the integral mean-value theorem, for $x \in X$ we have

$$\int_B f(s, x(s))\, ds = \sum_{j=1}^{m} \int_{I_j} f(s, x(s))\, ds \in \sum_{j=1}^{m} \operatorname{mes}(I_j)\, \overline{\operatorname{conv}}\left(\{f(s, x(s)) : s \in I_j\}\right) \subset$$

$$\subset \sum_{j=1}^{m} \operatorname{mes}(I_j)\, \overline{\operatorname{conv}}\, (f[I_j \times X_t]) .$$

Thus

$$\alpha(T[X](t)) \leqslant \alpha\left(\int_A f(s, X(s))\, ds\right) + \alpha\left(\sum_{j=1}^{m} \operatorname{mes}(I_j)\, \overline{\operatorname{conv}}\, (f[I_j \times X_t])\right) \leqslant$$

$$\leqslant 2 \cdot \sup\left\{\left\| \int_A f(s, x(s))\, ds \right\| : x \in X\right\} + \sum_{j=1}^{m} \operatorname{mes}(I_j)\, \alpha(f[I_j \times X_t]) \leqslant$$

$$\leqslant 2 \cdot \int_A G(s, r_0)\, ds + \sum_{j=1}^{m} \operatorname{mes}(I_j) L(s_j) h(\alpha(X_t)) < \varepsilon + h(\alpha(X_t)) \int_B L(s)\, ds +$$

$$+ \sum_{j=1}^{m} \int_{I_j} h(\alpha(X_t)) |L(s_j) - L(s)| ds < \varepsilon + h(\alpha(X_t)) + \varepsilon' t ,$$

and therefore $\alpha(T[X](t)) \leqslant \varepsilon + h(\alpha(X_t))$. Since

$$\alpha(X_t) = \sup\{\alpha(X(s)) : 0 \leqslant s \leqslant t\} \leqslant \Phi(X) ,$$

we obtain $\alpha(T[X](t)) \leqslant \varepsilon + h(\Phi(X))$; as ε is arbitrary, this implies $\alpha(T[X](t)) \leqslant h(\Phi(X))$. Hence $\Phi(T[X]) \leqslant h(\Phi(X))$, and consequently T has a fixed point in $\mathfrak{X}$. The proof is complete.

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