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# Minimal immersions of surfaces into 4-dimensional space forms

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# Minimal Immersions of Surfaces into 4-Dimensional Space Forms (*).

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SUMMARY - We find necessary and sufficient conditions for a real function defined on a surface $M$ to be the *normal curvature* function of a minimal immersion of $M$ into the 4-dimensional space $Q_c^4$ of constant curvature $c$. Moreover, we use these conditions to draw some geometrics conclusions. Also, we study deformations of minimal surfaces in $Q_c^4$ preserving the normal curvature function.

## 1. Introduction.

In the present paper we find necessary and sufficient conditions for a real function defined on a surface $M$ to be the *normal curvature* function of a minimal immersion of $M$ into the 4-dimensional space $Q_c^4$ of constant curvature $c$. Moreover, we use these conditions to draw some geometric conclusions. Also, we study deformations of minimal surfaces in $Q_c^4$ preserving the normal curvature function.

In the following it is convenient to use the notion of the *ellipse of curvature* studied by Little [13], Moore and Wilson [14] and Wong [18]. This is the subset of the normal space defined as $\{B(X, X): X \in T_pM, \|X\| = 1\}$, where $B$ is the second fundamental form of the

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immersion and $\| \ \|$ is the norm of the vectors of $T_pM$. Let $K$ and $K_N$ be the Gaussian and the normal curvature of $M$, and let $\Delta$ denote the Laplace-Beltrami operator of $M$.

THEOREM 1. Let $x$: $M \to Q_c^4$ be a minimal immersion of an oriented surface $M$ into an orientable 4-dimensional space form $Q_c^4$ of constant curvature $c$. Then, at points where the ellipse of curvature is not a circle, i.e. $(K-c)^2 - K_N^2 > 0$, we have

$$\Delta(\log|K_N - K + c|) = 2(2K - K_N) \tag{1.1}$$

and consequently

$$\Delta(\log|K_N + K - c|) = 2(2K + K_N) . \tag{1.2}$$

Conversely, if $K_N$ is a real function defined over a simply-connected surface satisfying (1.1), (1.2) and $(K-c)^2 - K_N^2 > 0$, then there exists a minimal isometric immersion into $Q_c^4$ with normal curvature $K_N$.

REMARKS 1. In the case $K_N = 0$ the conditions (1.1) and (1.2) are equivalent to the Ricci conditions, i.e. $K < 0$ and that the metric $d\tilde{s}^2 = \sqrt{c-K}\, ds^2$ be flat (see Chern and Osserman [9] and Lawson [12]).

**2.** R. Schoen has pointed out, that it follows from (1.1) and (1.2) that if $-2\chi_M < |\chi_N|$ then the ellipse of curvature is always a circle, where $\chi_M$ and $\chi_N$ are the Euler characteristics of the tangent bundle and the normal bundle, respectively.

COROLLARY 1. Let $x$: $M \to S^4(1)$ be a minimal immersion of a compact oriented surface $M$ of genus $g \leqslant 1$ into the unit sphere $S^4(1)$. If $2K \leqslant K_N$ or $2K \leqslant -K_N$, then either $x(M)$ is the Clifford torus in $S^3$ or $x(M)$ is the Veronese surface.

In the following we say that an immersion is *full* in $S^4$ if it does not lie in any totally geodesic submanifold. We say that a point $p \in M$ is *geodesic* if the second fundamental form vanishes at this point In [3], Bryant proved that there exist many minimal surfaces of all genera full in $S^4$. The following corollary shows that this is not so if we make some restrictions on the Gaussian and the normal curvatures.

COROLLARY 2. Let $x$: $M \to S^4(1)$ be a minimal immersion of a compact oriented suface $M$ of genus $g > 0$ into the unit sphere $S^4$

Suppose that $M$ has no geodesic points. Then

i) If $K_N$ does not change sign, $x(M)$ lies in $S^3$

ii) If $2K \geqslant K_N$ or $2K \geqslant -K_N$, $x(M)$ is the Clifford torus in $S^3$.

REMARK 3. The condition $g > 0$ in the corollary 2 is necessary, since using Chern [7] it is possible to construct many examples of minimal surfaces in $S^4$ satisfying $2K \geqslant K_N$. The Veronese surface is one of these examples, and if $K_N \geqslant 0$ it is the only example not totally geodesic. An analogous condition characterizes the generalized Veronese surfaces in higher codimensions (see do Carmo and Wallach [5] and Rodriguez and Guadalupe [16]). Moreover, it was proved by Chern [8] that the normal curvature $K_N$ of a minimal sphere in $S^4$ does not change sign.

The following results show that the existence of deformations of minimal surfaces preserving the normal curvature depends only on the property that the ellipse of curvature is not a circle everywhere.

THEOREM 2. Let $x$: $M \to Q_c^4$ be a minimal immersion of a simply-connected surface $M$ into an orientable 4-dimensional space $Q_c^4$ of constant curvature $c$. If the ellipse of curvature is not a circle everywhere then there exists a continuous deformation of $x$ by minimal isometric immersion $x_t$: $M \to Q_c^4$ such that $(K_N)_t = K_N$ for each $t \in [-\pi, \pi]$. Moreover, if $\tilde{x}$: $M \to Q_c^4$ is another minimal immersion with $\tilde{K}_N = K_N$, then there exists $\theta \in [-\pi/2, \pi/2]$ such that $\tilde{x}$ and $x_\theta$ coincide up to a rigid motion.

Theorems 1 and 2 are related th theorem 8 of Lawson [11].

In [4], Calabi proved that the Gaussian curvature of a minimal sphere $S^2$ in $Q_c^4$ satisfies

$$\Delta \log(c - K) = 2(3K - c) \tag{1.3}$$

The following theorem shows that this is a sufficient condition to have a local minimal immersion such that the ellipse of curvature is always a circle.

THEOREM 3. Let $ds^2$ be a riemannian metric defined over a simple-connected surface M. Suppose that the Gaussian curvature $K$ of this metric satisfies (1.3) and $K \neq c$. Then, there exists a minimal isometric immersion $x$: $M \to Q_c^4$ such that the ellipse of curvature is a circle everywhere. Moreover, if $\tilde{x}$: $M \to Q_c^4$ is another minimal isometric immersion such that the ellipse of curvature is a circle, then $x$ and $\tilde{x}$ coincide up to a rigid motion.

COROLLARY 3. Let $ds^2$ be a riemmanian metric defined over a surface $M$ of genus $g = 0$, such that the Gaussian curvature $K$ satisfies (1.3) and $K \neq 1$. Then, there exists a minimal isometric immersion of $M$ into $S^4$ unique up to rigid motions.

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## 2. Preliminaries.

Let $M$ be a surface immersed in a Riemannian manifold $Q^n$. For each $p$ in $M$, we use $T_pM$, $TM$, $N_pM$ and $NM$ to denote the tangent space of $M$ at $p$, the tangent bundle of $M$, the normal space of $M$ at $p$ and the normal bundle of $M$, respectively. Let $\nabla$ and $\bar{\nabla}$ be the covariant differentiations of $M$ and $Q^n$, respectively. Let $X$ and $Y$ be on $TM$, then the second fundamental form $B$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)\,. \tag{2.1}$$

It is well-known that $B(X, Y)$ is a simmetric bilinear form. For $\xi$ in $NM$, we write

$$\tilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi \tag{2.2}$$

where $-A_\xi(X)$ and $\nabla^\perp_X \xi$ denote the tangential and normal components of $\bar{\nabla}_X \xi$, respectively. Then we have

$$\langle A_\xi(X),\ Y\rangle = \langle B(X, Y),\ \xi\rangle \tag{2.3}$$

where $\langle\ ,\ \rangle$ denotes the scalar product in $TM$ and $NM$.

The mean curvature vector $H$ is defined by

$$H = \tfrac{1}{2}\,\mathrm{trace}\,B. \tag{2.4}$$

The immersion is said to be a *minimal immersion* if $H = 0$.

Let $R$ be the Riemannian curvature tensor associated with $\nabla$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2.5}$$

where $X$, $Y$, $Z$ are on $TM$. We note that $\langle R(X, Y)Y, X\rangle = K|X\wedge Y|^2$ where $|X\wedge Y|^2 = \langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2$. We define $R^{\perp}$, the curvature of $NM$ relative to $\nabla^{\perp}$ by the equation

$$(2.6) \qquad R^{\perp}(X, Y)\xi = \nabla^{\perp}_X\nabla^{\perp}_Y\xi - \nabla^{\perp}_Y\nabla^{\perp}_X\xi - \nabla^{\perp}{}_{[X,Y]}\xi$$

where $X$, $Y$ are on $TM$ and $\xi$ is on $NM$. With this notation, we can write the equations of Gauss and Ricci as following

$$(2.7) \qquad (K - c)\,|X\wedge Y|^2 = \langle B(X, X), B(Y, Y)\rangle - |B(X, Y)|^2$$

$$(2.8) \qquad R^{\perp}(X, Y)\xi = B(X, A_{\xi}Y) - B(A_{\xi}X, Y)\,.$$

For the second fundamental form $B$, we define the covariant derivate, denoted by $\overline{\nabla}_X B$, to be

$$(2.9) \qquad (\overline{\nabla}_X R)\,(Y, Z) = \nabla^{\perp}_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)\,.$$

Now, if $Q^n_c$ has constant curvature sectional $c$, the equation of Mainardi-Codazzi is given by

$$(2.10) \qquad (\overline{\nabla}_X B)\,(Y, Z) = (\overline{\nabla}_Y B)\,(X, Z)\,.$$

Suppose now that $Q^4_c$ has a given orientation. Then we can define the *N*ormal *C*urvature $K_N$ of $M$ by

$$(2.11) \qquad K_N = \langle R^{\perp}(X, H)\,e_4, e_3\rangle$$

where $\{X, Y\}$ and $\{e_3, e_4\}$ are orthonormal oriented bases of $T_pM$ and $N_pM$, respectively. Therefore $K_N > 0$ or $K_N < 0$ accord to the orientation of the normal bundle $NM$.

An interesting notion in the study of surfaces immersed with codimension two is that of the *ellipse of curvature* defined as $\{B(X, X) \in N_pM : \langle X, X\rangle = 1\}$. To see that it is an ellipse, we just have to look at the following formula, for $X = \cos\,\theta\;e_1 + \sin\,\theta\;e_2$

$$(2.12) \qquad B(X, X) = H + \cos 2\theta u + \sin 2\theta v$$

where $\mu = (B(e_1, e_1) - B(e_2, e_2))/2$, $v = B(e_1, e_2)$ and $\{e_1, e_2\}$ is a tangent frame. So we see that, as $X$ goes once around the unit tangent circle, $B(X, X)$ goes twice around the ellipse. Of course this ellipse could degenerate into a line segment or a point. Everywhere the ellipse is not a circle we can choose $\{e_1, e_2\}$ orthonormal such that $u$ and $v$ are perpendicular. When this happens they will coincide with the semi-axes of the ellipse.

## 3. Proof of Theorems.

PROOF OF THEOREM 1. By Itoh [10] there exist isothermal parameters $\{x_1, x_2\}$ such that putting $X_i = \partial/\partial x_i$, $i = 1, 2$ then $u = B(X_1, X_1) = -B(X_2, X_2)$ and $v = B(X_1, X_2)$ are on the semi-axes of the ellipse at every point where $(K - c)^2 - K_N^2 \neq 0$. Moreover we have $|X_i|^2 = E = ((K - c)^2 - K_N^2)^{-1/4}$, $i = 1, 2$. If we denote $\lambda = \langle u, u\rangle^{1/2}$ and $\mu = \langle v, v\rangle^{1/2}$ we have

$$\lambda^2 - \mu^2 = 1 \tag{3.1}$$

$$\lambda^2 + \mu^2 = -(K - c)E^2 \tag{3.2}$$

$$2\lambda\mu = |K_N|E^2 \tag{3.3}$$

where $\lambda > \mu \geqslant 0$ and $E = ((K - c)z - K_N^2)^{-1/4}$. We get (3.1) using ([10], p. 456). The equation (3.2) follows from (2.7). The equation (3.3) follows from (3.1) and (3.2).

Now we suppose $K_N > 0$. If $(K - c)^2 - K_N^2 > 0$ from (3.2) and (3.3) we obtain

$$\begin{aligned} \lambda + \mu &= (K_N - (K - c))^{1/2}((K - c)^2 - K_N^2)^{-1/4} \\ &= (K_N - K + c/K_N + K - c)^{1/4} \end{aligned} \tag{3.4}$$

Let $e_3 = \lambda^{-1}u$ and $e_4 = \mu^{-1}v$ be an oriented frame in $NM$. Then we have

$$\begin{cases} 2\langle\nabla^{\perp}_{X_1}v,\ v\rangle = X_1(\mu^2) \\ 2\langle\nabla^{\perp}_{X_2}u, u\rangle = X_2(\lambda^2)\,. \end{cases} \tag{3.5}$$

From Chen ([6], p. 103) we have $\nabla^{\perp}_{X_1}v = \nabla^{\perp}_{X_2}u$. Therefore using (3.5) we get

$$(3.6)\qquad \begin{aligned}\nabla^{\perp}_{X_1} v &= \lambda^{-2}\langle \nabla^{\perp}_{X_1} v, u\rangle\, u + \mu^{-2}\langle \nabla^{\perp}_{X_1} v, v\rangle v\\ &= 2^{-1}\lambda^{-2} X_2(\lambda^2)\, u + 2^{-1}\mu^{-2} X_1(\mu^2)\, v\,.\end{aligned}$$

Hence, from (3.1) and (3.6) we obtain

$$(3.7)\qquad \begin{aligned}\nabla^{\perp}_{X_1} e_4 &= X_1(1/\mu) v + (1/\mu)\nabla^{\perp}_{X_1} v\\ &= \big(X_1(1/\mu) + (1/2\mu^3) X_1(\mu^2)\big) v + \big(X_2(\lambda^2)/2\lambda^2\mu\big)\, u\\ &= \big(X_2(\mu^2)/2\lambda^2\mu\big)\, u = \big(X_2(\mu)/\lambda\big)\, e_3\,.\end{aligned}$$

Therefore we get

$$(3.8)\qquad \nabla^{\perp}_{X_1} e_4 = X_2(f)\, e_3 \quad \text{where } f = \log|\mu + \lambda|\,.$$

Similarly, we obtain

$$(3.9)\qquad \nabla^{\perp}_{X_2} e_4 = -X_1(f)\, e_3\,.$$

Hence, from (2.6), (2.11), (3.8) and (3.9) follows

$$(3.10)\qquad \begin{aligned}K_N &= \langle R^{\perp}(X_1, X_2)\, e_4, e_3\rangle E^{-1}\\ &= \big(-X_1 X_1(f) - X_2 X_2(f)\big) E^{-1}\\ &= -\tilde{\Delta}(f) E^{-1}\end{aligned}$$

where $\tilde{\Delta}$ denotes the Laplacian of the «flat» metric. We know $\tilde{\Delta}(f) = E\Delta(f)$, where $\Delta$ is the Laplacian of the surface. Hence, from (3.4) and (3.10) we get

$$(3.11)\qquad \Delta\big(\log|K_N - K + c/K_N + K - c|\big) = -4K_N\,.$$

Using $E = ((K-c)^2 - K_N^2)^{-1/4}$. and the Gaussian curvature $K$ given by the equation

$$(3.12)\qquad K = -\tfrac{1}{2} E^{-1}\tilde{\Delta} \log E$$

we obtain

$$(3.13)\qquad \Delta\big(\log|K_N - K + c|\big) + \Delta\big(\log|K_N + K - c|\big) = 8K\,.$$

From (3.11) and (3.13) we get the equations (1.1) and (1.2). Now if $\emptyset : Q_c^4 \to Q_c^4$ is an isometry that reverses the orientation then it reverses the sign of $K_N$. Therefore the case $K_N < 0$ reduces to the first one. At the points where $K_N = 0$ we get the equations by continuity.

Now we prove the converse. Since $M$ is simply-connected it is sufficient to work in small neighborhoods. Let $ds^2$ be the riemannian metric defined over $M$. From (1.1) and (3.12) it follows that the metric $d\tilde{s}^2 = \left((K-c)^2 - K_N^2\right)^{1/4} ds^2$ is flat. Hence, there exist coordinate sistems $(x_1, x_2)$ such that

$$ds^2 = E(dx_1^2 + dx_2^2) \qquad \text{where } E = \left((K-c)^2 - K_N^2\right)^{-1/4}.$$

We define real functions $\lambda, \mu$ with $\lambda > \mu \geqslant 0$ satisfying (3.1) and (3.2). Therefore from (3.1) and (3.2) it follows (3.3).

Suppose now that $K_N \geqslant 0$. Let $L(M)$ be a 2-plane bundle over $M$ equipped with a metric $\langle\, , \rangle$, where every fiber is generated by $\xi_3$, $\xi_4$ and the metric is defined by

$$\langle \xi_\alpha, \xi_\beta \rangle = \delta_{\alpha\beta}, \qquad \alpha, \beta = 3, 4. \tag{3.14}$$

We define a compatible connection $\nabla^\perp$ in $L(M)$ by

$$\begin{cases} \nabla^\perp_{X_1} \xi_4 = X_2(f)\xi_3 \\ \nabla^\perp_{X_2} \xi_4 = -X_1(f)\xi_3 \end{cases} \tag{3.15}$$

where $X_i = \partial/\partial x_i$, $i = 1, 2$ and $f = \log|\lambda + \mu|$.
Now, we define the second fundamental form $B_p$: $T_pM \times T_pM \to L_pM$ by

$$B_p(X_1, X_1) = \lambda\xi_3 = -B_p(X_2, X_2), \; B_p(X_1, X_2) = \mu\xi_4 \tag{3.16}$$

and let $A_\xi$: $T_pM \times L_pM \to T_pM$ be defined by

$$\langle A_\xi(X), Y \rangle = \langle B(X, Y), \xi \rangle \tag{3.17}$$

where $X$, $Y$ are on $T_pM$ and $\xi$ is on $L_pM$. Then by a straightforward calculation we can see that the Gauss and Mainardi-Codazzi equations (2.7) and (2.10) are satisfied. The Ricci equations (2.8) we get by reversing the proof of the equation (3.11). If $K_N < 0$ we change the sign in (3.15) and we define $B_p(X_1, X_2) = -\mu\xi_4$. The calculations follow similarly to the case $K_N \geqslant 0$. It is not hard to see that the connection and the form $B$ defined in this way are smooth. Hence, by Spivak ([17], p. 80) there exists a local isometric immersion $x\colon M \to Q_c^4$, in such a way that we may identify the normal bundle of the immersion with the bundle $LM$. Then the metric induced on the normal bundle coincides with the given bundle metric on $LM$, and the second fundamental form and the connection of the immersion coincide with $B$ and $\nabla^\perp$ respectively. Moreover the immersion is minimal with normal curvature $K_N$.

PROOF OF COROLLARY 1. First observe that we can choose the orientation of $S^4$ in such way that $2K \leqslant K_N$. It is well known that the differential form of degree 4, $\emptyset = (\|u\|^2 - \|v\|^2 - 2i\langle u, v\rangle)dz^4$ is holomorphic (see [7] or [16]). Hence it follows from the Riemann-Roch theorem that if $g = 0$, $\emptyset \equiv 0$ and if $g = 1$, $\emptyset \equiv 0$ or $\emptyset$ is never zero. Suppose $\emptyset \neq 0$. From (1.1) it follows that $\Delta(\log|K_N - K + 1|) \leqslant 0$. Since $\emptyset$ is never zero, $\log|K_N - K + 1|$ is subharmonic and bounded from above, so $|K_N - K + 1|$ has to be constant and therefore $2K = K_N$. This implies that $K$ and $K_N$ are constant. If $K_N \neq 0$ then by Asperti [2] $x(M)$ is homeomorphic to the sphere $S^2$. This is a contradiction. Therefore $K_N = 0$ everywhere and this implies that $x(M)$ is in $S^3$ (see [15]). Then by Lawson [11] $x(M)$ is the Clifford torus $S^1(1/\sqrt{2}) x S^1(1/\sqrt{2})$.

Suppose now $\emptyset \equiv 0$. In this case the ellipse of curvature is a circle everywhere and by [8] $K_N$ does not change of sign. So $K_N \geqslant 0$. Then by [16] $\Delta \log r = 2K - K_N$, where $r$ is the radius of the circle. We observe that if $r = 0$ at some point then $K_N = 0$ and $K = 1$ at this point. This contradicts the hypothesis $2K \leqslant K_N$. Thus $r$ is never zero. Hence $0 = \int_M \Delta \log r \, dM = \int_M (2K - K_N)\, dM$. Therefore $2K = K_N$ and this implies that $r$ is constant. Hence $K_N = 2r^2$ is constant. Finally by [10] $x(M)$ is the Veronese surface.

PROOF OF COROLLARY 2. Consider the holomorphic form $\emptyset$ as above. Observe that if $\emptyset \equiv 0$ in an immersed surface without geodesic points in $S^4$, then $K_N \neq 0$ everywhere. Hence $M$ has genus $g = 0$. So $\emptyset \not\equiv 0$. Since $\emptyset$ is holomorphic the possible zeros of $\emptyset$ are isolated points. If $K_N$ does not change of sign, we can suppose $K_N \geqslant 0$. It follows from theorem 1 or (3.11) that $\Delta(\log(|K_N - K + 1|/|K_N + K - 1|)) = -4K_N \leqslant 0$. So $\log(|K_N - K + 1|/|K_N + K - 1|)$ is superharmonic and bounded from below, since the surface has no geodesic points. Therefore the function $|K_N - K + 1|/|K_N + K - 1|$ is constant and $K_N \equiv 0$. This implies that $x(M)$ lies in $S^3$.

We now prove (ii). As in the proof of the first corollary, we can assume $2K \geqslant K_N$. Away from the points of $M$ where $\phi \equiv 0$ it follows from (1.1) that $\Delta(\log|K_N - K + 1|) \geqslant 0$. So $\log|K_N - K + 1|$ is subharmonic and bounded from above. Since $|K_N - K + 1| = 0$ at the possible isolated points causes no difficulties ([1], p. 135), we conclude that $|K_N - K + 1|$ is constant and therefore $2K = K_N$. So $K_N$ and $K$ are constant. Therefore if $g > 0$ we have $K_N = K = 0$. This implies that $x(M)$ is the Clifford torus in $S^3$.

**Proof of Theorem 2.** As in the proof of theorem 1, let $\{x_1, x_2\}$ be isothermal parameters such that $u$ and $v$ are on the semi-axes of the ellipse of curvature on $N_pM$. For each real function $\theta \in [-\pi, \pi]$ we define a form $B_\theta$ by

$$(3.18) \qquad \begin{cases} B_\theta(X_1, X_1) = \phantom{-}\cos 2\theta u + \sin 2\theta v = -B_\theta(X_2, X_2) \\ B_\theta(X_1, X_2) = -\sin 2\theta u + \cos 2\theta v = \phantom{-}B_6(X_2, X_1)\,. \end{cases}$$

We define $A_\alpha$ satisfying (2.3) by the equation

$$(3.19) \qquad A_\alpha(\theta) = R_\theta A_\alpha R_{-\theta}, \qquad \alpha = 3, 4$$

where $R_\theta$ is the rotation of angle $\theta$ in the tangent plane in the positive sense of the given orientation and $A_\alpha$ is the linear transformation corresponding to the form $B$.

It is easy to see that $B_\theta$ satisfies the Gauss equation. To verify the Ricci equation we observe that from (2.3), (2.8) and (2.11) we can prove $K_N = \langle [A_4, A_3](X), Y\rangle$. Hence, from (3.19) follows

$$\begin{aligned} (K_N)_\theta E &= \langle [A_4(\theta), A_3(\theta)](X_1), X_2\rangle \\ &= \langle (R_\theta A_4 R_{-\theta} R_\theta A_3 R_{-\theta} - R_\theta A_3 R_{-\theta} R_\theta A_4 R_{-\theta})(X_1), X_2\rangle \\ &= \langle R_\theta(A_4 A_3 - A_3 A_4) R_{-\theta}(X_1), X_2\rangle \\ &= \langle [A_4, A_3] R_{-\theta}(X_1), R_{-\theta}(X_2)\rangle = K_N E\,. \end{aligned}$$

Let's prove the Mainardi-Codazzi equation. By straightforward calculation we can see that

$$(3.20)\qquad B_\theta(X_j, \nabla_{X_i} X_k) = B_\theta(X_i, \nabla_{X_j} X_k), \qquad i, j, k = 1, 2 .$$

Hence, we need to prove only that

$$(3.21)\qquad \begin{cases} \nabla^{\perp}_{X_1} B_\theta(X_1, X_2) = \nabla^{\perp}_{X_2} B_\theta(X_1, X_1) \\ \nabla^{\perp}_{X_1} B_\theta(X_2, X_2) = \nabla^{\perp}_{X_2} B_\theta(X_1, X_2) . \end{cases}$$

From [6] and (3.18) we obtain

$$\begin{aligned} \nabla^{\perp}_{X_1} B_\theta(X_1, X_2) &= -\sin 2\theta \, \nabla^{\perp}_{X_1} u + \cos 2\theta \, \nabla^{\perp}_{X_1} v \\ &= \cos 2\theta \, \nabla^{\perp}_{X_1} u + \sin 2\theta \, \nabla^{\perp}_{X_1} v \\ &= \nabla^{\perp}_{X_2} B_\theta(X_1, X_1) . \end{aligned}$$

Similarly we prove the second equation. By [17] for each $\theta$ there exists a local isometric immersion $x_\theta\colon M \to Q_c^4$. From (3.18) we have that $x_\theta$ is minimal and from the Ricci equation we see that $(K_N)_\theta = K_N$. We get the deformation putting $x_\theta(p_0) = x(p_0)$ and $dx_\theta(p_0) = dx(p_0)$.

Now, suppose that $\tilde{x}\colon M \to Q_c^4$ is an other minimal immersion with $\tilde{K}_N = K_N$. Let $\{\tilde{x}_1, \tilde{x}_2\}$ be isothermal parameters such that $\tilde{u} = \tilde{B}(\tilde{X}_1, \tilde{X}_1) = -\tilde{B}(\tilde{X}_2, \tilde{X}_2)$ and $\tilde{v} = \tilde{B}(\tilde{X}_1, \tilde{X}_2)$ are on the semi-axes of the ellipse on $N_p\tilde{M}$. Let $\theta(p)$ the angle between $X_1$ and $\tilde{X}_1$ in $T_pM$ and $R_{\theta(p)}$ the rotation in $T_pM$ of angle $\theta(p)$. We know $[X_1, X_2] = [\tilde{X}_1, \tilde{X}_2] = 0$. This implies that $X_1(\theta) = X_2(\theta) = 0$. Hence $\theta$ is constant. By straightforward calculation we can see that $\{\tilde{X}_1, \tilde{X}_2\}$ diagonalize the ellipse of the above immersion $x_{-\theta}$. From (3.6) we see that the connection of the normal bundle depends only the functions $\lambda$ and $\mu$, which are the same for $\tilde{x}$ and $x_{-\theta}$. Now it is easy to show that there exists a bundle isomorphism preserving inner products, second fundamental forms, and normal connections. Then by [17] there is a rigid motion $L$ into $Q_c^4$ such that $\tilde{x} = L \circ x_{-\theta}$.

PROOF OF THEOREM 3. Let $\{x_1, x_2\}$ be a local isothermal parameters such that $ds^2 = E(dx_1^2 + dx_2^2)$ and define a real function $\lambda > 0$ satisfying (3.2) for $\lambda = \mu$, i.e. $\lambda^2 = 2^{-1}(c - K)E^2$. Now, we define the normal bundle, its connection and the second fundamental form, in a manner similar to the last part of proof of theorem 1. So the Gauss and Mainardi-Codazzi equations are satisfied. The Ricci equation follows from (1.3). The existence of the minimal immersion $x\colon M \to Q_c^4$ follows as in theorem 1.

Suppose now that $\tilde{x}\colon M \to Q_c^4$ is another minimal isometric immersion such that the ellipse of curvature is a circle. It is easy to see that the same function $\lambda$ satisfies (3.2) for $x$ and $\tilde{x}$. Now, observe that the connection of the normal bundle depends only on the function $\lambda$. The theorem follows as in the last part of theorem 2.

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