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Linearly Compact Rings and Selfcogenerators.

CLAUDIA MENINI (*)

0. Introduction.

0.1. Throughout this paper all rings are associative with identity $1 \neq 0$ and all modules are unitary.

In our previous work [5] we outlined the connection between linearly compact rings and quasi-injective modules. (Definitions and main results we got in [5] are listed below).

In this paper we give some applications of these results.

First of all, in section 1 we get a further characterization of linearly compact rings. More precisely we prove that a ring R admits a left linearly compact ring topology iff $R = \mathrm{End}(K_A)$ where A is a ring and K_A is a right A-module which is strongly quasi-injective, with essential socle and whose cyclic right A-submodules are linearly compact in the discrete topology. Moreover, in this case, R is linearly compact in the topology τ_* having as a basis of neighbourhoods of 0 the left ideals $\mathrm{Ann}_R(L)$ where L ranges among all linearly compact discrete submodules of K_A. This topology τ_* is the finest topology in the equivalence class of the K-tolology of R. This generalizes a previous result by F.L. Sandomierski ([9]), who proved it in the discrete case and gives a method to build linearly compact rings.

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The existence of a finest equivalent topology for a linearly compact ring has been recently proved by P.N. Ánh (see [1], Proposition 3.1.) in a rather different way and without the representation method.

In section 2 we deal with strictly linearly compact rings and, more in general, with topologically left artinian rings (a linearly topologized ring is called topologically left artinian—see [2] and [7]—if it has a basis of neighbourhoods of 0 consisting of left ideals with artinian residue). These rings have been extensively studied by Ballet in [2] and by A. Orsatti and the author in [7].

Here we point out the relation between such rings and Σ-quasi-injective modules (a module is Σ-quasi-injective iff any direct sum of copies of itself is quasi-injective). We prove that a left selfcogenerator ${}_RK$ over a ring R has a strictly linearly compact biendomorphism ring (which coincides with the Hausdorff completion $\hat{R}$ of R in its K-topology) iff K, regarded as a right module over its endomorphism ring, is Σ-quasi-injective.

Moreover we give an analogous version of characterization theorem above for strictly linearly compact rings. We prove that a ring R admits a left strictly linearly compact ring topology iff $R = \mathrm{End}(K_A)$ where A is a ring and K_A is a right A-module which is Σ-strongly quasi-injective, with essential socle and whose cyclic right A-submodules are linearly compact in the discrete topology. Theorem 2.9 describes those Σ-strongly quasi-injective left modules over a ring R which are also Σ-strongly quasi-injective as right modules over their endomorphism ring. Theorem 2.12 gives a characterization of topologically left artinian rings in terms of Σ-quasi injective modules. Finally in Proposition 2.14 we prove that a commutative linearly topologized ring (R, τ) is a topologically artinian ring iff τ coincides with the Leptin topology τ^* of τ and the minimal cogenerator of the hereditary pretorsion class associated with τ is Σ-strongly quasi-injective.

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0.2. We conclude this introduction giving some notations and recalling some definitions and results of [5].

Let R be a ring. R-Mod will denote the category of left R-modules and Mod-R that of right R-modules. The notation ${}_RM$ will be used to emphasize that M is a left R-module. Morphisms between modules will be written on the opposite side to that of the scalars and the composition of morphisms will follow this convention. For every $M \in R$-Mod,

$E_R(M)$, or simply $E(M)$, wull denote the injective envelope of M in R-Mod and Soc$({}_RM)$, or simply Soc(M), the socle of M.

$\mathbb{N}$ will denote the set of positive integers.

Let R be a ring and let $M \in R$-Mod. ${}_RM$ is *quasi-injective* (for short q.i.) if for every submodule $L \leqslant {}_RM$ and for every morphism $f: L \to {}_RM$, f extends to an endomorphism $\bar{f}$ of ${}_RM$. ${}_RM$ is a *selfcogenerator* if, for every $n \in \mathbb{N}$, given a submodule L of ${}_RM^n$ and an element $x \in M^n \setminus L$, there exists a morphism f: ${}_RM^n \to {}_RM$ such that $(L)f = 0$, and $(x)f \neq 0$. ${}_RM$ is called *strongly quasi-injective* (for short s.q.i.) if given any submodule B of ${}_RM$, a morphism f: $B \to {}_RM$ and an element $x \in M \setminus B$, f extends to an endomorphism $\bar{f}$ of ${}_RM$ such that $(x)\bar{f} \neq 0$. Clearly if ${}_RM$ is both quasi-injective and selfcogenerator, then ${}_RM$ is strongly quasi-injective. The converse is true as well (see [6] Corollary 4.5 and [3] Lemma 2.5).

Let ${}_RK_A$ be a bimodule. ${}_RK_A$ is *faithfully balanced* if $A \simeq$ End$({}_RK)$ and $R \simeq$ End(K_A) canonically.

Let R be a ring and let $M \in R$-Mod. The *M-topology of R* is the left linear ring topology defined by taking as a basis of neighbourhoods of 0 in R the annihilators in R of finite subsets of M.

Recall that a linearly topologized left module M over a discrete ring R is said to be *linearly compact* (for short l.c.) if M is Hausdorff and if any finitely solvable system of congruences $x \equiv x_i$, where the x_i are closed submodules of ${}_RM$, is solvable. We will say that a left R-module is *linearly compact discrete* (for short l.c.d.) iff it is linearly compact in the discrete topology.

All ring and module topologies are assumed to be linear. By a *topological ring* (R, τ) we mean a ring R endowed with a left linear topollgy τ. $\mathcal{F}_\tau$ denotes the filter of open left ideals of R and $\mathcal{C}_\tau$ the class of *τ-torsion* modules:

$$\mathcal{C}_\tau = \{M \in R\text{-Mod} : \forall x \in M,\ \mathrm{Ann}_R(x) \in \mathcal{F}_\tau\}$$

$\mathcal{C}_\tau$ is an hereditary pretorsion class of R-Mod.

For every left ideal I of R it is:

$$I \in \mathcal{F}_\tau \Leftrightarrow R/I \in \mathcal{C}_\tau$$

For every $M \in R$-Mod, $t_\tau(M)$ denotes the τ-torsion submodule of M. Analogous notations hold for a right linear topology.

The following two theorems are essentially the main results in [5]. We state them here in a slightly different way, which is more appropriate for our purposes.

0.3. THEOREM. *Let (R, τ) be a topological ring and let $(\hat{R}, \hat{\tau})$ the Hausdorff completion of (R, τ). The following statements are equivalent:*

(*a*) *$(\hat{R}, \hat{\tau})$ is linearly compact.*

(*b*) *If ${}_RK$ is a cogenerator of $\mathcal{C}_\tau$ and $A = \mathrm{End}({}_RK)$, then K_A is quasi-injective and ${}_{\hat{R}}K_A$ is faithfully balanced.*

(*c*) *There exists a cogenerator ${}_RK$ of $\mathcal{C}_\tau$ such that, setting $A = \mathrm{End}({}_RK)$, K_A* is *quasi-injective and ${}_{\hat{R}}K_A$ is faithfully balanced.*

(*d*) *Let ${}_RU$ be the minimal cogenerator of $\mathcal{C}_\tau$, $T = \mathrm{End}({}_RU)$. Then ${}_{\hat{R}}U_T$ is faithfully balanced.*

PROOF. It is easy too see that every left R-module in $\mathcal{C}_\tau$ has a natural structure of $\hat{R}$-module, that every morphism between two modules of $\mathcal{C}_\tau$ is an $\hat{R}$-morphism and that $\mathcal{C}_\tau = \mathcal{C}_{\hat{\tau}}$ (see [6], Proposition 6.5). It follows that a module ${}_RK \in \mathcal{C}_\tau$. is a cogenerator of $\mathcal{C}_\tau$ iff it is a cogenerator of $\mathcal{C}_{\hat{\tau}}$. Moreover the minimal cogenerator of $\mathcal{C}_\tau$ and that one of $\mathcal{C}_{\hat{\tau}}$ coincide. From these remarks and from the Main Theorem of [5] the proof follows.

0.4. THEOREM. *Let R be a ring, ${}_RK$ a strongly quasi-injective left R-module, $A = \mathrm{End}(({}_RK)$, $\hat{R}$ the Hausdorff completion of R in this K-topology. Then* Soc(K_A) *is essential in K_A, the bimodule ${}_{\hat{R}}K_A$ is faithfully balanced and the following conditions are equivalent:*

(*a*) *K_A is strongly quasi-injective.*

(*b*) *$\hat{R}$ is linearly compact in its K-topology and $\hat{R}$ separates points and submodules of K_A:*

(*c*) *$\hat{R}$ is linearly compact in its K-topology and* Soc$({}_RK)$ *is essential in ${}_RK$.*

Moreover, if these conditions hold, then A is linearly compact in its K-topology.

PROOF. If $_RK$ is s.q.i. and $A = \mathrm{End}(_RK)$, then by Proposition 6.10 of [6] $\mathrm{Soc}(K_A)$ is essential in K_A and by Corollary 7.4 of [6], $\mathrm{End}(K_A) \simeq \hat{R}$. Moreover it is easy to see (see [6], Theorem 6.7) that $_RK$ is s.q.i. iff $_{\hat{R}}K$ is s.q.i. Apply now Theorem 10 of [5].

1. A further characterization of linearly compact rings.

1.1. LEMMA. *Let R be a ring, $_RK$ a quasi-injective R-module, $A = \mathrm{End}(_RK)$. Let L ba a submodule of $_RK$, $\{a_1, \dots, a_n\}$ a finite subset of A, and set*

$$I = a_1A + \dots + a_nA + \mathrm{Ann}_A(L).$$

Then $I = \mathrm{Ann}_A \mathrm{Ann}_K(I)$.

PROOF. Straightforward.

1.2. LEMMA. *Let R be a ring, $_RK$ a selfcogenerator, $A = \mathrm{End}(_RK)$. Let L be a finitely generated A-submodule of an A-module M_A which is cogenerated by K_A.*

Then every morphism of L in K_A extends to a morphism of M in K_A.

PROOF. See Corollary 2.3 of [6].

Let R be a ring. A left ideal I of R is *completely irreducible* if R/I is an essential submodule of the injective envelope $E(S)$ of a left simple R-module S. A left R-module M is *finitely embedded* if its socle is finitely generated and it is essential in M.

1.3. LEMMA. *Let R be a ring, $_RK$ a quasi-injective left R-module with essential socle, $A = \mathrm{End}(_RK)$. Let I be a right ideal of A such that $I = \mathrm{Ann}_A \mathrm{Ann}_K(I)$. If $\mathrm{Ann}_K(I)$ is l.c.d., then for every right ideal H of A containing I it is $H = \mathrm{Ann}_A \mathrm{Ann}_K(H)$.*

PROOF. Let H be a left ideal of A containing I and let $a \in \mathrm{Ann}_A \mathrm{Ann}_K(H)$. Then $\mathrm{Ann}_K(a) \geqslant \mathrm{Ann}_K(H)$ and $L = \mathrm{Ann}_K(I) \geqslant \mathrm{Ann}_K(H)$. Thus $\mathrm{Ann}_L(a) \geqslant \mathrm{Ann}_L(H) = \bigcap_{h \in H} \mathrm{Ann}_L(h)$.

Since L is l.c.d., $L/\mathrm{Ann}_L(a) \simeq La$ is l.c.d. and since $\mathrm{Soc}(_RK)$ is essential in $_RK$, La is finitely embedded.

Thus, by Lemma 8 of [5], there exists $h_1, \ldots, h_n \in H$ such that

$$\mathrm{Ann}_L(a) > \bigcap_{i=1}^{n} \mathrm{Ann}_L(h_i)\,. \tag{1}$$

Since, by Lemma 1.1, the right ideal of A

$$J = (h_1A + \ldots + h_nA) + \mathrm{Ann}_A(L)$$

is closed in the K-topology of A and $J = \mathrm{Ann}_A \mathrm{Ann}_K(J)$, if $a \notin J$ there exists an $x \in \mathrm{Ann}_K(J)$ such that $xa \neq 0$. Since $\mathrm{Ann}_K(J) = \bigcap_{i=1}^{n} \mathrm{Ann}_K(h_i) \cap L$, by (1) this is impossible.

Let R be a ring, ${}_RK$ a left R-module. We will say that ${}_RK$ is *finitely linearly compact discrete* (for short f.l.c.d.) if every cyclic (and hence every finitely generated) left R-submodule of ${}_RK$ is l.d.c.

1.4. Lemma. *${}_RK$ is f.l.c.d. iff the Hausdorff completion $\hat{R}$ of R in its K-topology is linearly compact.*

Proof. Straightforward.

The following proposition unifies some known results ([9], Corollary 2 page 342, [6] Theorem 9.4, [8] Proposition 3.4 a)) which were proved by the use of the same technique.

1.5. Proposition. *Let R be a ring, ${}_RK \in R$-Mod a selfcogenerator, $A = \mathrm{End}({}_RK)$ and $L \leqslant {}_RK$. Then L is linearly compact discrete iff for any right ideal I of A and for any morphism $f\colon I \to K_A$ such that $\mathrm{Ker}(f) \geqslant \mathrm{Ann}_A(L)$, f extends to a morphism $\bar{f}\colon A \to K_A$. In particular ${}_RK$ is f.l.c.d. iff K_A is quasi-injective, while ${}_RK$ is l.c.d. iff K_A is injective.*

Proof. Assume that L is l.c.d. and let I be a right ideal of A and $f\colon I \to K_A$ a morphism such that $\mathrm{Ker}(f) \geqslant \mathrm{Ann}_A(L)$. Then f induces, in a natural way, a morphism $\bar{f}\colon I/\mathrm{Ann}_A(L) \to K_A$. Let $(H_j)_{j\in J}$ be the family of finitely generated A-submodules of $I/\mathrm{Ann}_A(L)$. Since $A/\mathrm{Ann}_A(L)$ is embeddable in $\prod_{x\in L} xA$ which is embeddable in K_A^L, by Lemma 1.2, for every $j \in J$, $\bar{f}_{|H_j}$ extends to a morphism $A/\mathrm{Ann}_A(L) \to K_A$ so that there exists an $x_j \in \mathrm{Ann}_K \mathrm{Ann}_A(L) = L$ such that $\bar{f}_{|H_j}$ coincides with the left multiplication by x_j. Now, write $H_j = I_j + \mathrm{Ann}_A(L)\,\mathrm{Ann}_A(L)$ where I_j is a finitely generated ideal of A. Then it is easy to prove that the system

(1) $$X \equiv x_j \quad \bmod \operatorname{Ann}_L(I_j) \quad j \in J$$

is finitely solvable in L and hence—since L is l.c.d.—it is solvable in L. Let $x \in L$ be a solution of (1). Then $x - x_j \in \operatorname{Ann}_L(I_j)$, for every $j \in J$, so that the left multiplication by x gives a morphism $A \to K_A$ which extends f.

Conversely assume that $L \leqslant_R K$ is such that for every right ideal I of A and any morphism $f\colon I \to K_A$ such that $\operatorname{Ker}(f) \geqslant \operatorname{Ann}_A(L)$, f extends to a morphism $A \to K_A$: Let

(2) $$X \equiv x_j \quad \bmod (L_j)$$

be a finitely solvable system of congruences in L.

Then the morphism

$$g\colon \sum_{j \in J} \operatorname{Ann}_A(L_j) \to K_A$$

defined by setting $g\left(\sum_{j \in F} a_j\right) = \sum_{j \in F} x_j$, where F is a finite subset of J and $a_j \in \operatorname{Ann}_A(L_j)$ for every $j \in F$, is well defined and $\operatorname{Ker}(g) \geqslant \operatorname{Ann}_A(L)$.

By hypothesis g extends to a morphism $A \to K_A$ so that there exists an $x \in K$ such that $x - x_j \in \operatorname{Ann}_K \operatorname{Ann}_A(L_j) = L_j$. Note that, since $x_j \in L$ and $L_j \leqslant L$, $x \in L$. Thus (2) is solvable in L.

The two last statements follows by the Baer's criterion for quasi-injectivity (see e.g. Proposition 6.6 of [6]) and that one for injectivity.

Let R be a ring. Recall that two left linear ring topologies on R are called *equivalent* if they have the same closed ideals.

1.6. THEOREM. *Let A be a ring. A admits a right linearly compact ring topology iff $A = \operatorname{End}({}_RK)$ where R is a ring and ${}_RK$ is a finitely linearly compact discrete and strongly quasi-injective left R-module with essential socle. In this case*

1) *A is linearly compact in the topology τ_* having the right annihilators of submodules of ${}_RK$ which are linearly compact discrete as a basis of neighbourhoods of* 0.
2) *τ_* is the finest topology in the equivalence class of the K-topology of A.*
3) *K_A is strongly quasi-injective with essential socle.*

PROOF. Assume that (A, τ) is right linearly compact and let U_A be an injective cogenerator of $\mathcal{C}_\tau$ with essential socle, $R = \text{End}(U_A)$. By Lemma 6 of [5] τ is equivalent to the U-topology τ_U of A so that (A, τ_U) is right linearly compact. Thus, by Theorem 0.4 $_RU$ is f.l.c.d. and s.q.i. with essential socle.

Conversely, assume R is a ring and $_RK$ is an f.l.c.d. and s.q.i. left R-module with essential socle. Thus by Lemma 1.4 and by Theorem 0.4 K_A is s.q.i. with essential socle and A is linearly compact in its K-topology τ. Now to prove statements 1) and 2) it is enough to show that τ_* is the finest topology in the equivalence class of τ. By Lemma 1.3 every open—and hence every closed—right ideal of τ_* is closed in τ. Let τ' be a topology equivalent to τ. To complete our proof let us show that every open right ideal I of τ' is open in τ_*. Since I is open in τ', which is equivalent to τ, A/I is l.c.d. and moreover every right ideal of A containing I is closed in τ. In particular I is closed in τ so that, as K_A is s.q.i., it is $I = \text{Ann}_A(L)$ where $L = \text{Ann}_K(I)$. Let us prove that L is l.c.d. To do this we use Proposition 1.5. Let H be a right ideal of A and let $f: H \to K_A$ be a morphism such that $\text{Ker}(f) \geq I$. Then, since A/I is l.c.d. and K_A has essential socle, Im(f) is finitely embedded. Thus, there exists a finite number $S_1, \dots, S_n$ of simple A-submodules of K_A such that $Im(f) \leq \bigoplus_{i=1}^{n} E(S_i)$ and hence f extends to a morphism $\bar{f}: A \to \bigoplus_{i=1}^{n} E(S_i)$. Let $x = \bar{f}(1)$. Then $x = x_1 + \dots + x_n$ where $x_i \in E(S_i)$ for every i, and $\bigcap_{i=1}^{n} \text{Ann}_A(x_i) = \text{Ann}_A(x) = \text{Ker}(\bar{f}) \geq \text{Ker}(f) \geq I$. Thus each $\text{Ann}(x_i)$ is closed in τ and, since it is completely irreducible, it is also open in τ. Thus $\text{Ker}(f) = H \cap \text{Ker}(\bar{f})$ is open in the relative topology on H of τ and, as K_A is q.i., f extends to a morphism $A \to K_A$. Hence, by Proposition 1.5., L is l.c.d.

1.7. COROLLARY (Theorem 3.10 of [9]). *A ring A is right linearly compact discrete if and only if $A = \text{End}(_RK)$ where $_RK$ is l.c.d. and s.q.i. with essential socle.*

1.8. COROLLARY (Proposition 4.3 of [8]). *Let R be a ring and let $_RK$ be an s.q.i. R-module, $A = \text{End}(_RK)$. The following conditions are equivalent:*

(*a*) *$_RK$ is l.c.d. and* $\text{Soc}(_RK)$ *is essential in $_RK$.*

(*b*) *K_A is an injective selfcogenerator.*

(*c*) *K_A is an injective cogenerator of Mod-A.*

(*d*) *A is linearly compact in the discrete topology which is equivalent to the K-topology of A and* Soc$({}_RK)$ *is essential in ${}_RK$.*

PROOF. (*a*) $\Leftrightarrow$ (*b*) follows by Lemma 1.4, Theorem 0.4 and Proposition 1.5.

(*c*) $\Rightarrow$ (*b*) is trivial.

(*a*) $\Rightarrow$ (*c*) and (*a*) $\Rightarrow$ (*d*). Since we already proved (*a*) $\Leftrightarrow$ (*b*), K_A is injective. By Theorem 1.6 A is linearly compact in the discrete topology which is equivalent to the K-topology of A. Thus K_A is a cogenerator of Mod-A. (*d*) $\Rightarrow$ (*a*). By Theorem 1.6.

2. A characterization of strictly linearly compact rings.

2.1. LEMMA. *Let R be a ring and let $(M_i)_{i\in I}$, $M_i \in R$*-Mod, *be a family of selfcogenerators. If $\bigoplus_{i\in I} M_i$ is quasi-injective, then it is strongly quasi-injective.*

PROOF. One easily sees that it is enough to give a proof for I finite. In this case a proof similar to that one of Lemma 2.5 in [3] works.

Let R be a ring and let $M \in R$-Mod. We will say that ${}_RM$ is *Σ-quasi-injective* (for short Σ-q.i.) if every direct sum of copies of ${}_RM$ is q.i. Moreover we will say that ${}_RM$ is *ω-quasi-injective* (for short ω-q.i.) if ${}_RM^{(\mathbf{N})}$ is quasi-injective. The definitions of *Σ-strongly quasi-injective* (for short Σ-s.q.i.) and *ω-strongly quasi-injective* (fort short ω-s.q.i.) module are given in an analogous way.

The following useful lemma is a trivial consequence of Lemma 2.1.

2.2. LEMMA *Let R be a ring and let $M \in R$-Mod be a selfcogenerator. Then ${}_RM$ is Σ-s.q.i. (ω-s.q.i.) iff ${}_RM$ is Σ-q.i. (ω-q.i.).*

2.3. PROPOSITION. *Let R be a ring, ${}_RK \in R$*-Mod *a selfcogenerator,* $A = \mathrm{End}({}_RK)$ *and $L \leqslant {}_RK$. The following statements are equivalent:*

(*a*) *${}_RL$ is artinian.*

(*b*) *For every right ideal I of A, for every set X, and for every morphism $f\colon I \to K_A^{(X)}$ such that $Ker(f) \geqslant \mathrm{Ann}_A(L)$, f extends to a morphism $A \to K_A^{(X)}$.*

(c) *For every right ideal I of A and for every morphism $f\colon I \to K_A^{(\mathbf{N})}$ such that* $\operatorname{Ker}(f) \geqslant \operatorname{Ann}_A(L)$, *$f$ extends to a morphism $A \to K_A^{(\mathbf{N})}$.*

PROOF. $(a) \Rightarrow (b)$. Assume $_RL$ is artinian and let $f\colon I \to K_A^{(X)}$ as in (b). Proceeding in a similar way as in Proposition 1.4 define $\bar{f}$: $I/\operatorname{Ann}_A(L) \to K_A^{(X)}$, $(H_j)_{j\in J}$ and I_j and note that for $j \in J$, $\bar{f}(H_j)$ is contained in an A-submodule M of $K_A^{(X)}$ which is a finite direct sum of copies of K_A. Thus since $_RK$ is a selfcogenerator, using Lemma 1.2 it is easy to prove that $\bar{f}_{|H_j}$ extends to a morphism $A/\operatorname{Ann}_A(L) \to M$. Hence, for every $j \in J$, there exists an $x_j \in K_A^{(X)}$ such that $\bar{f}_{|H_j}$ coincides with the left multiplication by x_j. It is $x_j(\operatorname{Ann}_A(L)) = 0$ and hence $x_j \in \operatorname{Ann}_{K^{(X)}} \operatorname{Ann}_A(L) = (\operatorname{Ann}_K \operatorname{Ann}_A(L))^{(X)} = L^{(X)}$. Hence the system

$$X \equiv x_j \mod \operatorname{Ann}_{L^{(X)}}(I_j) \qquad j \in J \tag{1}$$

is finitely solvable in $L^{(X)}$. Let $j_0 \in J$ be such that $\operatorname{Ann}_L(I_{j_0})$ is a minimal element of the non empty family $\{\operatorname{Ann}_L(I_j)\}_{j\in J}$ of submodules of the the artinian left R-module L. Then x_{j_0} is a solution of the system (1) so that left multiplication by x_{j_0} gives a morphism $A \to K^{(X)}$ which extends f.

$(b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (a)$ Assume $L_0 = L \geqslant L_1 \geqslant \ldots \geqslant L_n \geqslant \ldots$ is a strictly decreasing sequence of submodules of L and, for any $n \in \mathbf{N}$, let $y_n \in L_n \setminus L_{n+1}$. Since $_RK$ is a selfcogenerator and $A = \operatorname{End}(_RK)$, left multiplication by y_n defines a morphism $\mu_n\colon A \to K_A$ such that

$$\mu_n(\operatorname{Ann}_A(L_n)) = 0 \quad \text{and} \quad \mu_n(\operatorname{Ann}_A(L_{n+1})) \neq 0.$$

Let $I = \bigcup_{n\in\mathbf{N}} \operatorname{Ann}_A(L_n)$ and let $\mu\colon A \to K^{\mathbf{N}}$ be the diagonal morphism of the μ_n's. It is easy th check that $\mu(I) \leqslant K^{(\mathbf{N})}$ so that μ induces a morphism $\bar{\mu}\colon I \to K^{(\mathbf{N})}$. Note that, since $y_n \in L$ fo every $n \in \mathbf{N}$, $\operatorname{Ker}(\bar{\mu}) \geqslant \operatorname{Ann}_A(L)$. By hypothesis $\bar{\mu}$ extends to a morphism $A \to K^{(\mathbf{N})}$. Thus there exists an $x \in K^{(\mathbf{N})}$ such that $\operatorname{Im}(\bar{\mu}) \leqslant xA$. For every $n \in \mathbf{N}$, let $\pi_n\colon K^{(\mathbf{N})} \to K$ be the canonical projection. Then there is a $k \in \mathbf{N}$ such that $\pi_n \circ \bar{\mu} = 0$ for every $n \geqslant k$. Since $\pi_n \circ \bar{\mu} = \mu_n \neq 0$ for every n, we get a contradiction.

2.4. REMARK. Let R, $_RK$, L and A be as in Proposition 2.3 and let X be an infinite set. Assume that for every right ideal I of A and for every morphism $f\colon I \to K_A^{(X)}$ such that $\mathrm{Ker}(f) \geqslant \mathrm{Ann}_A(L)$, f extends to a morphism $A \to K_A^{(X)}$. Then L satisfies (*c*) of Proposition 2.3 and hence it is artinian.

We will say that a left R-module M is *artinian finitely generated* (for short artinian f.g.) if every cyclic, and hence every finitely generated, left R-submodule of M is artinian.

The definition of *noetherian finitely generated* (for short noetherian f.g.) module is given in an analogous way.

Recall (see [2] and [7]) that a topological ring (R, τ) is a *topologically left artinian ring* (for short (R, τ) is a TA-ring) if, for every $I \in \mathcal{F}_\tau$, R/I is left artinian.

The definition of *topologically left noetherian ring* (for short TN-ring) is given in an analogous way (see [1]).

(R, τ) is a *strongly topologically left artinian ring* (for short (R, τ) is an STA-ring) iff it is both a TA-ring and a TN-ring (see [7]).

Finally recall that (R, τ) is *strictly linearly compact* (for short s.l.c.) iff it is a complete and Hausdorff TA-ring.

For technical convenience we state the following lemma whose proof is straightforward.

2.5. LEMMA. *Let R be a ring, let ${}_RK \in R$-*Mod *and let τ be the K-topology of R. Denote by $(\hat{R}, \hat{\tau})$ the Hausdorff completion of (R, τ). Then:*

(*a*) *(R, τ) is a TA-ring $\Leftrightarrow$ $(\hat{R}, \hat{\tau})$ is a TA-ring $\Leftrightarrow$ ${}_RK$ is artinian f.g.*

(*b*) *(R, τ) is a TN-ring $\Leftrightarrow$ $(\hat{R}, \hat{\tau})$ is a TN-ring $\Leftrightarrow$ ${}_RK$ is noetherian f.g.*

(*c*) *(R, τ) is an STA-ring $\Leftrightarrow$ $(\hat{R}, \hat{\tau})$ is an STA-ring $\Leftrightarrow$ ${}_RK$ is both artinian f.g. and noetherian f.g.*

(*d*) *(R, τ) is strictly linearly compact $\Leftrightarrow$ $(R, \tau) = (\hat{R}, \hat{\tau})$ and ${}_RK$ is artinian f.g. $\Leftrightarrow$ (R, τ) is linearly compact and ${}_RK$ is artinian f.g.*

Moreover, if ${}_RK$ is a selfcogenerator then:

(*e*) *K_A artinian f.g. $\Rightarrow$ ${}_RK$ noetherian f.g.*

(*f*) *K_A noetherian f.g. $\Rightarrow$ ${}_RK$ artinian f.g.*

REMARK. Since ${}_RK$ has a natural structure of $\hat{R}$-module and since, with respect to this structure, every subgroup of K is an R-submodule iff it is an $\hat{R}$-submodule, it is clear that all the statements of lemma above hold if one writes ${}_{\hat{R}}K$ instead of ${}_RK$.

2.6. PROPOSITION. *Let (R, τ) be a linearly compact ring, ${}_RU$ an injective cogenerator of $\mathcal{C}_\tau$ with essential socle, $A = \mathrm{End}({}_RU)$. Let τ_* be the finest topology in the equivalence class of τ. Then (R, τ_*) is strictly linearly compact iff every linearly compact discrete submodule of U_A is finitely generated. In this case every topology equivalent to τ coincides with τ.*

PROOF. Recall (see Theorem 1.6) that ${}_RU_A$ is faithfully balanced and that U_A is f.l.c.d. and s.q.i. with essential socle. Moreover τ_* has the left annihilators of submodules of U_A which are linearly compact discrete as a basis of neighbourhoods of 0.

Now if (R, τ_*) is strictly linearly compact then it is clear that there is not any topology in the equivalence class of τ_* coarser than τ_*. Thus $\tau_* = \tau$ and every topology equivalent to τ coincides with τ. In particular τ_* coincides with the U-topology of R. Hence if L is a linearly compact discrete submodule of U_A, there exists a finitely generated submodule H of U_A such that $\mathrm{Ann}_R(L) \geqslant \mathrm{Ann}_R(H)$ so that $L \leqslant H$. As (R, τ) is s.l.c., ${}_RU$ is artinian f.g. and hence, by Proposition 2.5 *e*) U_A is noetherian f.g. It follows that L is a finitely generated A-module.

Conversely, assume that every linearly compact discrete submodule of U_A is finitely generated. Then τ_* coincides with the U-topology of R and U_A is noetherian f.g. By Lemma 2.5 *e*) and *a*) (R, τ_*) is strictly linearly compact.

The following result was suggested to me by D. Dikranjan.

2.7. COROLLARY. *Let (R, τ) be a linearly compact ring and assume that the Jacobson radical of R, $J(R)$, is zero. Then (R, τ) is strictly linearly compact.*

Consequentely (R, τ) is topologically isomorphic to a topological product $\prod_{\lambda \in \Lambda} \mathrm{End}_{D_\lambda}(V_\lambda)$ where, for every $\lambda \in \Lambda$, V_λ is a vector space over the division ring D_λ and $\mathrm{End}_{D_\lambda}(V_\lambda)$ is endowed with the finite topology.

PROOF. It is easy to see (cf. [5] Theorem 14) that for the minimal injective cogenerator ${}_RU$ of $\mathcal{C}_\tau$ we have, in this case,

$$ {}_RU = \bigoplus_{\lambda \in \Lambda} S_\lambda $$

where $(S_\lambda)_{\lambda\in\Lambda}$ is a system of representatives of the isomorphism classes of the left simple R-modules of $\mathcal{C}_\tau$. Thus each S_λ is fully invariant in ${}_RU$ and hence, setting $A = \mathrm{End}({}_RU)$, it is straightforward to prove that U_A is semisimple. Then every l.c.d. submodule of U_A has finite length.

By Proposition 2.6, (R, τ) is strictly linearly compact and, in particular, τ coincides with its Leptin topology. Thus the last assertion of the Corollary follows by the classical Leptin's result (see [5], Theorem 14).

In the following theorem we sum up all the main relations between the properties of a selfcogenerator ${}_RK \in R$-Mod and those of the Hausdorff completion of R in its K-topology.

2.8. THEOREM. *Let R be a ring, ${}_RK \in R$-Mod a selfcogenerator, $A = \mathrm{End}({}_RK)$, $(\hat{R}, \hat{\tau})$ the Hausdorff completion of R in its K-topology. Then*:

a) $(\hat{R}, \hat{\tau})$ *is l.c.* $\Leftrightarrow$ ${}_RK$ *is f.l.c.d.* $\Leftrightarrow$ K_A *is q.i.*

b) $(\hat{R}, \hat{\tau})$ *is s.l.c.* $\Leftrightarrow$ ${}_RK$ *is artinian f.g.* $\Leftrightarrow$ K_A *is Σ-q.i.* $\Leftrightarrow$ K_A *is ω-q.i.*

c) ${}_RK$ *is l.c.d.* $\Leftrightarrow$ K_A *is injective.*

d) ${}_RK$ *is artinian* $\Leftrightarrow$ K_A *is Σ-injective* $\Leftrightarrow$ K_A *is ω-injective.*

If moreover ${}_RK$ is s.q.i. then:

α) $(\hat{R}, \hat{\tau})$ *is l.c. and* $\mathrm{Soc}({}_RK)$ *is essential in* ${}_RK$ $\Leftrightarrow$ K_A *is s.q.i.*

β) $(\hat{R}, \hat{\tau})$ *is s.l.c.* $\Leftrightarrow$ K_A *is Σ-s.q.i.* $\Leftrightarrow$ K_A *is ω-q.i.*

γ) ${}_RK$ *is l.c.d. with essential socle* $\Leftrightarrow$ K_A *is an injective cogenerator of Mod-A.*

δ) ${}_RK$ *is artinian* $\Leftrightarrow$ K_A *is a Σ-injective cogenerator of Mod-A* $\Leftrightarrow$ $\Leftrightarrow$ A_A *is noetherian.*

PROOF. Assume ${}_RK$ is a selfcogenerator. Then

a) follows by Proposition 1.5,

b) follows by Proposition 2.3 and by Proposition 6.6 of [6] after observing that, for a non-empty set X, the K-topology and the $K^{(X)}$-topology of A coincide,

c) follows by Proposition 1.5,

d) follows by Proposition 2.3.

Assume now ${}_RK$ is s.q.i. Then

α) follows by Theorem 0.4,

β) follows by Theorem 0.4, Lemmata 2.5 and 2.2 and by *b*),

γ) follows by Corollary 1.8.

δ) The first equivalence of δ) follows by *d*) and γ). Now if Mod-A has a Σ-injective cogenerator then it is well known that A is right noetherian. Conversely if A is right noetherian then, as ${}_RK$ is a selfcogenerator and $A = \mathrm{End}({}_RK)$ it is straightforward to prove that ${}_RK$ is artinian.

Remark. Statement *c*) is Theorem 9.4 of [6].
Statement *d*) could be deduced from Theorem 9.4 of [6] and Proposition 3 of [4]. Statement δ) was already proved, in a different way, in [7] (see [7], Lemma 4.12).

It is natural to ask when, in the hypothesis of Theorem 2.8, ${}_RK$ and K_A are both Σ-s.q.i. Following theorem gives an answer to this question.

2.9. Theorem. *Let R be a ring and let ${}_RK \in R$-*Mod *be an s.q.i. module. Then, in the notations of Theorem* 2.6, *the following statements are equivalent*:

(*a*) *Every finitely generated submodule of ${}_RK$ has finite length.*

(*b*) *Every finitely generated submodule of K_A has finite length.*

(*c*) *$\hat{R}$ and A are both strictly linearly compact in their K-topologies.*

(*d*) *${}_RK$ and K_A are both Σ-strongly quasi-injective.*

Proof. (*a*) $\Rightarrow$ (*b*) Since ${}_RK$ is artinian f.g. by Theorem 2.8 K_A is Σ-s.q.i. Since ${}_RK$ is s.q.i., $\hat{R} = \mathrm{End}(K_A)$ by Theorem 0.4. Thus (*b*) follows by Lemma 2.5 *e*) and *f*).

(*b*) $\Rightarrow$ (*a*) Since ${}_R K$ is s.q.i., (*a*) follows by Lemma 2.5 *e*) and *f*).

(*a*) $\Rightarrow$ (*c*) Since (*a*) $\Leftrightarrow$ (*b*), (*c*) follows by Theorem 0.4.

(*c*) $\Rightarrow$ (*d*) As $\hat{R}$ is s.l.c. in its K-topology, by Theorem 0.4 ${}_{\hat{R}}K_A$ is faithfully balanced and K_A is s.q.i. (*d*) follows now from Theorem 2.8.

(*d*) $\Rightarrow$ (*a*) By Theorem 2.8 ${}_R K$ and K_A are both artinian f.g.

Since ${}_R K$ is s.q.i. (*a*) follows by Lemma 2.5 *e*).

2.10. PROPOSITION. *Let R be a ring and let ${}_R K \in R$-*Mod *be an f.l.c.d. and s.q.i. module with essential socle, $A = \mathrm{End}({}_R K)$. Then A is s.l.c. in its K-topology $\Leftrightarrow$ K_A is artinian f.g. $\Leftrightarrow$ ${}_R K$ is noetherian f.g. $\Leftrightarrow$ ${}_R K$ is Σ-q.i. $\Leftrightarrow$ ${}_R K$ is Σ-s.q.i.*

PROOF. By Lemma 1.4 and Theorem 0.4, ${}_{\hat{R}}K_A$ is faithfully balanced, K_A is an s.q.i. module and A is linearly compact in its K-topology. Thus by Lemma 2.5 *d*) A is s.l.c. in this topology $\Leftrightarrow$ K_A is artinian f.g. Now, by Lemma 2.5 *e*) and *f*) K_A is artinian f.g. $\Leftrightarrow$ ${}_R K$ is noetherian f.g. The other equivalences follow by Theorem 2.8.

As a corollary we get the following result which is analogous to Theorem 1.6.

2.11. COROLLARY. *Let A be a ring. A admits a right strictly linearly compact ring topology iff $A = \mathrm{End}({}_R K)$ where R is a ring and ${}_R K$ an f.l.c.d. Σ-strongly quasi-injective left R-module with essential socle. In this case A is s.l.c. in its K-topology.*

PROOF. Follows by Proposition 2.10 and Theorem 1.6.

The following theorem, which is analogous to Theorem 0.3, characterizes topologically left artinian rings.

2.12. THEOREM. *Let (R, τ) be a topological ring, $(\hat{R}, \hat{\tau})$ the Hausdorff completion of (R, τ) and τ^* the Leptin topology of τ. Then, with the notations introduced in* 0.2, *the following statements are equivalent:*

(*a*) *(R, τ) is a TA-ring (i.e. $(\hat{R}, \hat{\tau})$ is strictly linearly compact).*

(*b*) *If ${}_R K$ is a cogenerator of $\mathcal{C}_\tau$ and $A = \mathrm{End}({}_R K)$, then K_A is Σ-q.i. (ω-q.i.).*

(*c*) *$\tau = \tau^*$ and there exists a cogenerator K of $\mathcal{C}_\tau$ such that, setting $A = \mathrm{End}({}_R K)$, K_A is Σ-q.i. (ω-q.i.).*

(*d*) *Let* $_RU$ *be the minimal cogenerator of* $\mathcal{C}_\tau$, $T = \mathrm{End}(_RU)$.

Then U_T *is* Σ*-s.q.i.* (ω*-s.q.i.*) *and* $\tau = \tau^*$.

PROOF. (*a*) $\Rightarrow$ (*b*) Since (R, τ) is a *TA*-ring every module of $\mathcal{C}_\tau$ is artinian f.g. Now (*b*) follows from Theorem 2.8.

(*b*) $\Rightarrow$ (*a*) Let $_RK = \bigoplus_{I \in \mathcal{F}_\tau} t_\tau(E(R/I))$. Then $_RK$ is a cogenerator of $\mathcal{C}_\tau$. By Theorem 2.8 $_RK$ is artinian f.g. so that, for every $I \in \mathcal{F}_\tau$, R/I is left artinian.

(*a*) $\Rightarrow$ (*c*) Since (*a*) $\Leftrightarrow$ (*b*), this is trivial.

(*c*) $\Rightarrow$ (*a*) By Theorem 2.8 $_RK$ is artinian f.g. Since $_RK$ contains the minimal cogenerator of $\mathcal{C}_\tau$, by Lemma 5 of [5], (*a*) follows.

(*a*) $\Rightarrow$ (*d*) Since (*a*) $\Leftrightarrow$ (*b*), (*d*) follows by Theorem 0.3 and Lemma 2.2.

(*d*) $\Rightarrow$ (*c*) is trivial.

The following proposition characterizes *STA*-rings.

2.13. PROPOSITION. *In the hypothesis of Theorem* 2.1, *let* $_RU$ *be the minimal cogenerator of* $\mathcal{C}_\tau$, $T = \mathrm{End}(_RU)$. *The following statements are equivalent:*

(*a*) (R, τ) *is an STA-ring.*

(*b*) $\tau = \tau^*$ *and* $_RU$ *and* U_T *are both* Σ*-s.q.i.* (ω*-s.q.i.*).

(*c*) $_RU$ *is s.q.i. and* T, *endowed with its* U*-topology is a strongly topologically right artinian ring.*

(*d*) (R, τ) *is a TA-ring and* $_RU$ *is* Σ*-s.q.i.* (ω*-s.q.i.*).

PROOF. (*a*) $\Rightarrow$ (*b*) By Theorem 2.1 of [7] $_RU$ is s.q.i.
Thus Theorem 2.7 applies.

(*b*) $\Rightarrow$ (*c*) By Theorem 2.9.

(*c*) $\Rightarrow$ (*d*) By Theorem 2.9.

(*d*) $\Rightarrow$ (*b*) By Theorem 2.8.

(*b*) $\Rightarrow$ (*a*) By Theorem 2.9.

Commutative TA-rings were extensively studied in [7] (see [7], section 6). Following proposition gives a further characterization of these rings.

2.14. PROPOSITION. *Let* (R, τ) *be a commutative topological ring,* ${}_RU$ *the minimal cogenerator of* $\mathcal{C}_\tau$. *The following statements are equivalent:*

(*a*) (R, τ) *is a (strongly) topologically artinian ring.*

(*b*) $\tau = \tau^*$ *and* ${}_RU$ *is* Σ-*s.q.i.*

(*c*) $\tau = \tau^*$ *and* (R, τ) *is a topologically noetherian ring.*

PROOF. First of all note that (R, τ) is a topologically artinian ring iff it is a strongly topologically artinian ring (see Proposition 3.9 of [7]). Thus

(*a*) $\Rightarrow$ (*b*) follows from Proposition 2.13.

(*b*) $\Rightarrow$ (*c*) Since ${}_RU$ is Σ-s.q.i. it is straightforward to prove that (R, τ) is a topologically noetherian ring (see Theorem 15 of [1]).

(*c*) $\Rightarrow$ (*a*) Let $x \in {}_RU$, $x \neq 0$, $I = \mathrm{Ann}_R(x)$. Then Rx is noetherian and thus the ring R/I is a commutative noetherian ring. Since Rx is a finitely embedded R/I-module, from the general theory of commutative noetherian rings, it follows that Rx is artinian.

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