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BMO Continuity for Some Heat Potentials.

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0. Introduction.

The problem discussed in this note arises from the study of the temperatures of a nonsmooth domain D in $\mathbf{R}^n$, controlled by surface data which may be discontinuous and also unbounded. By temperatures, we mean solutions of the (linear) heat equation

$$Lu(X, t) \equiv \Delta_X u - D_t u = 0, \qquad \text{for all } (X, t) \in D \times \mathbf{R}^+ ,$$

$\mathbf{R}^+ = (0, \infty)$. The bounded domain D is assumed to be a C^1 domain (as in [1], [3], etc.) with boundary ∂D. Given $0 < T < \infty$, consider the Initial-Dirichlet Problem

$$(\text{I.D.P.}) \qquad \begin{cases} Lu(X, t) = 0 & \text{in the cylinder } D \times (0, T)\,, \\ \lim\limits_{t \to 0} u(X, t) = 0 & \text{uniformly on compacts in } D, \\ u(X, t) \to f(P, s) & \text{a.e. on the lateral surface } \partial D \times (0, T) \end{cases}$$

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as $(X, t) \in D\times(0, T)$ tends to the boundary point (P, s) from within a suitable approach-region with vertex (P, s); see [3], [4]. For smoother domains and continuous data f such problems are classical. For C^1 domains and L^p data, if $1 < p < \infty$ then Fabes and Riviere [3] obtained unique solutions u of (I.D.P.) given by *double-layer heat potentials* of a suitable transform of the boundary data.

An attempt to extend these potential methods and results to a more general class of integrable data f on $\partial D\times(0, T)$, as done for the Laplace equation in [2], should start with th study of (I.D.P.) for a *dual* set of data on the lateral surface. Such a class, might turn out to be the subspace B_0MOC—defined below—of the class BMOC of functions with *caloric bounded mean oscillation* on $\partial D\times(0, T)$. The main result in this work is the boundedness of the caloric singular integral operator J (see [3]) on this subspace. It is presented in order to show certain differences from the corresponding steady-state situation (i.e., the Dirichlet Problem for $\Delta u = 0$), and to point out the direction of some new research already in progress. We warmly thank Professor Eugene Fabes for his generous help and continuing interest in this project.

1. Definitions and preliminaries.

We shall consider in space-time $\mathbb{R}^n\times\mathbb{R}^+$, $n\geq 2$, the *cylinders*

$$D_+ = D\times\mathbb{R}^+ \quad \text{and} \quad D_T = D\times(0, T), \quad \text{for } 0 < T < \infty,$$

with *lateral boundaries*

$$S_+ = \partial D\times\mathbb{R}^+ \quad \text{and} \quad S_T = \partial D\times(0, T)$$

and cross-section D, a bounded C^1 domain in $\mathbb{R}^n$. Capital letters X, Y will denote points in D (or $\mathbb{R}^n$), while P, Q will denote points of ∂D. Letters t and s are used for time variables in $\mathbb{R}^+$. Along ∂D, we have a continuous vector field N_Q, the *inner unit normal* at $Q\in\partial D$. For all $(X, t)\in\mathbb{R}^n\times\mathbb{R}^+$, we let

$$\Gamma(X, t) = (\pi t)^{-n/2}\exp\left(-|X|^2/4t\right)$$

denote the fundamental solution of the heat equation. The kernel

$K(X, t)$ of the double-layer heat potential is just the (spatial) normal derivative of Γ; that is,

$$K(X, t) = \langle \nabla_X \Gamma(X, t), N_Q \rangle \tag{1.0}$$

where $\langle\ ,\ \rangle$ is the euclidean inner product. Consequently, from well-known estimates for Γ and its first partials it follows that (with various positive constants C which may depend also on n)

$$|K(X, t)| \leqslant |\nabla_X \Gamma(X, t)| \leqslant C \begin{cases} t^{-(n+1)/2} \\ |X|^{-(n+1)} \end{cases} \tag{1.1}$$

$$|\nabla_X K(X, t)| \leqslant C \begin{cases} t^{-(n+2)/2} \\ |X|^{-(n+2)} \end{cases} \tag{1.2}$$

and

$$|D_t K(X, t)| \leqslant C \begin{cases} t^{-(n+3)/2} \\ |X|^{-(n+3)} \end{cases} \tag{1.2'}$$

and so on.

The kernel $k(P, Q) = \langle P - Q, N_Q \rangle |P - Q|^{-n}$ of the *harmonic* double-layer potential satisfies two basic properties on the boundary ∂D of a C^1 domain (see [1]):

(i) the truncations $\int\limits_{|P-Q|>\varepsilon} k(P, Q)\, d\sigma(Q)$ are uniformly bounded for all $P \in \partial D$ and $\varepsilon > 0$;

(ii) $\text{p.v.}\int\limits_{\partial D} k(P, Q)\, d\sigma(Q) = \lim\limits_{\varepsilon \to 0} \int\limits_{|P-Q|>\varepsilon} k(P, Q)\, d\sigma(Q) = \frac{\omega_n}{2}$

for all $P \in \partial D$, where ω_n = area of the unit sphere in $\mathbf{R}^n$. These properties are very helpful in the study of the boundary values of double layer heat potentials as well.

To begin with, let us consider the singular integrals

$$\begin{cases} J_1(P, t) = \lim\limits_{\varepsilon \to 0} \int\limits_0^{t-\varepsilon} \int\limits_{\partial D} \dfrac{\langle P - Q, N_Q \rangle}{(t-s)^{n/2+1}} \exp\left(\dfrac{-|P-Q|^2}{4[t-s]}\right) dQ\, ds \\ \quad \equiv \lim\limits_{\varepsilon \to 0} J_{1,\varepsilon}(P, t) \end{cases} \tag{1.3}$$

for each $(P, t) \in \partial D \times \mathbb{R}^+$ and any $0 < \varepsilon < t$, where (for brevity) we let $dQ = d\sigma(Q)$ denote the surface measure on ∂D. Here, we have

$$J_{1.\varepsilon}(P, t) = \lim_{\eta \to 0} \int_{|P-Q|>\eta} \langle P-Q, N_Q \rangle \left\{ \int_0^{t-\varepsilon} \exp\left(\frac{-|P-Q|^2}{4[t-s]} \right) \frac{ds}{(t-s)^{n/2+1}} \right\} dQ .$$

For each $(P, t) \in S_+$ and $Q \in \partial D$, the change of variable

$$\tau = (t-s)\, |P-Q|^{-2}, \quad \text{in} \quad \{...\} \text{ above,}$$

leads to the formula

$$\text{(iii)} \quad J_{1,\varepsilon}(P, t) = \lim_{\eta \to 0} \int_{|P-Q|>\eta} k(P, Q) \left\{ \int_{\varepsilon|P-Q|^{-2}}^{t|P-Q|^{-2}} \exp\left(\frac{-1}{4\tau} \right) \tau^{n/2-1} d\tau \right\} dQ$$

where, as before, $k(P, Q) = \langle P-Q, N_Q \rangle\, |P-Q|^{-n}$. Note that for all $0 < \varepsilon < t$, the inner integrals are positive and less than

$$H(0) = \int_0^\infty \exp\left(-\frac{1}{4\tau} \right) \tau^{n/2-1}\, d\tau < \infty .$$

Since $|k(P, Q)| \leqslant |P-Q|^{1-n}$, it follows that for each fixed $\eta > 0$ the surface integral in (iii) is absolutely convergent. Thus, we may let $\varepsilon \to 0$ inside $\{...\}$ and redefine the function $J_1(P, t)$ as the (conditionally convergent) iterated integral

$$(1.4) \qquad J_1(P, t) = \int_{\partial D} k(P, Q) \left\{ \int_0^{t|P-Q|^{-2}} \exp(-1/4\tau)\, \tau^{n/2-1} d\tau \right\} dQ .$$

LEMMA 1.0. The function $J_1(P, t)$ is bounded on $S_+ = \partial D \times \mathbb{R}^+$.

PROOF. Letting $s = 1/4\tau$ in (1.4) yields

$$\text{(iv)} \quad J_1(P, t) = c_n \int_{\partial D} k(P, Q) \left\{ \int_{|P-Q|^2/4t}^{\infty} \exp(-s)\, s^{n/2-1} ds \right\} dQ$$

where all inner integrals are positive and less than $\Gamma(n/2)$. For each $t > 0$, we split the surface integral into two parts:

$$\int_{|P-Q|>2\sqrt{t}} \dots dQ + \int_{|P-Q|<2\sqrt{t}} \dots dQ \equiv A + B .$$

To estimate A, note that

$$\exp(-s) < \exp(-3s/4) \exp(-|P-Q|^2/4t)$$

if $s > |P-Q|^2/4t$, so that—with new constant $c_n > 0$—

$$|A| \leqslant c_n \int_{|P-Q|>2\sqrt{t}} |P-Q|^{1-n} \exp\left(\frac{-|P-Q|^2}{4t}\right) dQ .$$

Let now $r_0 =$ the radius of the balls giving local coordinates for D in the definition of C^1 domain (e.g. [1]). We distinguish two cases.

Case 1. If $|P-Q| \geqslant r_0$, then $|P-Q|^{1-n} \leqslant r_0^{1-n}$ while $\exp(\dots) < < \exp(-1)$ on the region of integration. Hence,

$$|A| \leqslant c_n r_0^{1-n} \exp(-1) \sigma(\partial D)$$

which is a geometric constant depending only on D and n.

Case 2. If $|P-Q| < r_0$, then we are inside a coordinate neighborhood with $P = (x, \varphi(x))$, $Q = (y, \varphi(y))$ etc. and we have again

$$|A| \leqslant c_n (2\sqrt{t})^{1-n} \int_{\mathbf{R}^{n-1}} \exp\left(\frac{-|x-y|^2}{4t}\right) dx = C_n$$

since the Gauss-Weierstrass kernel is an approximate identity.

In order to estimate B, we can first rewrite it in the form

$$B = \Gamma(n/2) \int_{|P-Q|<2\sqrt{t}} k(P, Q)\, dQ - \int_{|P-Q|<2\sqrt{t}} k(P, Q) \left\{ \int_0^{|P-Q|^2/4t} \exp(-s)\, s^{n/2-1}\, ds \right\} dQ$$

$$= B_1 - B_2, \text{ say.}$$

But, by formulas (i) and (ii) above, we have

$$\Gamma(n/2)^{-1}B_1 = \frac{\omega_n}{2} - \int\limits_{|P-Q|>2\sqrt{t}} k(P,Q)\,dQ$$

and the truncated integrals are uniformly bounded in $\varepsilon = 2\sqrt{t}$. For B_2, integrating $s^{n/2-1}ds$ yields (with another $c_n > 0$)

$$|B_2| \leqslant c_n \int\limits_{|P-Q|<2\sqrt{t}} |P-Q|^{1-n} \frac{|P-Q|^n}{(2\sqrt{t})^n}\,dQ = c_n(2\sqrt{t})^{-n} \int\limits_{|P-Q|<2\sqrt{t}} |P-Q|\,dQ$$

$$\leqslant 2c_n(2\sqrt{t})^{1-n}\sigma(\partial D)\,, \qquad \text{another geometric constant.} \qquad \text{QED}.$$

The anisotropic BMO space corresponding to the $|X|^2/t$ homogeneity of the heat equation is defined in terms of integral averages over the following sets in $S_+ = \partial D \times \mathbb{R}^+$: if $(Q,t) \in S_+$ and $r > 0$, then

$$(1.5) \qquad \Delta = \Delta_r(Q,t) = \{(P,s) \in S_+ : |P-Q| < r,\ |s-t| < r^2\}$$

is called a *caloric surface disc* with center (Q,t) and radius r.

Definition 1.1. A function $f \in L^1_{\mathrm{loc}}(S_+)$ has *caloric bounded mean oscillation* on S_+—in symbols, $f \in \mathrm{BMOC}(S_+)$—if

$$(1.6) \qquad \|f\|_* = \sup_{\Delta} \left\{ |\Delta|^{-1} \int\limits_{\Delta} |f - f_\Delta|\,d\sigma_+ \right\} < \infty$$

where the sup is taken over all discs Δ as in (1.5), $|\Delta| = \sigma_+(\Delta)$, and

$$f_\Delta = |\Delta|^{-1} \int\limits_{\Delta} f\,d\sigma_+ = |\Delta|^{-1} \int\limits_{\Delta} f\,dQ\,dt$$

letting $d\sigma_+ = d\sigma(Q)\,dt = dQ\,dt$, as before.

As usual, BMOC functions are determined up to additive constants. Since (see below) the integral operator J to be studied does not preserve constants, but transforms them into bounded functions, we are led to the following notion.

DEFINITION 1.2. A function $f \in L^1_{\mathrm{loc}}(S_+)$ has *bounded behavior at* $t = 0$ if

$$B_0(f) = \sup_{\Delta} \left| r^{-n-1} \int_0^{r^2} \int_{S_r} f \, d\sigma \, dt \right| < \infty \tag{1.7}$$

where the sup is taken over all surface balls

$$S_r = \{P \in \partial D : |P - P_0| < r\} \quad \text{in } \partial D, P_0 \in \partial D .$$

The function space

$$B_0MOC(S_+) = \{f \in BMOC(S_+) : B_0(f) < \infty\} \tag{1.8}$$

of all caloric BMO functions with bounded behavior at $t = 0$, will be equipped with the complete norm

$$\|f\|_{0,*} = B_0(f) + \|f\|_* . \tag{1.8'}$$

Averaging the estimate $|f|^p \leqslant 2^{p-1}[|f - f_\Delta|^p + |f_\Delta|^p]$ only over those surface discs $\Delta = S_r \times (0, r^2)$, as in (1.7), it is easy to see that $B_0(f)$ is finite if and only if

$$C_p(f) = \left\{ |\Delta|^{-1} \iint_{\Delta} |f|^p \, d\sigma \, dt \right\}^{1/p} < \infty, \quad \text{for any } 1 \leqslant p < \infty . \tag{1.9}$$

In fact, the anisotropic John-Nirenberg Inequality for BMOC implies that $\|f\|_*$ is equivalent to

$$\|f\|_{*,p} = \sup_{\Delta} \left\{ |\Delta|^{-1} \int_{\Delta} |f - f_\Delta|^p \, d\sigma_+ \right\}^{1/p} .$$

Consequently,

$$\|f\|_{p,*} \equiv C_p(f) + \|f\|_{*,p}, \quad 1 \leqslant p < \infty, \tag{1.10}$$

are all equivalent norms for B_0MOC. In particular, for $p = 1$, we see that the property of bounded behavior of an $f \in BMOC$ is not due to eventual cancellations of $f(P, t)$ as $t \to 0^+$, since f could be replaced by $|f|$ in formula (1.7).

2. Continuity of the heat potential on B_0MOC.

The boundary values of the double-layer heat potential, with L^p density f on $S_+ = \partial D \times \mathbf{R}^+$, give rise to the singular integral operator

$$(2.0) \qquad \begin{cases} [Jf](P,t) = \int_0^t \int_{\partial D} K(P,Q;t-s)\, f(Q,s)\, dQ\, ds \\ \equiv \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_{\partial D} \ldots = \lim_{\varepsilon \to 0} [J_\varepsilon f](P,t) \end{cases}$$

with kernel

$$(2.1) \qquad K(P,Q;t-s) = c_n \frac{\langle P-Q, N_Q \rangle}{(t-s)^{n/2+1}} \exp\left(\frac{-|P-Q|^2}{4[t-s]}\right)$$

for all P, Q in ∂D and $0 < s < t < \infty$. In [3], it is shown—explicitly if $p = 2$, but the general case then follows by standard techniques of singular integrals—that J defines a bounded operator on $L^p(S_+)$ for all $1 < p < \infty$. Using this result, we will show here that J is also bounded on B_0MOC(S_+).

Note first that for constant data $f = c$, we have

$$[Jc]\,(P,t) = c[J(1)]\,(P,t) = cJ_1(P,t)$$

the function in (1.4) which, by Lemma 1.0, is bounded on S_+. Let us rewrite it in the form

$$(2.2) \qquad J_1(P,t) = c_n \int_{\partial D} k(P,Q) \left\{ \int_{|P-Q|^2/4t}^{\infty} \exp(-s)\, s^{n/2-1} ds \right\} dQ$$

where, as before

$$(2.2') \qquad k(P,Q) = \langle P-Q, N_Q \rangle\, |P-Q|^{-n}.$$

Let us show that, for large times, $J_1(P,t)$ becomes asymptotically flat, at a given rate over caloric surface discs of $\partial D \times \mathbf{R}^+$.

LEMMA 2.1. For each $\Delta = S_r \times (t_0, t_0 + r^2)$ there exists a constant $C(\Delta) > 0$ such that, if $t_0 > r^2$ then

$$(2.3) \qquad |J_1(P, t) - C(\Delta)| \leqslant c_n r / t_0^{\frac{1}{2}}, \qquad \text{for all } (P, t) \in \Delta\,,$$

with some c_n independent of P, t and Δ.

PROOF. Let $t_0 > r^2$ and split J_1 as follows:

$$J_1(P, t) = [J_1(P, t) - J_1(P, t_0)] + J_1(P, t_0)\,.$$

For all $(P, t) \in \Delta$, $t_0 < t < 2t_0$. So, by (2.2), (2.2′) and the Mean Value Theorem we have—for varoius constants c_n—that

$$|J_1(P, t) - J_1(P, t_0)| \leqslant c_n \int_{\partial D} |P - Q^{1-n} \left\{ \int_{|P-Q|^2/t}^{|P-Q|^2/t_0} \exp(-s)\, s^{n/2-1} ds \right\} dQ$$

$$\leqslant c_n \int_{\partial D} |P - Q|^{1-n} \exp\left(-|P - Q|^2/t_0\right) \left(\frac{|P - Q|^2}{t_0}\right)^{n/2-1} |P - Q|^2 (t_0^{-1} - t^{-1})\, dQ \leqslant$$

$$\leqslant c_n \frac{r^2}{t_0} \left\{ \frac{1}{t_0^{(n-1)/2}} \int_{\partial D} \frac{|P - Q|}{t_0^{\frac{1}{2}}} \exp\left(\frac{-|P - Q|^2}{t_0}\right) dQ \right\}.$$

Since $\dim(\partial D) = n - 1$, an affine change of variables shows that the term $\{\ldots\}$ is uniformly bounded in P and t_0. Hence, since $r^2 < t_0$, we see that for each $P \in \partial D$ and $t \in (t_0, t_0 + r^2)$,

$$\text{(i)} \qquad |J_1(P, t) - J_1(P, t_0)| \leqslant c_n r^2 / t_0 < c_n r / t_0^{\frac{1}{2}}.$$

Let $S_r = \{P \in \partial D\colon |P - P_0| < r\}$ and recall that

$$\text{p.v.} \int_{\partial D} k(P, Q)\, dQ = \frac{\omega_n}{2} \qquad \text{and} \qquad \int_0^\infty \exp(-s)\, s^{n/2-1} ds = \Gamma\left(\frac{n}{2}\right).$$

From (2.2) and (2.2′), we may write, with $C_n = \Gamma(n/2)\, \omega_n / 2$,

$$J_1(P, t_0) = C_n - \int_{\partial D} k(P, Q) \left\{ \int_0^{|P-Q|^2/t_0} \exp(-s)\, s^{n/2-1} ds \right\} dQ$$

and then split the surface integral over $|P_0 - Q| < 5r$ and $|P_0 - Q| > 5r$, say. Setting

$$A(P) \equiv \int_{|P_0-Q|<5r} k(P,Q) \left(\int_0^{|P-Q|^2/t_0} \exp(-s)\, s^{n/2-1}\, ds \right) dQ \,,$$

$$B(P) \equiv \int_{|P_0-Q|>5r} k(P,Q) \left(\int_0^{|P_0-Q|^2/t_0} \ldots ds \right) dQ \,, \text{ and}$$

$$C(P) \equiv \int_{|P_0-Q|>5r} k(P,Q) \left(\int_0^{|P-Q|^2/t_0} \ldots ds - \int_0^{|P_0-Q|^2/t_0} \ldots ds \right) dQ$$

we have to estimate

(ii) $J_1(P, t_0) = C_n - A(P) - B(P) - C(P)$, with $|P - P_0| < r$.

For $A(P)$, since $|P - Q| < 6r$ and $|k(P,Q)| \leqslant |P - Q|^{1-n}$, integrating $s^{n/2-1}\, ds$ we see (for various constants c_n) that

$$|A(P)| \leqslant c_n \int_{|P-Q|<6r} |P - Q|^{1-n} \frac{|P-Q|^n}{t_0^{n/2}}\, dQ \leqslant c_n \frac{r}{t_0^{n/2}} \int_{|P-Q|<6r} dQ \leqslant c_n \frac{r^n}{t_0^{n/2}} < c_n \frac{r}{t_0^{1/2}}$$

since $(r/t_0^{\frac{1}{2}}) < 1$ here. For $C(P)$, noting that $|P - P_0| < r$ whereas $|Q - P_0| > 5r$, and using the Mean Value Theorem, we see that $|C(P)|$ is majorized by

$$c_n \int_{|P_0-Q|>5r} |P_0 - Q|^{1-n} \left(\exp(-|P_0 - Q|^2/t_0) \left(\frac{|P_0 - Q|^2}{t_0} \right)^{n/2-1} \right) \frac{|P_0 - Q|}{t_0} \cdot$$
$$\cdot |P - P_0|\, dQ \,.$$

Hence, as before, we obtain

$$|C(P)| \leqslant c_n \frac{r}{t_0^{1/2}} \left\{ \frac{1}{t_0^{(n-1)/2}} \int_{\partial D} \exp\left(\frac{-|P_0 - Q|^2}{t_0} \right) dQ \right\} \leqslant c_n \frac{r}{t_0^{1/2}} \,.$$

Next, in view of (ii) and of the desired estimate (2.3), we induce the

« constant »

$$C(\Delta) \equiv C_n - \int_{|P_0-Q|>5r} k(P_0, Q) \left\{ \int_0^{|P_0-Q|^2/t_0} \exp(-s)\, s^{n/2-1}\, ds \right\} dQ$$

which depends also on the parameters P_0, t_0 and r. Thus, it remains to estimate the integral

$$\int_{|P_0-Q|>5r} [k(P, Q) - k(P_0, Q)] \left\{ \int_0^{|P-Q|^2/t_0} \dots ds \right\} dQ =$$

$$= \int_{5r<|P_0-Q|<5\sqrt{t_0}} [\dots]\{\dots\} + \int_{|P_0-Q|\geqslant 5\sqrt{t_0}} [\dots]\{\dots\} \equiv B_1 + B_2 .$$

Estimating $\{\dots\}$ as done for $A(P)$ and $[\dots]$ by the Mean Value Theorem, we see that

$$|B_1| \leqslant c_n |P - P_0| \int_{|P_0-Q<5\sqrt{t_0}} |P_0 - Q|^{-n} \frac{|P_0 - Q|^n}{t_0^{n/2}} dQ \leqslant c_n \frac{r}{t_0^{\frac{1}{2}}}$$

for some new c_n. Finally, we have

$$|B_2| \leqslant c_n |P - P_0| \int_{|P_0-Q|>5\sqrt{t_0}} |P_0 - Q|^{-n} \left\{ \int_0^\infty \dots ds \right\} dQ < c_n r \Gamma(n/2) \int_{|P_0-Q|>5\sqrt{t_0}} |P_0 - Q|^{-n} dQ \leqslant c_n \frac{r}{t_0^{\frac{1}{2}}}$$

for another c_n, and the proof of the lemma is complete. QED.

For any $g \in L^2_{\text{loc}}(S_+)$, we set (as in (1.9) above)

$$C_2(g) = \sup \left\{ |\Delta|^{-1} \iint_\Delta |g|^2 d\sigma\, dt \right\}^{\frac{1}{2}} \tag{2.4}$$

with sup taken over all $\Delta = S_r \times (0, r^2)$ and $0 < 2r \leqslant \operatorname{diam}(\partial D)$. Then, the operator J in (2.0) satisfies the following property.

LEMMA 2.2. There is a constant $M > 0$ such that

$$C_2(Jf) \leqslant M C_2(f), \qquad \text{for all } f \in L^2_{\text{loc}}(S_+) . \tag{2.5}$$

PROOF. Fix such a Δ and denote by χ_1 the characteristic function of $\Delta^* = S_{2r}\times(0, 4r^2)$. Then,

$$r^{-n-1}\int_0^{r^2}\int_{S_r}|J(f\chi_1)|^2\,d\sigma\,dt \leqslant r^{-n-1}\iint_{S_+}|J(f\chi_1)|^2\,d\sigma\,dt \leqslant C^2\,r^{-n-1}\int_0^{4r^2}\int_{S_{2r}}|f|^2\,d\sigma\,dt$$

by the L^2-estimate in [3], Theorem (1.1). Hence, (2.5) holds for $f\chi_1$. Since $f = f\chi_1 + f\chi_2$ with $\chi_2 = 1 - \chi_1$, it suffices to show that there is a constant $M > 0$ such that

$$(2.5') \qquad |J(f\chi_2)(P, t)| \leqslant MC_2(f), \qquad \text{for all } P \in S_r \text{ and } 0 < t < r^2 .$$

If S_r has center $P_0 \in \partial D$, then $|P_0 - Q| > 2r$ for all $(Q, s) \notin \Delta^*$ with $0 < s < t$, and also $|P - Q| \approx |P_0 - Q|$. Hence, the left-side of (2.5′) is dominated by

$$\int_0^t\int_{|P_0-Q|>2r}|P_0 - Q|\exp\left(\frac{-|P_0-Q|^2}{t-s}\right)(t-s)^{-(n+2)/2}|f(Q, s)|\,dQ\,ds .$$

For each Q here, let $A = |P_0 - Q|^2 > 4r^2$ and $\tau = (t - s)$, so $0 < \tau < r^2$) Since $\exp(-A/\tau)\,\tau^{-(n+2)/2}$ is strictly increasing on $0 < \tau < 2A/(n+2)$ and $2A/(n+2) > 8r^2/(n+2) = c_n r^2$, it follows that on $S_r\times(0, r^2)$

$$|J(f\chi_2)| \leqslant cr^{-n-1}\int_0^{r^2}\int_{|P_0-Q|>2r}(|P_0-Q|r^{-1})\exp(-|P_0-Q|^2r^{-2})|f(Q, s)|\,dQ\,ds \leqslant$$

$$\leqslant c\sum_k r^{-n-1}\int_0^{r^2}\int_{|P_0-Q|\approx 2^k r}\ldots\,dQ\,ds \leqslant c\sum_k 2^{k(n+2)}\exp(-4^k)\left\{(2^k r)^{-n-1}\int_0^{(2^k r)^2}\int_{|P_0-Q|<2^k r}|f|\,dQ\,ds\right\}.$$

So, by Schwarz's inequality, we obtain

$$|J(f\chi_2)(P, t)| \leqslant cC_2(f)\sum_k 2^{k(n+2)}\exp(-4^k) \equiv MC_2(f) \qquad \text{QED} .$$

THEOREM 2.3 The operator J is bounded on $\mathrm{B_0MOC}(S_+)$.

PROOF. Let $f \in \mathrm{B_0MOC}(S_+)$ and fix any disc $\Delta = S_r \times (t_0, t_0 + r^2)$ in S_+. If $t_0 \leqslant 8r^2$, say, we observe that, by (2.5) above,

$$r^{-n-1}\int_{t_0}^{t_0+r^2}\int_{S_r}|Jf|^2 \leqslant \frac{c_n}{r^{n+1}}\int_0^{9r^2}\int_{S_{3r}}|Jf|^2 \leqslant C_n M^2 C_2(f)^2 .$$

Hence, for all such «initial discs» Δ, we have

$$|\Delta|^{-1}\iint_{\Delta}|Jf|\, d\sigma\, dt \leqslant C\cdot C_2(f) \leqslant C_0\|f\|_{0,*} \tag{2.6}$$

for some constant $C_0 > 0$ independent of f and Δ. In particular, Jf has bounded behavior at $t = 0$.

If $t_0 > 8r^2$, it is more convenient to write Δ in the form:

$$\Delta = \Delta_r(P_0, t_0) = S_r \times (t_0 - r^2, t_0 + r^2) ,$$

where
$$S_r = \{Q \in \partial D\colon |P_0 - Q| < r\}$$

with $P_0 \in \partial D$, and to let $\Delta^* = \Delta_{2r}(P_0, t_0) = S_{2r} \times (t_0 - 4r^2, t_0 + 4r^2)$. Applying Lemma 2.1 (with an obvious change of notation), there exists a constant $C(\Delta^*)$ such that, $\forall (P, t) \in \Delta$ and with various $c_n > 0$,

$$|f_{\Delta^*}||J_1(P, t) - C(\Delta^*)| \leqslant \frac{C_n}{\sqrt{t_0}}(2r)^{-n}\iint_{\Delta^*}|f| \leqslant \frac{C_n}{\sqrt{t_0}}\left\{\iint_{\Delta^*}|f|^{n+1}\right\}^{1/(n+1)}$$

by Hölder's inequality (with $p = n + 1$, so $1/p' = n/(n + 1)$ and $|\Delta^*|^{n/(n+1)} \approx (2r)^n$). Since $\sqrt{2t_0} > 2r$, it follows that

$$t_0^{-\frac12}\left\{\iint_{\Delta^*}|f|^{n+1}\right\}^{1/(n+1)} \leqslant \left\{t_0^{-(n+1)/2}\int_0^{2t_0}\int_{|P_0-Q|<2\sqrt{t_0}}|f|^{n+1}dQ\, ds\right\}^{1/(n+1)} \leqslant C_p(f)$$

with $p = (n + 1)$. Therefore, for all Δ in question, we have

$$\sup_{(P,t)\in\Delta}\{|f_{\Delta^*}|\cdot|J_1(P, t) - C(\Delta^*)|\} \leqslant C_1\|f\|_{0,*} \tag{2.7}$$

for some $C_1 > 0$, independent of f and Δ.

Next, we let $\chi_1 =$ the characteristic function of Δ^*,

$$f_1 = [f - f_{\Delta^*}]\,\chi_1,\; f_2 = [f - f_{\Delta^*}]\,(1 - \chi_1)\,,$$

and choose constant $Jf(\Delta)$ given by

$$Jf(\Delta) = Jf_2(P_0, t_0) + f_{\Delta *}C(\Delta^*)\,. \tag{2.8}$$

Since $f = f_{\Delta^*} + f_1 + f_2$ and $J(c) = cJ_1(P, t)$, we have

$$Jf(P, t) - Jf(\Delta) = f_{\Delta^*}[J_1(P, t) - C(\Delta^*)] + Jf_1 + [Jf_2 - Jf_2(P_0, t_0)]\,.$$

As before, by Theorem (1.1) of [3],

$$|\Delta|^{-1}\iint_{\Delta} |Jf_1|^2\, d\sigma\, dt \leqslant C|\Delta|^{-1}\iint_{\Delta^*} |f - f_{\Delta^*}|^2\, d\sigma\, dt \leqslant C_n\|f\|_*^2\,.$$

Hence, by (2.7) and Schwarz's inequality,

$$|\Delta|^{-1}\iint_{\Delta} |Jf - Jf(\Delta)|\, d\sigma\, dt \leqslant C_1\|f\|_{0,*} + C_2\|f\|_* + B$$

where, with $dP = d\sigma(P)$ as usual,

$$B = |\Delta|^{-1}\iint_{\Delta} |Jf_2(P, t) - Jf_2(P_0, t_0)|\, dP\, dt\,. \tag{2.8'}$$

In remains to show that $B \leqslant C_3\|f\|_*$, for some $C_3 > 0$ independent of Δ and f.

The integrand of (2.8′) is majorized by

$$\int_{\substack{S_+ \setminus \Delta^* \\ 0<s<t}} |K(P, Q; t - s) - K(P_0, Q; t_0 - s)|\,|f_2(Q, s)|\, dQ\, ds$$

where the kernel $K(P, Q; t - s)$ is defined in (2.1). Adding and subtracting $K(P, Q; t_0 - s)$ and using the Mean Value Theorem, we see that

$$|K(P, Q; t - s) - K(P_0, Q; t_0 - s)| \leqslant$$
$$\leqslant |D_t K(P, Q; \tilde{t} - s)| \cdot |t - t_0| + |\nabla_P K(\tilde{P}, Q; t_0 - s)| \cdot |P - P_0| \equiv B_1 + B_2$$

for some $\tilde{t}$ between t and t_0, and some intermediate point $\tilde{P}$ between P_0 and P. Thus, substituting into (2.8′) and using Minkowski's integral inequality, we get

$$B \leqslant \int_{\substack{S^+ \setminus \Delta^* \\ 0<s<t}} |f - f_{\Delta^*}| \left\{ |\Delta|^{-1} \iint_{\Delta} (B_1 + B_2)\, dP\, dt \right\} dQ\, ds \tag{2.9}$$

To get some uniform estimates on B_1 and B_2, note that for any $(P, t) \in \Delta$, $|t - t_0| < r^2$ and $|P - P_0| < r$. So, by (1.2) and (1.2′), we have:

$$|t - t_0| < r^2 \qquad \text{implies } B_1 \leqslant C_1 \begin{cases} r^2 |\tilde{t} - s|^{-(n+3)/2}, & \text{for } |P - Q| < r \\ r^2 |P - Q|^{-(n+3)}, & \text{for } |P - Q| \geqslant r, \end{cases}$$

and

$$|P - P_0| < r \qquad \text{implies } B_2 \leqslant C_2 \begin{cases} r |t_0 - s|^{-(n+2)/2}, & \text{for } |P_0 - Q| < 2r \\ r |\tilde{P} - Q|^{-(n+2)}, & \text{for } |P_0 - Q| \geqslant 2r. \end{cases}$$

But, for all $(P, t) \in \Delta$ and all $(Q, s) \notin \Delta^*$, if $|P - Q| < r$ or if $|P_0 - Q| \geqslant 2r$ then $|t_0 - s| > 4r^2$. From this, it follows easily that in all cases

(i) $B_i \leqslant C_i r^{-n-1}$, for all $(P, t) \in \Delta$ and $(Q, s) \notin \Delta^*$, where $i \in \{1, 2\}$.

Now, for all $j \in \mathbf{N}$, consider the dyadic dilations $\Delta_j \equiv \Delta_{2^j r}(P_0, t_0)$, so that $\Delta_1 = \Delta^*$ and, if $j \geqslant 2$, Δ_j is the concentric caloric surface disc with sizes $(2^j r; 2^{2j} r^2)$. As before, we note that, for all $j \geqslant 2$,

$$\text{if } |P_0 - Q| \geqslant 2^{j-1} r, \quad \text{then } |\tilde{P} - Q| \geqslant (2^{j-1} - 1) r,$$

whereas

$$\text{if } |P_0 - Q| < 2^{j-1} r \quad \text{but } (Q, s) \notin \Delta_{j-1}, \quad \text{then } |t_0 - s| \geqslant (2^{j-1} r)^2.$$

Moreover,

$$\text{if } |P - Q| < 2^{j-1} r, \quad \text{then } |\tilde{t} - s| \geqslant [2^{2(j-1)} - 1] r^2.$$

So, for all $j \geqslant 2$,

$$\text{(ii)} \quad \begin{cases} B_1 \leqslant C_1 r^2 [2^{2(j-1)} - 1]^{-(n+3)/2} r^{-(n+3)} \\ B_2 \leqslant C_2 r [2^{j-1} - 1]^{-(n+2)} r^{-(n+2)} \end{cases}$$

for all $(P, t) \in \Delta$ and $(Q, s) \notin \Delta_{j-1}$. Simplifying constants, (ii) yields

(iii) $B_1 + B_2 \leqslant C 2^{-(j-1)(n+2)} r^{-n-1}$, for all $(P, t) \in \Delta$ and $(Q, s) \notin \Delta_{j-1}$.

Substituting (iii) into (2.9) and recalling that $\Delta^* = \Delta_1$, we obtain

$$B \leqslant \sum_{j \geqslant 2} \int_{\Delta_j \setminus \Delta_{j-1}} |f - f_{\Delta^*}| \left\{ |\Delta|^{-1} \iint_{\Delta} (B_1 + B_2) \, dP \, dt \right\} dQ \, ds$$

$$\leqslant C \sum_{j \geqslant 2} \{2^{-(j-1)(n+2)} r^{-n-1}\} \int_{\Delta_j} |f - f_{\Delta_1}| \, dQ \, ds .$$

Since the discs Δ_j have measure equivalent to $2^{n+1} 2^{(j-1)(n+1)} r^{n+1}$, we deduce that, for some new $C > 0$,

$$B \leqslant C \sum_{j \geqslant 2} 2^{-(j-1)} |\Delta_j|^{-1} \int_{\Delta_j} |f - f_{\Delta_1}| \, dQ \, ds . \tag{2.10}$$

Moreover, by standard BMO argument, we have that

$$|f - f_{\Delta_1}| \leqslant |f - f_{\Delta_j}| + |f_{\Delta_j} - f_{\Delta_1}| \leqslant |f - f_{\Delta_j}| + (j-1) 2^{n+1} \|f\|_* .$$

Thus, substituting into (2.10), we conclude at once that $B \leqslant C_3 \|f\|_*$. This completes the proof of Theorem 2.3.

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