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Periodic solutions of Liénard equations

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Periodic Solutions of Liénard Equations.

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1. Introduction.

In this paper we extend and sharpen ealier results ([23], [38]) concerning the existence of p-periodic solutions of the Liénard vector equation in $\mathbf{R}^m$

$$(1.1) \qquad x''(t) + (d/dt)\varphi(x(t)) + Ag(x(t)) = h(t)\,, \qquad (' = d/dt)$$

under the following basis hypotheses which will be assumed throughout the paper: $\varphi\colon \mathbf{R}^m \to \mathbf{R}^m$ *is a* C^1*-map*, $g\colon \mathbf{R}^m \to \mathbf{R}^m$ *is continuous*, A *is a* $m\times m$ (possibly singular) *constant matrix*, $h\colon \mathbf{R} \to \mathbf{R}^m$ *is continuous and* p*-periodic.*

It is well known that Liénard equations are considered in several problems in mechanics, engineering and electrical circuits theory. Although the question of the existence of periodic solutions represents only a first step in the study of the nonlinear oscillations of (1.1), nevertheless the problem has been extensively studied in the literature, for its physical significance [34], [33], [28], [31], [32]. The present work provides an extension, to differential systems, of the following classical result, concerning the existence of p-periodic solutions of the scalar equation

$$(1.1') \qquad x''(t) + f(x(t))x'(t) + g(x(t)) = h(t)\,,$$

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with $f, g, h\colon \mathbb{R} \to \mathbb{R}$ continuous and h p-periodic. Obviously (1.1′) is a special case of (1.1), with $\varphi(x) := \int_0^x f(s)\, ds$.

THEOREM (Mizohata and Yamaguti [21]). *Let us assume*

$$\lim_{|x|\to\infty} \varphi(x)\cdot \operatorname{sign}(x) = +\infty(-\infty) \quad \textit{and} \quad g(x)\cdot\operatorname{sign}(x) > 0, \qquad \textit{for } |x| \geqslant d\,.$$

Then (1.1′) *has a p-periodic solution for any h with* $\int_0^p h(s)\, ds = 0$.

Related results, for scalar equations, were obtained in the fifties, by Reuter [29] and by Levinson [13], De Castro [33], Newman [22], Cartwright and Littlewood [4], Opial [25] and others (see [33] or [28] for a thorough survey). All these authors prove, at first, the boundedness in the future, or in the past, of the solutions of the Cauchy Problems; then they apply the Massera Theorem, or similar statements, in order to get the existence of p-periodic solutions of (1.1′) (see [11], [36] for the standard boundedness results).

More recently, topological degree tools have been used, even under dissipativity type conditions for (1.1) or (1.1′), by Mawhin [15], [18], [17], [32, Ch. XI], Bebernes and Martelli [1], Cesari and Kannan [5], [6], Reissig [26] and Ward [35].

In our main result (Theorem 1, below) we consider a growth assumption on the damping term, we already used in [23], [38] for less general situations. Then we are able to extend the Mizohata-Yamaguti Theorem and to improve a theorem of Mawhin [17], [32, Ch. XI, Th. 6.6] which was not contained in our previous papers (quoted above).

The basic tool in the proofs is the following continuation type theorem by Mawhin [16, Th. 4, p. 944], [32 (2), Th. 3.17], recalled here, for reader's convenience, in a simpler form.

THEOREM 0. *Let* $f = f(t, x, y; \lambda)\colon \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m\times[0, 1] \to \mathbb{R}^m$ *be continuous and p-periodic in t. Let us assume there is $K > 0$ such that*

(α) $|x|_\infty + |x'|_\infty \leqslant K$, *for any* $(x, \lambda) \in C^2_p\times]0, 1[$, *solution of* $x''(t) = \lambda f(t, x(t), x'(t); \lambda)$.

(β) $\bar f(z) \neq 0$ *for* $|z| \geqslant K$ *and* $\deg(\bar f, B(0, r), 0) \neq 0$ *for every* $r \geqslant K$, *where* $\bar f(z) := (1/p)\int_0^p f(t, z, 0; 0)\, dt$.

Then the equation $x'' = f(t, x, x'; 1)$ *has a* p*-periodic solution.*

In subsequent sections, using further assumptions on the nonlinear terms of equation (1.1), we allow a dependence of φ and g on the time variable and we treat also the case of a nonconservative restoring field (not considered in Theorem 1). Conditions ensuring the uniqueness of the solutions are considered too. At last, using a version of the above recalled Mawhin's Continuation Theorem, suitable to be applied to functional differential equations (see [18, Th. 4, p. 248]), we examine the case of the Liénard equation with delays

$$x''(t) + (d/dt)\varphi(x(t)) + \int_{-r}^{s} d\eta(\theta) g(x(t+\theta)) = h(t) \tag{1.2}$$

and obtain a variant of our previously given main result, which improves [18, Th. 6], [37], [23] and [9, p. 326].

Although we restrict ourselves to the «classical» case, we point out that all the results hold under Carathéodory-type assumptions [20].

2. Notations and the main result.

Throughout the paper, the following notations are used. $\mathbb{R}^m$ is the m-dimensional real euclidean space with inner product $(\cdot|\cdot)$ and norm $|\cdot|$. $\|\cdot\|$ denotes the norm of a matrix, thought as a linear operator in $\mathbb{R}^m$ (with respect to the usual orthonormal basis), i.e. $\|\cdot\|$ is the spectral norm for matrices. $B(x, r)$ is the open ball centered at $x \in \mathbb{R}^m$ with radius $r > 0$, and $\operatorname{cl} D$ denotes the closure of a set $D \subset \mathbb{R}^m$. $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree in $\mathbb{R}^m$. $C^k(\mathbb{R}^d, \mathbb{R}^m)$ is the vector space of all the continuous maps $\mathbb{R}^d \to \mathbb{R}^m$ of class C^k and C_p^k denotes the vector space of the continuous and p-periodic vector valued functions $\mathbb{R} \to \mathbb{R}^m$, of class C^k; moreover, for $x \in C_p^0$, $|x|_\infty := \sup\{|x(t)| : t \in [0, p]\}$. In C_p^0 also the L^q-norm $|x|_q = \left(\int_0^p |x(s)|^q ds\right)^{1/q}$ $(q \geqslant 1)$ and the L^2-scalar product $(u, v)_2 = \int_0^p (u(s)|v(s))\, ds$ will be considered. Moreover, if $x \in C_p^0$, $\bar{x} := (1/p)\int_0^p x(s)\, ds$ (the mean value of x). $\mathbb{R}_+$ is the set of the nonnegative reals.

We consider the space

$$\Gamma := \{\psi \in C^0(\mathbf{R}^m, \mathbf{R}^m) | (\exists L \geqslant 0)(\exists k \geqslant 0): \\ |\psi(x) - \psi(y)| \leqslant L|x - y| + k, \text{ for all } x, y \in \mathbf{R}^m\} .$$

For a map $\psi \in \Gamma$, we define

$$v(\psi) := \inf \{L \geqslant 0 | (\exists k \geqslant 0): |\psi(x) - \psi(y)| \leqslant \\ \leqslant L|x - y| + k, \text{ for all } x, y \in \mathbf{R}^m\} .$$

We observe that Γ contains, for instance, all the uniformly continuous maps $\mathbf{R}^m \to \mathbf{R}^m$, since Γ contains any map $\psi: \mathbf{R}^m \to \mathbf{R}^m$ such that, for every $r > 0$, $\sup \{|\psi(x) - \psi(y)|: |x - y| < r\} < +\infty$.
If A is a $m \times m$ constant matrix, we define

$$A_s = (A + A^T)/2 \qquad \text{and } A_a = (A - A^T)/2 ,$$

where A^T is the transpose matrix of A.

For any fixed $h \in C^0(\mathbf{R}, \mathbf{R}^m)$, let $\Sigma(h) \subset C^1(\mathbf{R}, \mathbf{R}^m)$ be the set of all the functions H, such that $H'(t) = h(t)$, for every $t \in \mathbf{R}$. It is well known that $\Sigma \subset C^1_p$, provided that $h \in C^0_p$ and $\bar{h} = 0$ $\left(\text{i.e., } \int_0^p h(s)\, ds = 0\right)$. In this case we can define

$$d(h) = \min \{|H|_\infty: H \in \Sigma(h)\} = |H_1|_\infty ,$$

where H_1 is the unique function of lowest norm in $\Sigma(h)$. It is easily checked that the following estimate holds

$$d(h) \leqslant \sup_t \left| \int_0^t h(s)\, ds \right| \leqslant \int_0^p |h(s)|\, ds = |h|_1 .$$

Finally, for every number $M > 0$, we define

$$W(M) := \{x \in \mathbf{R}^m | (\forall z \in \mathbf{R}^m)(\exists |y_z| \leqslant M): (g(x + y_z)|z) = 0\} ,$$

where $g \in C^0(\mathbf{R}^m, \mathbf{R}^m)$ is the (nonlinear) vector field in equation (1.1). $W(M)$ is a closed set which has the following geometrical meaning:

a point $x \in \mathbf{R}^m$ belongs to $W(M)$ if and only if the continuum (i.e. compact and connected set) $g(\operatorname{cl} B(x, M))$ has non-empty intersection with every hyperplane of $\mathbf{R}^m$ passing through the origin.

With the above positions, we can state our main result.

THEOREM 1. *Let us assume* $g = \operatorname{grad} G$, *with* $G\colon \mathbf{R}^m \to \mathbf{R}$, *of class* C^1, *and* $\int_0^p h(s)\,ds = 0$. *Let* $\varphi(x) = \varphi_1(x) + \varphi_2(x)$ *and suppose* $\varphi_1 = \operatorname{grad} F$, *with* $F\colon \mathbf{R}^m \to \mathbf{R}$, *of class* C^1, *and* $\varphi_2 \in \Gamma$, *such that*

$$v(\varphi_2) < \omega \qquad (\omega = 2\pi/p)\,.$$

Moreover, let us suppose that, for every $x \in \mathbf{R}^m$,

$$\text{(j)} \qquad (\varphi(x)|g(x)) \geqslant a|g(x)|^2 + b|g(x)| - c\,,$$

holds, with

$$a \geqslant \|A_d\|/\omega\,, \qquad b > d(h)\,, \qquad c \geqslant 0\,.$$

Finally, let us assume

(w) *for any* $M > 0$, *either* $W(M)$ *is bounded, or there exists a map*

$$\psi = \psi_M \in \Gamma \quad \text{such that} \quad \lim_{\substack{|x| \to \infty \\ x \in W(M)}} |(g(x)|\psi(x))| = \infty\,,$$

and

(d) $\deg(g, B(0, R), 0) \neq 0$, *for each* $R \geqslant r > 0$.

Then equation (1.1) *has a p-periodic solution.*

PROOF. Let us fix a number $\varepsilon > 0$, such that the matrix A has no eigenvalue in the interval $]0, \varepsilon]$. Then the matrix $A(\lambda) := A - (1 - \lambda)\varepsilon I$ (with I the $m \times m$ identity matrix) is nonsingular, for every $\lambda \in [0, 1[$. Now we set, for $t \in \mathbf{R}$, $x, y \in \mathbf{R}^m$, $\lambda \in [0, 1]$,

$$f(t, x, y; \lambda) := -\varphi'(x) \cdot y - A(\lambda) g(x) + h(t)\,,$$

where $\varphi'(x)$ denotes the Jacobian matrix of φ evaluated at x ($\varphi'(x) = \varphi_2'(x) + \operatorname{Hess} F(x)$, when F is of class C^2) and observe that equation (1.1) is equivalent to

$$x'' = f(t, x, x'; 1) . \tag{2.1}$$

Moreover, we note that (by definition of $W(M)$),

$$\{z \in \mathbf{R}^m \,|\, g(z) = 0\} \subset W(M) , \quad \text{for each } M > 0 .$$

Consequently, assumption (w) ensures that $g(x) \neq 0$, for any $z \in \mathbf{R}^m$, whose norm is large enough.

Hence, by definition of $A(\lambda)$, we have

$$\bar{f}(z) := (1/p)\int_0^p f(t, z, 0; 0)\, dt =$$

$$= -A(0)\cdot g(z) = (\varepsilon I - A) g(z) \neq 0 ,$$

for any $z \in \mathbf{R}^m$, whose norm is large enough.

Then, according to (d), we get

$$|\deg(\bar{f}, B(0, R), 0)| = |\deg(g, B(0, R), 0)| \neq 0 , \tag{2.2}$$

for $R \geqslant r > 0$.

In order to prove the existence of p-periodic solutions of (2.1) (i.e. of (1.1)) we apply Mawhin's Continuation Theorem (Theorem 0); in virtue of (2.2), we only need to find a constant $K > 0$ (independent of $x(\cdot)$ and λ) such that

$$|x|_\infty + |x'|_\infty < K \tag{2.3}$$

holds, for any pair $(x, \lambda) \in C^2_p \times]0, 1[$, solution of the equation

$$x'' + \lambda (d/dt)\varphi(x) + \lambda A(\lambda) g(x) = \lambda h , \tag{1.1$_\lambda$}$$

which is equivalent to $x'' = \lambda f(t, x, x'; \lambda)$.

Let $x \in C^2_p$ be a solution of (1.1$_\lambda$) for some $\lambda \in]0, 1[$. Taking the mean value of (1.1$_\lambda$) (a device used in [18, Th. 6], for $g(x) = x$), we observe that

$$\int_0^p A(\lambda)g(x(t))\,dt = A(\lambda)\int_0^p g(x(t))\,dt = 0$$

Then we get

$$\int_0^p g(x(t))\,dt = 0\,, \tag{2.4}$$

because $A(\lambda)$ is nonsingular, for every $\lambda \in]0, 1[$.

Therefore a function $y \in C_p^1$ exists such that $\int_0^p y(t)\,dt = 0$ and $y'(t) = g(x(t))$, for every $t \in \mathbf{R}$.

We take now the inner product of (1.1_λ) by $y(t)$ and integrate between 0 and p, that is, we take the L^2-scalar product of (1.1_λ) by y. By integrating by parts, we obtain (see [38, proof of Th. 1]).

$$(x'', y)_2 = -(x', g(x))_2 = 0 \quad (\text{since } g = \operatorname{grad} G)\,, \tag{2.5}$$

$$((d/dt)\varphi(x), y)_2 = -(\varphi(x), g(x))_2 \tag{2.6}$$

and, for $H_1 \in \Sigma(h)$, such that $|H_1|_\infty = d(h)$, we get

$$|(h, y)_2| = |(H_1, g(x))_2| \leqslant d(h)\cdot|g(x)|_1\,. \tag{2.7}$$

At last, let us note that $(A(\lambda))_d = A_d$ (i.e. independent of λ) and $(A(\lambda))_s = A_s(\lambda)$. Therefore we have, since $A_s(\lambda)$ is symmetric,

$$\begin{aligned}(A(\lambda)g(x), y)_2 &= (A(\lambda)y', y)_2 = \\ &= ((A(\lambda))_d y', y)_2 + ((A(\lambda))_s y', y)_2 = \\ &= (A_d y', y)_2 + (A_s(\lambda)y', y)_2 = \\ &= (A_d g(x), y)_2 - (A_s(\lambda)y, y')_2 = \\ &= (A_d g(x), y)_2 + 0 = (A_d g(x), y)_2\,.\end{aligned} \tag{2.8}$$

Via Cauchy-Schwarz and Wirtinger inequalities, we obtain

$$|(A_d g(x), y)_2| \leqslant \|A_d\|\,|y'|_2^2/\omega = \|A_d\|\,|g(x)|_2^2/\omega\,. \tag{2.9}$$

From equation (1.1_λ) and using the estimates (2.5)-(2.9), we have

$$(\varphi(x), g(x))_2 \leqslant \|A_d\| |g(x)|_2^2/\omega + d(h)|g(x)|_1$$

and hence the assumption (j) provides the existence of an a priori bound (independent of x and λ) for the L^1-norm of $g(x)$, namely

$$|g(x)|_1 \leqslant K_1 := c \cdot p/(b - d(h)) . \tag{2.10}$$

Now we write the equation (1.1_λ) in the equivalent form

$$(d/dt)(x' + \lambda\varphi(x)) = \lambda z , \tag{2.11_λ}$$

where $z(\cdot) = z(\cdot, \lambda) := -A(\lambda)g(x(\cdot)) + h(\cdot)$.

Let us observe that $\int_0^p z(t, \lambda)\, dt = 0$ and, by (2.10), z is bounded in the L^1-norm:

$$|z(\cdot, \lambda)| \leqslant K_2 := (\|A\| + \varepsilon) \cdot K_1 + |h|_1 . \tag{2.12}$$

We take now the L^2-scalar product of (2.11_λ) by the function $-u(\cdot) := -x(\cdot) + \bar{x}$ ($\bar{x}$ being the mean value of $x(\cdot)$) and get

$$\begin{aligned} |x'|_2^2 = |u'|_2^2 &\leqslant \lambda|(\varphi(x), u')_2| + \lambda|(z, u)_2| \leqslant \\ &\leqslant |(\varphi_1(x), x')_2| + |(\varphi_2(x), u')_2| + |z|_1 |u|_\infty \leqslant \\ &\leqslant 0 + |(\varphi_2(x) - \varphi_2(\bar{x}), u')_2| + K_2(p/12)^{\frac{1}{2}}|x'|_2 . \end{aligned} \tag{2.13}$$

In the above estimates we used the hypothesis $\varphi_1 = \operatorname{grad} F$, an inequality in [32, Ch. XI, 7.8, p. 216] and (2.12).

Moreover, the assumption $v(\varphi_2) < \omega$ implies the existence of two constants $0 \leqslant L < \omega$ and $k \geqslant 0$ such that

$$|\varphi_2(x) - \varphi_2(\bar{x})| \leqslant L|x - \bar{x}| + k . \tag{2.14}$$

Therefore, using (2.13), (2.14) and Wirtinger inequality, we obtain

$$|x'|_2^2 \leqslant (L/\omega)|x'|_2^2 + (p^{\frac{1}{2}}k + (p/12)^{\frac{1}{2}}K_2)|x'|_2 .$$

Hence, we easily get a bound for the L^1-norm of $x' = u'$:

$$|x'|_2 \leqslant K_3 := \omega(p^{\frac{1}{2}}k + (p/12)^{\frac{1}{2}}K_2)/(\omega - L) , \tag{2.15}$$

and, as a consequence, we find $|u|_\infty$ bounded too

$$|u|_\infty \leqslant (p/12)^{\frac{1}{2}} K_3 := K_4 . \tag{2.16}$$

From Jensen inequality, if follows, for all $t, s \in \mathbb{R}$, $|x(t) - x(s)| \leqslant \leqslant (p/2)^{\frac{1}{2}}|x'|_2$, for any $x \in C^1_p$. Therefore, if x is a solution of (1.1_λ), we get, in virtue of (2.15),

$$|x(t) - x(s)| \leqslant M := (p/2)^{\frac{1}{2}} K_3 . \tag{2.17}$$

Now, recalling (2.4) and the definition of the set $W(M)$, we have

$$x(t) \in W(M) , \quad \textit{for every } t \in [0, p] , \tag{2.18}$$

with M as in (2.17).

Let us take any map $\psi \in \Gamma$, i.e. $|\psi(x) - \psi(y)| \leqslant L_\psi|x - y| + k_\psi$, and let $x(\cdot) \in C^2_p$ be any solution of (1.1_λ). We have

$$\left|\int_0^p \big(g(x(t))\,|\,\psi(x(t))\big)\,dt\right| = |(g(x), \psi(x))_2| =$$

(using (2.4)) $\quad = |(g(x), \psi(x) - \psi(\bar{x}))_2| \leqslant$

$\quad \leqslant |g(x)|_1 L_\psi |x - \bar{x}|_\infty + p^{\frac{1}{2}} k_\psi |g(x)|_1 \leqslant$

(using (2.10)) $\quad \leqslant K_1 L_\psi |x - \bar{x}|_\infty + p^{\frac{1}{2}} k_\psi K_1 \leqslant$

(using (2.16)) $\quad \leqslant K_5 := K_1(L_\psi K_4 + p^{\frac{1}{2}} k_\psi) .$

According to the mean value theorem, a point $t_1 = t_1(\lambda, x, \psi) \in [0, p]$ exists such that

$$\left|\big(g(x(t_1))\,|\,\psi(x(t_1))\big)\right| \leqslant K_6 := K_5/p . \tag{2.19}$$

The assumption (w), together with the estimates (2.18) and (2.19), implies the existence of $t_0 = t_0(x, \lambda) \in [0, p]$ such that

$$|x(t_0)| \leqslant K_7 , \tag{2.20}$$

with K_7 a positive constant independent of x and λ.

From (2.17) we obtain now that $|x|_\infty$ is bounded

$$|x|_\infty \leqslant K_7 + M . \tag{2.21}$$

Finally, by a standard bootstrap argument, making use of (2.15), (2.21) and the continuity of φ' and g, (2.3) is proved for a suitable choice of the constant K. □

3. Remarks and corollaries.

It is easy to see that the hypothesis (j) in Theorem 1 can be changed into

$$(\varphi(x)|g(x)) \leqslant - a|g(x)|^2 - b|g(x)| + c , \tag{j'}$$

with a, b and c satisfying the same estimates as in (j).

Indeed, if (j') holds, then (j) is verified with respect to $\gamma(x) := - g(x)$, that is, $(\varphi(x)|\gamma(x)) \geqslant a|\gamma(x)|^2 + b|\gamma(x)| - c$. Then our claim is proved, since $Ag(x) = (-A)\gamma(x)$ and $\|(-A)_d\| = \|-A_d\| = \|A_d\|$.

In previous papers (see [23], [38], [24]) we examined some sign conditions, less general than (w), in connection with nonlinear vector fields for Liénard systems, delayed Liénard systems and higher order differential equations. In particular we proved various sufficient criteria which imply the validity of (w) $\cap$ (d). For instance, (w) $\cap$ (d) holds in each of the following cases:

(i) *g is a homeomorphism,* (see [38]);

(ii) *There is a $m \times m$ non-singular matrix U such that* (see [24])

(1) $g_i(x)(Ux)_i > 0$ *for* $|x_i| \geqslant r$ $(i = 1, \ldots, m)$, *or*

(2) $\liminf\limits_{|x| \to \infty} (g(x)|Ux)/|g(x)||Ux| > 0$, *or*

(3) $\lim\limits_{|x| \to \infty} (g(x)|Ux) = +\infty$.

We also remark that the hypothesis

(w′) *For any* $M > 0$, *either* $W(M)$ *is bounded or* $\lim_{\substack{|x|\to\infty \\ x\in W(M)}} |g(x)| = \infty$,

which was assumed in [23] and [38], is a particular case of (w), with the choice $\psi_M(x) = \psi(x) := g(x)/(|g(x)| + 1)$ (for each M).

At last, we point out that our sign condition (w) *is invariant with respect to* (linear) *changes of coordinates.*

For a detailed examination of these conditions and of other more general ones, see [24, Sec. 5].

Let us observe that, if we change (j) (respectively (j′)) of Theorem 1 into

(jj) $\liminf_{|x|\to\infty} (\varphi(x)|g(x))/|g(x)|^2 = a$

((jj′) $\limsup_{|x|\to\infty} (\varphi(x)|g(x))/|g(x)|^2 = -a$, respectively), *with* $a > \|A_d\|/\omega$,

then the existence of p-periodic solutions of (1.1) is achieved, for any h with mean value zero, provided that the other conditions ((w), (d), ...) are assumed.

We notice that the estimate $a > \|A_d\|/\omega$ is *sharp* (at least for a class of Liénard systems containing the second order linear differential equations) in the sense that it cannot be relaxed, without further hypotheses of nonlinearity on φ and g, as the following example in $\mathbf{R}^2$ shows. Let us consider the coupled linear system

$$\begin{aligned} x_1'' + \alpha x_1' + \omega^2 x_1 - \alpha\omega x_2 &= \cos(\omega t) \\ x_2'' + \alpha x_2' + \omega^2 x_2 + \alpha\omega x_1 &= \sin(\omega t) \end{aligned} \qquad (\alpha > 0)$$

which does not possess p-periodic solutions (as elementary arguments prove). Here, $\varphi(x) = \varphi_1(x) = \alpha\cdot x$, $\varphi_2(x) = 0$ $(v(\varphi_2) = 0)$, $g(x) = x$, $A = \omega^2 I + \alpha\omega J$, $A_d = \alpha\omega J$, $(\varphi(x)|g(x)) = \alpha|x|^2$, $\|A_d\|/\omega = \alpha$ (where I is the identity matrix and J is the skew-symmetric matrix in $\mathbf{R}^2$ defined by $J(x_1, x_2) = (-x_2, x_1)$).

A simple linear example with $\varphi_1 = 0$, $\varphi_2 = \omega J$, $A = 0$ shows that the hypothesis $v(\varphi_2) < \omega$ is sharp too.

Now we briefly discuss some consequences of our Theorem 1, yet we will not examine in detail the corollaries that can be also directly deduced from our previous results (like Theorem 1 in [38]).

We think that it is fairly clear that the present result is a full extension of the one produced in [38], where the simplest case $A = I$, $\varphi_2 = 0$ and the less general sign condition (w′) (in place of (w)) were considered. Now, let us observe that, in the scalar case ($m = 1$), the matrix A is symmetric ($A_a = 0$) and φ admits an integral. Thus, if we take $\varphi = \varphi_1$, $\varphi_2 = 0$ and $v(\varphi_2) = 0$, we immediately prove the following

COROLLARY 1 ([38, Cor. 3]): ($m = 1$). *Assume there are two constants* $b > |h - \bar{h}|_1$ *and* $d > 0$ *such that, for any* $|x| \geqslant d$,

$$\left(\int_0^x f(t)\,dt\right)\operatorname{sign}(x) \geqslant b \qquad (or \ \leqslant -b)$$

and

$$(g(x) - \bar{h})\operatorname{sign}(x) > 0 \qquad (or \ < 0)$$

hold. Then equation (1.1′) *has a p-periodic solution.*

It is easily checked that Corollary 1 fits out Mizohata-Yamaguti's Theorem as well as it extends lots of the existence results recalled in the introduction; for instance [29], [12], [26], [8], [15, th. 5.4], [2, Th. 2, $c \neq 0$] [35], [6, Case I] are all contained in our theorem.

Our result generalizes also Theorem 6.6 in [32, Ch. XI] for vector equations (this fact was not obtained in [38]). Indeed, in [32, Ch. XI, Th. 6.6] it is assumed A nonsingular and $g(x) = x$, $\varphi_2(x) = 0$, $(\varphi(x)|x) = (\varphi_1(x)|x) = (\operatorname{grad} F(x)|x) \geqslant k|x|^{2q}$ ($k > 0$, $q > 1$) for $|x|$ large enough. Hence (w) and (d) are trivially satisfied as well as (jj) holds for $a = +\infty$.

At last we present a corollary of Theorem 1 in which the sign condition (w) is fulfilled as a consequence of an assumption (condition (rr) below) on the recession function J_g. We recall, following [3, Ch. II], the definition of the recession function associated to a vector field g:

$$J_g(z) := \liminf_{\substack{t \to +\infty \\ y \to z}} (g(ty)|y)\,, \qquad (t \in \mathbf{R},\ y, z \in \mathbf{R}^m);$$

(see [3], [30] for the main properties of J_g).

For the sake of simplicity, we restrict ourselves to the case $\varphi_2 = 0$ (so that $v(\varphi_2) = 0$).

COROLLARY 2. *Let us suppose* $\bar{h} \in \mathrm{Im}(A)$ *and assume there is a vector* $z \in \mathbf{R}^m$, *with* $Az = \bar{h}$, *such that*

$$\text{(r)} \qquad \liminf_{|x|\to\infty} (\varphi(x)|g(x) - z)/|g(x)|^2 > \|A_d\|/\omega$$

and

$$\text{(rr)} \qquad J_g(y) > (z|y), \text{ for each } y \neq 0$$

hold.

Then the vector Liénard equation (1.1), *with* $\varphi = \mathrm{grad}\, F$, $g = \mathrm{grad}\, G$, *has a p-periodic solution* ($\bar{h}$ is the mean value of h).

We observe that the ratio considered in (r) is defined, at least for $|x|$ large enough (this is a consequence of (rr)). A sufficient condition, for the validity of (rr), is, for instance,

$$\inf \{J_g(y) \colon |y| = 1\} > |z| \, .$$

Then it seems convenient to choose as z the element of minimum norm in $A^{-1}(\bar{h})$.

PROOF. We shall apply Theorem 1 to the equation

$$x'' + (d/dt)\varphi(x) + A\big(g(x) - z\big) = h(t) - \bar{h}$$

which is equivalent to (1.1). Let us set $\tilde{g}(x) := g(x) - z$, $\tilde{h}(t) := h(t) - \bar{h}$. Obviously, $\tilde{g}(x) = \mathrm{grad}\, \tilde{G}(x)$, with $\tilde{G}(x) = G(x) - (z|x)$ and $\tilde{h}$ has mean value zero. We evaluate now $\big(\varphi(x)|g(x)\big)$ in order to prove (j) of Theorem 1 (recall $v(\varphi_2) = 0$).

From hypothesis (r) and elementary properties of the « lim inf » we have that there are two constants $\delta > \|A_d\|/\omega$ and $\varrho \geq 0$ such that

$$\big(\varphi(x)|\tilde{g}(x)\big) = \big(\varphi(x)|g(x) - z\big) \geq \delta|g(x)|^2 - \varrho$$

holds for every $x \in \mathbf{R}^m$. Hence we have

$$\begin{aligned}(\varphi(x)|\tilde{g}(x)) &\geqslant \delta|\tilde{g}(x) + z|^2 - \varrho = \\ &= \delta|\tilde{g}(x)|^2 + 2\delta(z|\tilde{g}(x)) + \delta|z|^2 - \varrho \geqslant \\ &\geqslant \delta|\tilde{g}(x)|^2 - 2\delta|z||\tilde{g}(x)| + \delta|z|^2 - \varrho \geqslant \\ &\geqslant a|\tilde{g}(x)|^2 + b|\tilde{g}(x)| - c\,,\end{aligned}$$

with $b > |\tilde{h}|_1 = |h - \bar{h}|_1$ and a suitable choice of $a \in [\|A_d\|/\omega, \delta[$ and $c \geqslant 0$.

The hypothesis (rr), together with the lower semicontinuity of J_g (see [3, p. 259]) implies that $\min_{|y|=1}\{J_g(y) - (z|y)\} \geqslant \varepsilon > 0$. From the definition of $\tilde{g}$ we have $J_{\tilde{g}}(y) = J_g(y) - (z|y)$. Then we immediately obtain $\liminf_{t\to+\infty} (\tilde{g}(ty)|y) \geqslant J_{\tilde{g}}(y) \geqslant \varepsilon > 0$ for each $|y| = 1$. Hence it follows the validity of the sign condition (ii-3) which, as already stated, implies (w) and (d). The proof is therefore complete. □

At the end of this section we give a variant of Theorem 1 in which the conservative component φ_1 of the dissipative term φ possesses a potential function F which appears as a « guiding function » [10] for the restoring field g. Indeed, we can prove the following

Corollary 3. *Let us assume, as in Theorem* 1, $g = \operatorname{grad} G$, $\varphi = \varphi_1 + \varphi_2$ *with* $\varphi_1 = \operatorname{grad} F$, $\varphi_2 \in \Gamma$ *and* $v(\varphi_2) < \omega$. *Moreover, let us suppose*

$$(\mathrm{s}_i)\quad \lim_{|x|\to\infty} (\varphi(x)|g(x))/|g(x)|^i = +\infty\ (-\infty),$$

with $i = 1$ *if* $A_d = 0$, $i = 2$ *otherwise.*

Finally, let

$$\lim_{|x|\to\infty} F(x) = +\infty\ (-\infty) \text{ and } \lim_{|x|\to\infty} (\operatorname{grad} F(x)|g(x)) = +\infty\ (-\infty)\,.$$

Then equation (1.1) *has a p-periodic solution for any h with mean value zero.*

Proof. At first we note that $(\operatorname{grad} F(x)|g(x)) = (\varphi_1(x)|g(x)) > 0$, (< 0), for $|x|$ sufficiently large and so, by Poincaré-Bohl Theorem [20

Prop. II.19], $|\deg(g, B(0, R), 0)| = |\deg(\varphi_1, B(0, R), 0)|$ for $R \geqslant r > 0$. From a result in [10] (see also [20, Prop. II.21, p. 23]) we have $\deg(\operatorname{grad} F, B(0, R), 0) \neq 0$, for each R large enough and therefore (d) is proved.

Now we repeat the proof of Theorem 1 (as far as step (2.17)) and look for a priori bounds for the solutions of (1.1_λ).

From assumptions (s_i) $(i = 1, 2)$ and the, previously proved, estimates (2.5)-(2.9), it follows that there is a constant $C_1 \geqslant 0$ such that

$$|(\varphi(x), g(x))_2| \leqslant C_1 ,$$

for any $x \in C^2_p$ solution of (1.1_λ), for some $\lambda \in]0, 1[$.

Hence, we also get, using (2.4), (2.14), (2.10) and (2.16),

$$\begin{aligned} |(\varphi_1(x), g(x))_2| &\leqslant |(\varphi(x), g(x))_2| + |(\varphi_2(x), g(x))_2| \leqslant \\ &\leqslant C_1 + |(\varphi_2(x) - \varphi_2(\bar{x}), g(x))_2| \leqslant \\ &\leqslant C_1 + L|x - \bar{x}|_\infty |g(x)|_1 + kp^{\frac{1}{2}}|g(x)|_1 \leqslant \\ &\leqslant C_2 := (LK_4 + kp^{\frac{1}{2}})K_1 + C_1 . \end{aligned}$$

Then, according to the mean value theorem, a point $t_0 = t_0(\lambda, x) \in [0, p]$ exists such that

$$\Big|\big(\varphi_1(x(t_0))\,|\,g(x(t_0))\big)\Big| \leqslant C_3 := C_2/p . \tag{3.1}$$

From (3.1) and the growth assumption on $(\operatorname{grad} F(x)|g(x))$, we get (2.20) (of the proof of Theorem 1) again. Then the result follows through the same arguments already employed. □

4. Related results.

In this section we deal with a more general situation than in the previous chapters. Indeed, we examine now the case of a Liénard-type system in which the restoring field g is not necessarily a conservative map. Moreover, we allow an explicit dependence on time both in g and in the dissipative term φ.

Accordingly, we consider the vector equation

$$x''(t) + (d/dt)\varphi(t, x(t)) + Ag(t, x(t)) = h(t) , \tag{4.1}$$

where A is a $m\times m$ *constant matrix*, $\varphi \in C^1(\mathbf{R}\times\mathbf{R}^m, \mathbf{R}^m)$ and $g\in C^0(\mathbf{R}\times\mathbf{R}^m, \mathbf{R}^m)$ are *p-periodic in the first variable* and $h \in C^0_p$. For the sake of simplicity, throughout the present section, we also suppose $\int_0^p h(t)\,dt = 0$.

In order to get the existence of p-periodic solutions to (4.1), a restriction on the growth of g with respect to φ (see (k) below) will be assumed, together with a sign condition on g (the (k_δ) below) which is closely related to the rate of growth of g too. We remark that the results in this section do not extend Theorem 1 (Sec. 2), even if they cover some situations more general than the foregoing ones. We shall give only the mean features of the proofs, since they follow arguments which are similar to those developed in the preceding sections.

Now we list some conditions we are going to use in the following.

(k) *There are a constant $a > 0$ and a continuous map $b\colon [0, p]\times\mathbf{R}^m \to \mathbf{R}_+$ such that, for all $x\in\mathbf{R}^m$ and $t\in[0, p]$,*

$$\big(\varphi(t, x)|g(t, x)\big) \geqslant a|g(t, x)|^2 - b(t, x) .$$

Assumption (*k*) is a growth restriction on g which recalls a similar one considered by Brézis and Nirenberg in [3] (with $\varphi(x) = x$). A further hypothesis on the map $b(\cdot,\cdot)$, involved in (*k*), is made

(b_γ) *There is a constant $\gamma \geqslant 0$ such that*

$$\lim_{|x|\to\infty} b(t, x)/|x|^\gamma = 0 ,$$

uniformly with respect to $t\in[0, p]$.

Finally, a sign condition is given

(k_δ) *There are a constant $\delta > 0$ and a non-singular matrix U, such that*

$$\liminf_{|x|\to\infty} (g(t, x)|Ux)/|x|^\delta = G(t) ,$$

uniformly in t, with $\int_0^p G(t)\,dt > 0$.

Now we can state the following

THEOREM 2. *Let us assume* $\varphi(t,x) = \varphi_1(x) + \varphi_2(t,x)$ *with* $\varphi_1 = \operatorname{grad} F$, $F \in C^1(\mathbf{R}^m, \mathbf{R})$ *and* $\varphi_2 \in C^0(\mathbf{R}\times\mathbf{R}^m, \mathbf{R}^m)$ *such that, for each* $x, y \in \mathbf{R}^m$, $t \in [0, p]$,

(kk) $$|\varphi_2(t,x) - \varphi_2(0,y)| \leqslant L|x-y| + k$$

holds, with $0 \leqslant L < \omega$ *and* $k \geqslant 0$.

Moreover, suppose that (k), (b_γ) *and* (k_δ) *hold with*

(kkk) $\gamma \leqslant \min\{2, \delta\}$.

Then equation (4.1) *admits a p-periodic solution provided that*

(kv) $$a > (\|A\|\omega + \|A_d\|\sigma)/\sigma\omega \qquad (\sigma := \omega - L)\,.$$

PROOF. We just outline the main steps of the proof. As in Theorem 1, we define

$$f(t, x, y; \lambda) := -(\partial\varphi/\partial x)(t,x)\cdot y - (\partial\varphi/\partial t)(t,x) - A(\lambda)g(t,x) + h(t)\,,$$

with $A(\lambda) := A - (1-\lambda)\varepsilon I$, $\varepsilon > 0$ small enough.
Condition (k_δ) implies

$$|\deg(\bar f, B(0,R), 0)| = |\deg(\bar g, B(0,R), 0)| = |\deg(U, B(0,R), 0)| \neq 0,$$

for $R \geqslant r > 0$; with $\bar f$ as in Theorem 1 and $\bar g(z) := (1/p)\int_0^p g(t,z)\,dt$, $z \in \mathbf{R}^m$ and therefore (β) of Theorem 0 holds.
In order to verify (α) we are looking for an a priori bound for the (possible) solutions $(x,\lambda) \in C_p^2 \times]0,1[$ of the equation

(4.1_λ) $$x''(t) = \lambda f(t, x(t), x'(t); \lambda)\,,$$

(which, for $\lambda = 1$, is equivalent to (4.1)).

Observe

$$p\cdot\overline{g\circ x}=\int_0^p g(t,x(t))\,dt=0\,. \tag{4.2}$$

Let $y\in C_p^1$, with $\bar{y}=0$ and $y'(t)=g(t,x(t))=(g\circ x)(t)$, and take the L^2-scalar product of (4.1_λ) by $y(\cdot)$. We easily get

$$(\varphi\circ x, g\circ x)_2=-(1/\lambda)(x', g\circ x)_2+(A_d(g\circ x), y)_2-(h, y)_2\,. \tag{4.3}$$

On the other hand, taking the L^2-scalar product of (4.1_λ) by $-u(\cdot)=$ $=-x(\cdot)+\bar{x}$, and using (kk), we have

$$|x'|_2^2=-\lambda(\operatorname{grad}F(x), x')_2-\lambda(\varphi_2\circ x, x')_2+\lambda(A(\lambda)g\circ x, u)_2-\lambda(h, u)_2\leq$$
$$\leq 0+\lambda|(\varphi_2(\cdot, x)-\varphi_2(0,\bar{x}), x')_2|+\lambda|A(\lambda)g\circ x|_2|u|_2+\lambda|h|_2|u|_2\leq$$
$$\leq(L/\omega)|x'|_2^2+\lambda k|x'|_1+\lambda\big((\|A\|+\varepsilon)/\omega\big)|g\circ x|_2|x'|_2+(\lambda/\omega)|h|_2|x'|_2\,.$$

Hence, the following estimate is obtained

$$|x'|_2\leq\lambda\big((\|A\|+\varepsilon)/\sigma\big)|g\circ x|_2+\lambda(k\omega p^{\frac{1}{2}}+|h|_2)/\sigma\,, \tag{4.4}$$

with $\sigma:=\omega-L$.

Now, inserting (4.4) into (4.3), dividing by $\lambda>0$ and using (k), we have

$$a|g\circ x|_2^2-|b\circ x|_1\leq(\varphi\circ x, g\circ x)_2\leq\big((\|A\|+\varepsilon)/\sigma+\|A_d\|/\omega\big)|g\circ x|_2^2+c_1|g\circ x|_2\,,$$

where $c_1:=(k\omega p^{\frac{1}{2}}+|h|_2)/\sigma+|h|_2/\omega$.

Making use of (kv) we get

$$|g\circ x|_2^2\leq c_2|b\circ x|_1+c_3\,, \tag{4.5}$$

and, from (4.4) and (4.5), also

$$|x'|_2^2\leq c_4|b\circ x|_1+c_5\,, \tag{4.6}$$

with c_2, c_3, c_4, c_5 suitable positive constants.

Hence, by means of (b_γ), we obtain

$$(4.7) \quad (\forall \eta > 0)(\exists K_\eta > 0): \quad |x(s) - x(t)| \leqslant (p/2)^{\frac{1}{2}} |x'|_2 \leqslant \eta \big| |x|^{\gamma/2} \big|_2 + K_\eta \,,$$

for all $t, s \in [0, p]$ *and*

$$|g \circ x|_2 \leqslant \eta \big| |x|^{\gamma/2} \big|_2 + K_\eta \,.$$

Now, let us set $\varrho := \min \{x(t)| : t \in [0, p]\} = |x(t^*)|$ and $v(t) = x(t) - x(t^*)$. Then an easy computation gives

$$(4.8) \quad (\forall \eta > 0)(\exists K'_\eta > 0):$$
$$|v|_\infty \leqslant \eta \cdot \varrho^{\gamma/2} + K'_\eta \quad \text{and} \quad |g \circ x|_1 \leqslant \eta \cdot \varrho^{\gamma/2} + K'_\eta \,.$$

We claim: *there is a constant* $M > 0$, *such that, for any* $x \in C^2_p$, *solution of* (4.1λ),

$$\min \{|x(t)| : t \in [0, p]\} = \varrho \leqslant M \,.$$

Hence, we can conclude that $|v|_\infty$, $|x|_\infty$ and $|x'|_2$ are bounded, via (4.8) and (4.7), and the thesis follows as in the proof of Theorem 1. We prove now the claim. Let us assume it does not hold: there is a sequence $(x_n(\cdot))_n$ of solutions of (4.1_{λ_n}) such that $\varrho_n \to +\infty$. Let us compute

$$(4.9) \quad \int_0^p \big(g(t, x_n(t)) | x_n(t)\big) / \varrho_n^\delta \, dt = \varrho_n^{-\delta} (g \circ x_n, x_n)_2 =$$
$$= \varrho_n^{-\delta} (g \circ x_n, v_n)_2 \leqslant \varrho_n^{-\delta} |g \circ x_n|_1 |v_n|_\infty \leqslant \varrho_n^{-\delta} (\eta \cdot \varrho_n^{\gamma/2} + K'_\eta)^2 \,.$$

Let us observe that $\lim_{n \to \infty} |x_n(t)| \varrho_n^{-1} = 1$, uniformly in $t \in [0, p]$.

Then, passing to the lim inf into (4.9), using the Fatou's Lemma and taking into account $\delta \geqslant \gamma$, we have

$$\eta^2 \geqslant \liminf_{n \to \infty} \int_0^p (g \circ x_n | x_n) / \varrho_n^\delta \, dt \geqslant \int_0^p \Big(\liminf_{n \to \infty} (g \circ x_n | x_n) / \varrho_n^\delta \Big) \, dt =$$
$$= \int_0^p \Big(\liminf_{n \to \infty} \big(g(t, x_n(t)) | x_n(t)\big) / |x_n(t)|^\delta \Big) \cdot \Big(\lim_{n \to \infty} |x_n(t)| / \varrho_n^\delta \Big) \, dt = \int_0^p G(t) \, dt > 0$$

and a contradiction is obtained for a choice of η small enough. $\square$

We remark that Theorem 2 holds too, if we replace (k_δ) with the following (classical) sign condition, we already examined in Section 3,

(ii-1′) *There is a nonsingular matrix U such that*

$$g_i(t, x)\cdot(Ux)_i > 0\,, \quad \textit{for } |x_i| \geqslant R > 0 \qquad (i = 1, \dots, m)$$

and every $t \in [0, p]$.

In this case (kkk) simply becomes: $\gamma \leqslant 2$.

In order to prove this result, we repeat the proof of Theorem 1 as far as the step (4.7); then, following the same argument given in [19 proof of Th. 1], we achieve the thesis.

Besides we point out that condition (b_γ) may be weakened (if either (k_δ) or (ii-1′) is assumed) taking

There is $\alpha_0 \geqslant 0$ such that $b(t, x) \leqslant \alpha_0 |x|^2$, for $|x| \geqslant R > 0$ and every $t \in [0, p]$,

where α_0 is a (small) suitable constant, which may be computed by means of the coefficients of the equation (4.1).

Now we state a variant of Theorem 2, which can be regarded as a counterpart of Corollary 3, in the present situation. Indeed, the potential map F, such that $\operatorname{grad} F = \varphi_1$, will appear as a «guiding function» for the time dependent restoring field g. Accordingly, let us consider the following condition

(e_δ) *There is a constant $\delta > 0$ such that*

$$\liminf_{|x| \to \infty} (\operatorname{grad} F(x) | g(t, x)) / |x|^\delta = G(t)\,,$$

uniformly in t, with $\int_0^p G(t)\,dt > 0$.

Corollary 4. *Let us assume, as in Theorem 2, $\varphi(t, x) = \varphi_1(x) + \varphi_2(t, x)$, with $\varphi_1 = \operatorname{grad} F$, $F \in C^1(\mathbf{R}^m, \mathbf{R})$ and $\varphi_2 \in C^0(\mathbf{R} \times \mathbf{R}^m, \mathbf{R}^m)$ satisfying* (kk), *with $0 \leqslant L < \omega$ and $k \geqslant 0$. Moreover, let $b \in C^0([0, p] \times \mathbf{R}^m, \mathbf{R}_+)$*

and suppose

(ee) $\lim\limits_{|x|\to\infty}\big((\varphi(t,x)|g(t,x)) + b(t,x)\big)/|g(t,x)|^2 = +\infty$,

uniformly in t.

Finally, let (b_γ) *and* (e_δ) *hold, together with*

(eee) $\lim\limits_{|x|\to\infty} F(x) = +\infty\ (-\infty)$.

Then equation (4.1) *has a* p*-periodic solution, provided that*

(kkk) $\gamma \leqslant \min\{2, \delta\}$.

Proof. Let us define $f(t, x, y; \lambda)$ as in Theorem 2. First, we note that, by assumption (e_δ),

$$p(\varphi_1(z)|\bar{g}(z)) = \int_0^p (\operatorname{grad} F(z)|g(t,z))\,dt \geqslant \Big(\int_0^p G(t)\,dt - \varepsilon p\Big)|z|^\delta > 0\,,$$

($\varepsilon > 0$, small enough), for $|z|$ sufficiently large. Then, using some arguments already employed (i.e. Poincaré-Bohl Theorem, condition (eee) and Lemma 6.5 in [10]), we have $|\deg(\bar{f}, B(0,R), 0)| = |\deg(\bar{g}, B(0,R), 0)| = |\deg(\varphi_1, B(0,R), 0)| \neq 0$ and (β) of Theorem 0 follows.

Now we carry on the proof of Theorem 2 as far as the step (4.8); only the final claim needs a lightly different proof. Indeed, from (4.3), (4.5) and (4.6), we have

$$|(\varphi\circ x, g\circ x)_2| \leqslant d_1|b\circ x|_1 + d_2\,,$$

with d_1, d_2 two suitable positive constants. Then, using (kk) as in Theorem 2, it follows, from (4.5) and (4.6),

$$|(\varphi_1\circ x, g\circ x)_2| \leqslant |(\varphi\circ x, g\circ x)_2| + |(\varphi_2\circ x, g\circ x)_2| \leqslant d_3|b\circ x|_1 + d_4\,.$$

Consequently, (b_γ) implies

$$(\forall\eta > 0)(\exists K''_\eta > 0):\qquad |(\varphi_1\circ x, g\circ x)_2| \leqslant \eta\cdot\varrho^\gamma + K''_\eta\quad (\varrho \text{ as in } (4.8))\,,$$

and hence

$$(4.10)\qquad \left|\int_0^p \left((\varphi_1\circ x|g\circ x)/\varrho^\delta\right)dt\right| = \varrho^{-\delta}|(\varphi_1\circ x, g\circ x)_2| \le \eta\cdot\varrho^{\gamma-\delta} + \varrho^{-\delta}\cdot K''_\eta$$

holds (compare to (4.9)). Finally, (e_δ) gives the result, through the same device employed in the proof of the claim in Theorem 2. □

We point out that in all previous results we have also implicitly proved that the set of the solutions to (1.1) and (4.1) is compact (non-empty) in the space C^1_p endowed with the norm $x \mapsto |x|_\infty + |x'|_\infty$. More detailed informations about the structure of the solution set may be obtained, assuming further conditions on the coefficients of the equations (see [2], [27]).

We end this section stating the following uniqueness result concerning the equation

$$(4.11)\qquad x'' + (d/dt)\varphi(t, x) + Ax = h(t)\,,$$

which is a particular case of (4.1) with $g(t, x) = x$.

PROPOSITION 1. *Let A be non-singular and assume*

(u) $(\varphi(t, x) - \varphi(t, y)|x - y) > (\|A_d\|/\omega)|x - y|^2$

for each t and $x \ne y$.

Then equation (4.11) *has at most one p-periodic solution.*

PROOF. Let $x, y \in C^2_p$ be two p-periodic solutions to (4.11). Then, by difference,

$$(4.12)\qquad (x - y)'' + (d/dt)\big(\varphi(\cdot, x) - \varphi(\cdot, y)\big) + A(x - y) = 0\,.$$

We set $z(t) := x(t) - y(t)$. Passing to the mean value in (4.12), we have $\overline{Az(\cdot)} = A\bar z = 0$; hence $\bar z = 0$, for A is nonsingular. Let $Z \in C^1_p$ such that $Z' = z$, $\bar Z = 0$. Taking the L^2-scalar product of (4.12) by Z, with obvious computations, we get

$$(\varphi\circ x - \varphi\circ y, z)_2 \le \|A_d\|\cdot|z|_2^2/\omega\,,$$

which, together with (u), gives $x(t) = y(t)$, for each t. □

Proposition 1, in connection with an existence theorem for equation (4.11) extends to the systems a classical result for the scalar case [14, p. 10]. A linear example can be easily found in order to show that the assumption (u) is sharp.

5. Functional differential equations.

Now we present a version of Theorem 1, concerning the existence of p-periodic solutions to the functional differential system of Liénard type

$$x''(t) + (d/dt)\varphi(x(t)) + \int_{-r}^{s} d\eta(\theta) g(x(t+\theta)) = h(t)\,, \tag{5.1}$$

where, following [18, § 7], $\eta(\theta)$, $-r \leqslant \theta \leqslant s$, *is a* $m \times m$ *matrix whose elements have bounded variation,* $\varphi\colon \mathbf{R}^m \to \mathbf{R}^m$ *is a* C^1*-map,* $g\colon \mathbf{R}^m \to \mathbf{R}^m$ *is continuous,* $h\colon \mathbf{R} \to \mathbf{R}^m$ *is continuous and* p*-periodic.* For sake of simplicity, we shall assume $\int_0^p h(t)\,dt = 0$.

We define $M := \int_{-r}^{s} d\eta(\theta)$.

THEOREM 3. *Let us suppose, as in Theorem* 1, $g = \operatorname{grad} G$, $\varphi = \varphi_1 + \varphi_2$, *with* $\varphi_1 = \operatorname{grad} F$, $\varphi_2 \in \Gamma$ *and* $v(\varphi_2) < \omega$. *Moreover, let us assume*

(s$_2$) $\lim_{|x|\to\infty} (\varphi(x)|g(x))/|g(x)|^2 = +\infty\ (-\infty)\,.$

Finally, let us suppose that (w) *and* (d) *hold. Then equation* (5.1) *has a* p*-periodic solution.*

PROOF. Let us fix a number $\varepsilon > 0$ such that the matrix M has no eigenvalue in the interval $]0, \varepsilon]$. Setting, for $\lambda \in [0, 1]$, $\eta_\lambda(\theta) := \eta(\theta) - (\varepsilon(1-\lambda)\theta/(r+s))I$ (I, the identity matrix), $M(\lambda) := \int_{-r}^{s} d\eta_\lambda(\theta) = M - \varepsilon(1-\lambda)I$, we have (by our choice of ε) the matrix $M(\lambda)$ nonsingular for any $\lambda \in [0, 1[$.

Let us consider the equation

$$(5.1_\lambda)\qquad x''(t) = -\lambda(d/dt)\varphi(x(t)) - \lambda\int_{-r}^{s} d\eta_\lambda(\theta)g(x(t+\theta)) + \lambda h(t)\,,$$

which, for $\lambda = 1$, is (5.1). We define

$$(5.2)\qquad f\colon \mathbf{R}\times(\mathbf{R}^m)^3\times[0,1]\to\mathbf{R}^m\,,$$

$$f(t,x,y,z;\lambda) := -\varphi'(x)\cdot y - z + \varepsilon(1-\lambda)g(x) + h(t)\,;$$

f is continuous and p-periodic in t.

$$(5.3)\quad T\colon (C^0_p, |\cdot|_\infty)\to(C^0_p, |\cdot|_\infty)\,,\qquad (Tx)(t) := \int_{-r}^{s} d\eta(\theta)g(x(t+\theta))\,;$$

T is continuous and maps bounded sets into bounded sets. Moreover, $Tc = Mg(c)$, for any constant map $c\in\mathbf{R}^m$.

Under the above positions, (5.1_λ) is equivalent to

$$(5.4_\lambda)\qquad x''(t) = \lambda f(t, x(t), x'(t), (Tx)(t);\lambda)\,.$$

Now, in order to get a p-periodic solution of (5.4_1), we can employ a version of Theorem 0, suitable to be applied to functional differential equations, which can be directly deduced from Mawhin's Generalized Continuation Theorem [7, Th. IV. 1], [20, Th. IV. 1].

Indeed, there is a constant K such that

$$(5.5)\qquad |x|_\infty + |x'|_\infty \leqslant K\,,$$

for any $(x,\lambda)\in C^2_p\times]0,1[$ solution of (5.4_λ).

Moreover, for each $R\geqslant r_0>0$,

$$(5.6)\qquad \deg(\bar f, B(0,R), 0)\neq 0\,,$$

with $\bar f(c) := p^{-1}\int_0^p f(t,c,0,Tc;0)\,dt = -M(0)g(c)\neq 0$, for $|c|$ large enough (by (w)).

The statement (5.6) follows immediately from (d). For proving (5.5) we repeat the argument in Theorem 1, making use of some devices introduced in [18, proof of Th. 6]. In detail, at first we observe

$$\int_0^p (Tx)(t)\,dt = \int_0^p \Big(\int_{-r}^s d\eta(\theta) g(x(t+\theta))\Big)\,dt = \int_{-r}^s d\eta(\theta)\Big(\int_0^p g(x(t+\theta))\,dt\Big) =$$

$$= \int_{-r}^s d\eta(\theta)\Big(\int_0^p g(x(t))\,dt\Big) = pM\overline{g\circ x}\,.$$

Then, passing to the mean value in (5.1_λ), $\lambda \in]0,1[$, we have $M(\lambda)\,\overline{g\circ x} = 0$ and therefore

$$(5.7) \qquad \int_0^p g(x(t))\,dt = 0\,.$$

Let $y \in C^1_p$ with $y'(t) = g(x(t))$, $\bar{y} = 0$. We take now the L^2-scalar product of (5.1_λ) by y. From Lemma 3 in [18], we have

$$(5.8) \qquad |(Tx, y)_2| \leqslant N|g\circ x|_2|y|_2 \leqslant (N/\omega)|g\circ x|_2^2\,,$$

where $N = \max\limits_{i,j=1,\dots,m} \Big\{\sum\limits_{k=1}^2 (\eta_{ij}^{(k)}(s) - \eta_{ij}^{(k)}(-r))\Big\}$, with $n_{ij}^{(1)}(\theta) - \eta_{ij}^{(2)}(\theta) = \eta_{ij}(\theta)$, $\eta_{ij}^{(1)}, \eta_{ij}^{(2)}$ increasing functions. Hence, through the same computations as in Theorem 1, we obtain:

$$(5.9) \qquad |(\varphi\circ x, g\circ x)_2| \leqslant \omega^{-1}(\varepsilon + N)|g\circ x|_2^2 + d(h)|g\circ x|_1\,.$$

From assumption (s_2), a bound for $|g\circ x|_2$ is achieved. The remainder of the proof is straightforward. □

REMARKS. (1) In the particular case where it is assumed M nonsingular $g(x) = x$, $\varphi_2(x) = 0$, $(\varphi(x)|x) = (\varphi_1(x)|x) \geqslant k|x|^{2q}$, $(k > 0,\ q > 1)$, for $|x|$ large enough, we obtain Theorem 6 in [18].

(2) Whenever the matrix η is chosen in such a way that

$$\int_{-r}^s d\eta(\theta) w(t+\theta) = \mathrm{col}\Big(\sum_{j=1}^m a_{ij} w_j(t-\tau_j)\Big)\,, \qquad (w \in C^0_p)$$

with $\tau_j \in \mathbb{R}$ $(j = 1, \ldots, m)$ and $A = ((a_{ij}))$, a $m \times m$ constant matrix $(A \neq 0)$, we can relax (s_2) into

(j) $(\varphi(x)|g(x)) \geqslant a|g(x)|^2 + b|g(x)| - c,$

with $a \geqslant \|A\|/\omega$, $b > d(h)$, $c \geqslant 0$.

This can be proved as above, defining $M(\lambda) := \int_{-r}^{s} d\eta_\lambda(\theta)$, $\eta_\lambda(\theta) := (1 - \varepsilon(1-\lambda)\|A\|^{-1})\eta(\theta) - (\varepsilon(1-\lambda)\theta/(r+s))I$, with $0 < \varepsilon \leqslant \|A\|/2$ such that $A = M$ has no eigenvalue in the interval $]0, 2\varepsilon]$. Then, a direct computation shows that $\|A(\lambda)\| \leqslant \|A\|$, for $\lambda \in [0, 1]$, and that (5.8), (5.9) become, respectively,

$$|(Tx, y)_2| \leqslant (\|A\|/\omega)|g \circ x|_2^2,$$

$$|(\varphi \circ x, g \circ x)_2| \leqslant (\|A\|/\omega)|g \circ x|_2^2 + d(h)|g \circ x|_1. \quad \square$$

In this version, Theorem 3 extends results in [37], [23] and [9, p. 326]. Furthermore, the estimate $a \geqslant \|A\|/\omega$ is sharp, as a linear counterexample shows.

(3) A functional differential version of the other results in Sections 3 and 4 could be derived as well (with the only restriction of considering in all cases $g = g(x)$, i.e. g not dependent on the time variable).

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