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Generalized V-rings and von Neumann regular rings

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Generalized V-Rings and Von Neumann Regular Rings.

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0. Introduction and notations.

Throughout the present paper all rings are associative with $0 \neq 1$ and all modules are unitary. Given a ring R, we shall denote with $J(R)$ the Jacobson radical of R. $\mathbf{S}$ will be a choosen set of representatives of all simple right R-modules, while $\mathbf{P}$ is the subset of $\mathbf{S}$ of the representatives of all simple projective right R-modules. Given a right R-module M, we shall write $E(M)$ for the injective envelope of M; the notation $N \leqslant M_R$ (resp. $N \trianglelefteq M_R$) will mean that N is an R-submodule (resp. an essential R-submodule) of M. For a given subset $\mathbf{A} \subset \mathbf{S}$, we shall denote by $\mathrm{Soc}_{\mathbf{A}}(M)$ the $\mathbf{A}$-homogeneous conponent of the socle Soc M of M; we shall write $\mathrm{Soc}_U(M)$ instead of $\mathrm{Soc}_{\{U\}}(M)$, for a given $U \in \mathbf{S}$. For every right (resp. left) R-module M, $r_R(M)$ (resp. $l_R(M)$) will be the annihilator in R of M.

A ring R is a *right V-ring* (resp. *right GV-ring*) if all simple (resp. all simple singular) right R-modules are injective. We say that R is *fully right idempotent* (*FRI*, for short) if $\mathfrak{a} = \mathfrak{a}^2$ for every right ideal $\mathfrak{a}$ of R or, equivalently, if ${}_R(R/\mathfrak{h})$ is flat for every two-sided ideal $\mathfrak{h}$ of R. Several authors studied the connections between the above concepts and the concept of a (Von Neumann) regular ring (see [2], [7], [13], [14], [16], [17], [18], [19], [20]). The present paper is intended

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to give further contributions to the study of the above classes of rings (it is well known that, in the commutative case, all these coincide with the class of regular rings).

In the first section we concentrate our attention to the class of those rings R for which $\mathrm{Soc}\, R_R$ is projective and $R/\mathrm{Soc}\, R_R$ is fully right idempotent; we call them *GFRI* rings (generalized *FRI* rings). This class properly includes the class of right *GV*-rings and, of course, the class of *FRI* rings. We prove, among others, that a *GFRI* ring has a regular centre; moreover, if $\mathrm{Soc}\, R_R \cap J(R) = 0$, then *GFRI* implies *FRI* (actually, we shall prove also that, under the same hypothesis, if $R/\mathrm{Soc}\, R_R$ is regular, then R is regular). In sect. 2 right *GV*-rings are characterized by means of the behavior of the projective components of the socle of each right module. We also consider the class of right *SI*-rings introduced by K.R. Goodearl in [8] (R is a right *SI*-ring provided each singular right R-module is injective; R is then a right *GV*-ring). We prove that R is a regular right *SI*-ring iff R is semiprime and $R/\mathrm{Soc}\, R_R$ is semisimple; thus, for regular rings, the « *SI* » condition is right/left symmetrical.

In sect. 3 we consider the class of regular rings which are strongly regular modulo their socle; we call them *ASR* rings. R. Yue Chi Ming considered in [19, 20] some classes of rings, namely MELT left p-V'-rings, semiprime MELT left p-V-rings, MELT regular rings and semiprime MELT rings whose simple right modules are flat (MELT means that all maximal essential left ideals are two-sided). We prove that all these classes coincide with the class of *ASR* rings. Finally we prove that a ring R is a self-injective *ASR* ring iff $R = S' \times S'' \times T$, where S' is a semisimple ring, S'' is a direct product of division rings and T is a strongly regular self-injective ring with zero socle. This gives a refinement of [20, Theor. 27].

In the last section we discuss some examples.

1. Rings which are fully right idempotents modulo the right socle.

Throughout this section R will be a given ring. Recall that if $\mathfrak{k}$ is a two-sided ideal of R, then the following conditions are eqivalent: (a) ${}_R(R/\mathfrak{k})$ is flat, (b) for each $x \in \mathfrak{k}$, $x \in x\mathfrak{k}$, (c) for each module M_R and submodule $N \leqslant M$, $N \cap M\mathfrak{k} = N\mathfrak{k}$, ($d$) for each simple module $S_{R/\mathfrak{k}}$, $E(S_{R/\mathfrak{k}}) = E(S_R)$ (the proof of the equivalence between (d) and the other conditions can be found in [2, Theor. 1.1]). We shall make extensive use of the following result, which was originally proved in [3, Prop. 1.7, Coroll. 1.5 and 1.11]:

PROPOSITION 1.1. *For every ring* R *one has* $\mathrm{Soc}_{\boldsymbol{P}}(R_R) = (\mathrm{Soc}\, R_R)^2$. *Moreover, if* $\mathfrak{k}$ *is a two-sided ideal contained in* $\mathrm{Soc}\, R_R$, *then the following conditions are equivalent*:

(1) $\mathfrak{k}^2 = \mathfrak{k}$.

(2) ${}_R(R/\mathfrak{k})$ *is flat.*

(3) *There is a subset* $\boldsymbol{A} \subset \boldsymbol{P}$ *such that* $\mathfrak{k} = \mathrm{Soc}_{\boldsymbol{A}}(R_R)$.

If these conditions hold, then for each module M_R *one has* $\mathrm{Soc}_{\boldsymbol{A}}(M) = M\mathfrak{k}$.

It is well known that if R is semiprime, then $\mathrm{Soc}\, R_R = \mathrm{Soc}\, {}_RR$ is projective on both sides. Moreover, if $\mathfrak{n}$ is a minimal right ideal, then $\mathfrak{n} = eR$ for an idempotent $e \in R$ and $\mathrm{Soc}_{\mathfrak{n}}(R_R) = ReR$ is a minimal two-sided ideal.

LEMMA 1.2. *Let* C *be the centre of the ring* R. *Given* $P \in \boldsymbol{P}$ *and set* $\mathfrak{k} = \mathrm{Soc}_P(R_R)$, *for every* $a \in C$ *the following conditions are equivalent*:

(1) $a\mathfrak{k} \neq 0$.

(2) $a\mathfrak{k} = \mathfrak{k}$.

(3) $\mathfrak{k} \subset aR$.

PROOF. (1) $\Rightarrow$ (2). There is an idempotent $e \in R$ such that $P \cong eR$ and $\mathfrak{k} = ReR$. We have then $0 \neq a\mathfrak{k} = aReR = ReaR$ and hence $eaR \neq 0$, from which $eaR = eR$. It follows that $a\mathfrak{k} = ReaR = ReR = \mathfrak{k}$.

(2) $\Rightarrow$ (3): it is obvious.

(3) $\Rightarrow$ (1). Being ${}_R(R/\mathfrak{k})$ flat, we have $0 \neq \mathfrak{k} = \mathfrak{k} \cap aR = aR\mathfrak{k} = a\mathfrak{k}$. ∎

PROPOSITION 1.3. *Let* $P \in \boldsymbol{P}$ *and* $\mathfrak{k} = \mathrm{Soc}_P(R_R)$. *Then* $\mathfrak{k}$ *is generated by a central idempotent if and only if* $\mathfrak{k} \cap \mathrm{Cen}(R) \neq 0$.

PROOF. The «only if» part is clear. Suppose that $0 \neq a \in \mathfrak{k} \cap \mathrm{Cen}(R)$. Since ${}_R(R/\mathfrak{k})$ is flat, we have $0 \neq aR = aR \cap \mathfrak{k} = a\mathfrak{k}$ and then, by Lemma 1.2, $\mathfrak{k} = a\mathfrak{k} \subset aR \subset \mathfrak{k}$, so that $\mathfrak{k} = aR$. We claim that $\mathfrak{k} \cap J(R) = 0$, from which we will conclude that $\mathfrak{k}$ is generated by a central idempotent, according to [3, Theor. 2.7] and being $\mathfrak{k}_R$ finitely generated. From the above and Prop. 1.1 we have $aR = \mathfrak{k} = \mathfrak{k}^2 = aRaR = a^2R$, therefore $a = a^2b$ for some $b \in R$. Suppose that $x \in \mathfrak{k} \cap J(R)$ and let $y \in R$ be such that $x = ay$. Then we have $y = ay = a^2by = ba^2y = baay = bax = 0$, because $ba = ab \in \mathfrak{k}$. Thus $x = 0$. ∎

PROPOSITION 1.4. *Given a ring R, the following conditions are equivalent:*

(1) $R/(\mathrm{Soc}\, R_R)^2$ *is a FRI-ring.*

(2) $r_R(S) = r_R(E(S_R))$ *for each simple singular right R-module S.*

(3) $\mathrm{Soc}\, R_R$ *is projective and if $\mathfrak{a}$ is a right ideal of R such that $\mathfrak{a} \cap \mathrm{Soc}\, R_R = \mathrm{Soc}_A(R_R)$ for some $A \subset P$, then $\mathfrak{a} = \mathfrak{a}^2$.*

PROOF. First, observe that a simple right R-module is singular if and only if it is annihilated by $\mathrm{Soc}\, R_R$. According to Prop. 1.1, ${}_R(R/(\mathrm{Soc}\, R_R)^2)$ is flat and hence, given a simple singular right R-module S, the $R/(\mathrm{Soc}\, R_R)^2$-injective envelope of S is the same as $E(S_R)$. Thus the equivalence of conditions (1) and (2) is a consequence of [2, Theor. 3.2].

(2) $\Rightarrow$ (3). We shall prove that $E(S_{R/\mathrm{Soc}\, R_R}) = E(S_R)$ for each singular simple right R-module S, from which it will follow that ${}_R(R/\mathrm{Soc}\, R_R)$ is flat and hence $\mathrm{Soc}\, R_R$ will be projective by Prop. 1.1. Indeed, if S_R is simple and singular, then $S(\mathrm{Soc}\, R_R) = 0$ and therefore (2) implies that $(E(S_R))(\mathrm{Soc}\, R_R) = 0$. Taking into account of [15, Prop. 2.27] we get

$$E(S_{R/\mathrm{Soc}\, R_R}) = \{x \in E(S_R) | x(\mathrm{Soc}\, R_R) = 0\} = E(S_R)\,.$$

Now, assume that $\mathfrak{a}$ is a right ideal of R such that $\mathfrak{a} \cap \mathrm{Soc}\, R_R = \mathrm{Soc}_A(R_R)$ for some subset $A \subset P$ and let $x \in \mathfrak{a}$. By the equivalence of conditions (1) and (2), there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{a}$ and $z \in (\mathrm{Soc}\, R_R)^2 = \mathrm{Soc}\, R_R$ such that $x = x_1y_1 + \ldots + x_ny_n + z$. This implies that $z \in \mathfrak{a} \cap \mathrm{Soc}\, R_R = \mathrm{Soc}_A(R_R)$. Since the latter is idempotent we infer that $z \in \mathfrak{a}^2$ and we conclude that $x \in \mathfrak{a}^2$.

(3) $\Rightarrow$ (1): it is clear. ∎

If R satisfies the equivalent conditions of Prop. 1.4, then we call R a *GFRI-ring* (generalized *FRI*-ring). Thus R is *GFRI* exactly when $\mathrm{Soc}\, R_R$ is projective and $R/\mathrm{Soc}\, R_R$ is a *FRI*-ring.

PROPOSITION 1.5. *If R is a GFRI-ring, then the centre C of R is a regular ring.*

PROOF. Assume that R is *GFRI*. We claim that, for every $a \in C$, there exists $b \in R$ such that $a = a^2 b$. Then the regularity of C will follow by the same argument as in the proof of [13, Lemma 2.3(d)]. Given $a \in C$, for every $P \in \boldsymbol{P}$ we have $aR \cap \mathrm{Soc}_P(R_R) = a(\mathrm{Soc}_P(R_R))$ by the flatness of ${}_R(R/\mathrm{Soc}_P(R_R))$. It follows from Lemma 1.2 that either $\mathrm{Soc}_P(R_R) \subset aR$ or $\mathrm{Soc}_P(R_R) \cap aR = 0$. Inasmuch as $\mathrm{Soc}\, R_R$ is projective by Prop. 1.4, we infer that, in any case, $aR \cap \mathrm{Soc}\, R_R = \mathrm{Soc}_A(R_R)$ for some subset $\boldsymbol{A} \subset \boldsymbol{P}$ (possibly $\boldsymbol{A} = \emptyset$). Thus, according to Prop. 1.4, $aR = aRaR = a^2R$ and hence $a = a^2 b$ for some $b \in R$. ∎

A module M_R is said to be p-injective if for every principal right ideal $\mathfrak{a}$ of R, all R-homomorphisms from $\mathfrak{a}$ to M extends to R. Equivalently, M is p-injective if $\mathrm{Ext}^1_R(R/xR, M) = 0$ for each $x \in R$. p-injective modules, expecially simple p-injective modules, were studied extensively by R. Yue Chi Ming, who also gave the following definition: the ring R is a *right p-V-ring* (resp. *p-V'-ring*) if every simple (resp. simple singular) right R-module is p-injective (see [16, 17, 18, 19]). If R is a right *p-V-ring*, then R is a *FRI*-ring (see [16, Lemma 1]); moreover R is regular iff each right R-module is p-injective (see [17, Lemma 2]).

PROPOSITION 1.6. *Given a ring R, the following conditions are equivalent:*

(1) *R is a right p-V'-ring.*

(2) $\mathrm{Soc}\, R_R$ *is projective and* $R/\mathrm{Soc}\, R_R$ *is a right p-V-ring.*

Thus a right p-V'-ring is *GFRI*.

PROOF. (1) $\Rightarrow$ (2). If $\mathfrak{n}$ is a p-injective minimal right ideal of R, then $\mathfrak{n}$ is generated by an idempotent. This shows that $\mathrm{Soc}\, R_R$ is projective. Since the simple singular right R-modules are exactly the simple right $R/\mathrm{Soc}\, R_R$-modules, it is easily seen that if R is a right p-V'-ring, then $R/\mathrm{Soc}\, R_R$ is a right p-V-ring.

(2) $\Rightarrow$ (1). Assume that (2) holds and set $\mathfrak{k} = \mathrm{Soc}\, R_R$. Given a simple singular right R-module S, we have $S\mathfrak{k} = 0$. Being ${}_R(R/\mathrm{Soc}\, R_R)$

flat, for each $x \in R$ we have

$$\mathrm{Ext}^1_R(R/xR, S) \cong \mathrm{Ext}^1_{R/\mathfrak{k}}((R/xR)\otimes_R(R/\mathfrak{k}), S) \cong \mathrm{Ext}^1_{R/\mathfrak{k}}(R/xR + \mathfrak{k}, S) = 0$$

(we are using the fact that $R/xR + \mathfrak{k} \cong (R/\mathfrak{k})/(xR + \mathfrak{k}/\mathfrak{k})$ and $xR + \mathfrak{k}/\mathfrak{k}$ is a principal right ideal of $R/\mathfrak{k}$). Thus R is a right p-V'-ring. ■

PROPOSITION 1.7. *Let $\mathfrak{k}$ be a two-sided ideal of the ring R contained in* $\mathrm{Soc}\, R_R$ *and suppose that $\mathfrak{k} \cap J(R) = 0$. Then the following are true:*

(i) *R is FRI if and only if $R/\mathfrak{k}$ is FRI.*

(ii) *R is regular if and only if $R/\mathfrak{k}$ is regular.*

PROOF. In both statements (i) and (ii) the « only if » part is trivial, so we must prove the « if » part. Firstly, by the hypothesis and [3, Prop. 2.1], both $_R(R/\mathfrak{k})$ and $(R/\mathfrak{k})_R$ are flat and $\mathfrak{k} = \mathrm{Soc}_{\boldsymbol{A}}(R_R)$ for some subset $\boldsymbol{A} \subset \boldsymbol{P}$. (i). Assume that $R/\mathfrak{k}$ is *FRI*. We claim that for each simple right R-module S the equality $r_R(S) = r_R(E(S_R))$ holds, from which it will follow that R is *FRI* by [2, Theor. 3.1]. Suppose that $S \in \boldsymbol{A}$ and set $\mathfrak{h} = \mathrm{Soc}_S(R_R)$. Since $\mathfrak{h} \cap J(R) = 0$, taking into account of [4, Theor. 1.3] we get $r_R(S) = r_R(\mathfrak{h}) = r_R(E(\mathfrak{h}_R)) \subset r_R(E(S_R)) \subset r_R(S)$, whence $r_R(S) = r_R(E(S_R))$. Assume that $S \in \boldsymbol{S} - \boldsymbol{A}$. Then $S\mathfrak{k} = 0$ and, being $_R(R/\mathfrak{k})$ flat, we have that $E(S_{R/\mathfrak{k}}) = E(S_R)$. Since $R/\mathfrak{k}$ is *FRI*, then $r_{R/\mathfrak{k}}(S) = r_{R/\mathfrak{k}}(E(S_{R/\mathfrak{k}})) = r_{R/\mathfrak{k}}(E(S_R))$ and we infer, again, $r_R(S) = r_R(E(S_R))$.

(ii). Suppose that $R/\mathfrak{k}$ is regular. In order to prove that R is regular we must prove that every module M_R is flat. Inasmuch as $\mathfrak{k}_R$ is projective, then it is flat; thus, given any module M_R, we get the exact sequence

$$0 \to M\mathfrak{k} \to M \to M/M\mathfrak{k} \to 0\,. \tag{1}$$

According to Prop. 1.1, we have $M\mathfrak{k} = \mathrm{Soc}_{\boldsymbol{A}}(M)$ and hence $M\mathfrak{k}$ is flat, being projective. Furthermore $M/M\mathfrak{k}$ is flat as a right $R/\mathfrak{k}$-module, because $R/\mathfrak{k}$ is regular. Being $(R/\mathfrak{k})_R$ flat, it follows from [1, Prop. 10] that $(M/M\mathfrak{k})_R$ is flat. Finally, from the exact sequence (1) we conclude that M_R is flat. ■

As in [4], we call the ring R *right weakly semiprime* (right w.s. for short) if $(\mathrm{Soc}\, R_R)^2 \cap J(R) = 0$. Thus the foregoing proposition allows us to state the following corollary, the first statement of which generalizes [18, Prop. 6], according to Prop. 1.6.

COROLLARY 1.8. *Suppose that R is right w.s.. Then the following are true:*

(i) *R is FRI if and only if R is GFRI.*

(ii) *R is regular if and only if $R/(\mathrm{Soc}\, R_R)^2$ is regular.* ■

2. Applications to generalized V-rings.

In [14, Theor. 3.3] a right GV-ring R is characterized in terms of the behavior of the Jacobson radical of each right R-module. We want to give an alternative description of R which involves the right socle of R. It is clear that a right GV-ring with zero right socle is a right V-ring.

PROPOSITION 2.1. *For every ring R the following conditions are equivalent:*

(1) *$R/\mathrm{Soc}\, R_R$ is a right V-ring.*

(2) *If M is a right R-module, then every essential submodule of M is an intersection of maximal submodules of M.*

PROOF. (1) $\Rightarrow$ (2). Let M be a right R-module and let L be an essential submodule of M. Since $M(\mathrm{Soc}\, R_R) \subset \mathrm{Soc}(M)$, then $M(\mathrm{Soc}\, R_R) \subset \subset L$. If $R/\mathrm{Soc}\, R_R$ is a right V-ring, then it follows from [13, Theor. 2.1] that $L/M(\mathrm{Soc}\, R_R)$, as a right $R/\mathrm{Soc}\, R_R$-submodule of $M/M(\mathrm{Soc}\, R_R)$, is an intersection of maximal $R/\mathrm{Soc}\, R_R$-submodules of $M/M(\mathrm{Soc}\, R_R)$. This is enough to conclude that L is an intersection of maximal submodules of M.

(2) $\Rightarrow$ (1). Let S be a simple right $R/\mathrm{Soc}\, R_R$-module, let $\mathfrak{a}$ be a right ideal of R, with $\mathrm{Soc} R_R \subset \mathfrak{a}$ and $\mathfrak{a}/\mathrm{Soc}\, R_R$ essential in $R/\mathrm{Soc}\, R_R$, and let $f\colon \mathfrak{a}/\mathrm{Soc}\, R_R \to S$ be a non-zero $R/\mathrm{Soc}\, R_R$-homomorphism. We claim that if $\pi\colon \mathfrak{a} \to \mathfrak{a}/\mathrm{Soc}\, R_R$ is the canonical epimorphism and $\mathfrak{b} = \mathrm{Ker}(f\pi)$, then $\mathfrak{b} \trianglelefteq \mathfrak{a}_R$. If not, then, by the definition of $\mathfrak{b}$, there is a minimal right ideal $\mathfrak{n}$ of R such that $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{n}$, in contradiction with the fact that $\mathrm{Soc}\, R_R \subset \mathfrak{b}$. Inasmuch as $\mathfrak{a}/\mathrm{Soc}\, R_R$ is essential in $R/\mathrm{Soc}\, R_R$, $\mathfrak{a}$ is essential in R and hence $\mathfrak{b}$ is essential in R. Since $\mathfrak{b} \neq \mathfrak{a}$

from (2) it follows that there is a maximal right ideal $\mathfrak{m}$ of R such that $\mathfrak{b} \subset \mathfrak{m}$ and $\mathfrak{a} \not\subset \mathfrak{m}$. Being $\mathfrak{b}$ maximal in $\mathfrak{a}$, we have that $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{m}$ and therefore $\mathrm{Ker}(f) = \mathfrak{b}/\mathrm{Soc}\, R_R = (\mathfrak{a} \cap \mathfrak{m})/\mathrm{Soc}\, R_R = (\mathfrak{a}/\mathrm{Soc}\, R_R) \cap (\mathfrak{m}/\mathrm{Soc}\, R_R)$. From this and by the « 3×3 » lemma it follows that f extends to an $R/\mathrm{Soc}\, R_R$-homomorphism from $R/\mathrm{Soc}\, R_R$ to S. This shows that $S_{R/\mathrm{Soc}\, R_R}$ is injective. ■

THEOREM 2.2. *Given a ring R, the following conditions are equivalent*:

(1) *R is a right GV-ring.*

(2) *$\mathrm{Soc}\, R_R$ is projective and $R/\mathrm{Soc}\, R_R$ is a right V-ring.*

(3) *If M is a right R-module, then $Z(M) \cap M(\mathrm{Soc}\, R_R) = 0$ and every essential submodule of M is an intersection of maximal submodules.*

PROOF. (1) $\Leftrightarrow$ (2). The proof is similar to the proof of Prop. 1.6. (2) $\Leftrightarrow$ (3). This equivalence is a consequence of Prop. 2.1, taking into account that if $\mathrm{Soc}\, R_R$ is projective, then, for each module M_R, we have $M(\mathrm{Soc}\, R_R) = \mathrm{Soc}_{\boldsymbol{P}}(M)$ by Prop. 1.1, whence $Z(M) \cap M(\mathrm{Soc}\, R_R) = 0$. ■

Since a right GV-ring is $GFRI$, from Coroll. 4.8 we obtain the following two results (recall that if R is a FRI-ring, then $J(R) = 0$).

PROPOSITION 2.3. *For a right GV-ring R the following conditions are equivalent*:

(1) *R is FRI.*

(2) *R is right w.s..*

(3) $J(R) = 0$. ■

PROPOSITION 2.4. *A ring R is a regular right GV-ring if and only if $J(R) = 0$ and $R/\mathrm{Soc}\, R_R$ is a regular V-ring.* ■

Combining Prop. 2.3 with [2, Theor. 4.5], we obtain the following theorem which generalize [14, Theor. 4.8] (see also [11, Prop. 1.3]).

THEOREM 2.5. *Given a ring R, the following conditions are equivalent*:

(1) *R is a semiprime right GV-ring with finite right (resp. left) Goldie dimension.*

(2) *R is a semiprime right GV-ring and R is a right (resp. left) Goldie ring.*

(3) *R is isomorphic to a direct product of finitely many simple right V-rings which are right (resp. left) Goldie rings.* ■

We point out that the condition (2) of [14, Theor. 4.8] is not congruent with the others stated there. Indeed, if S is a semisimple ring and T is a simple right Noetherian V-ring which is not Artinian (see [6]), then $S \times T$ is a semiprime right Noetherian right GV-ring and, obviously, it is neither simple nor Artinian. However, in view of our Theor. 2.5, that condition should be replaced with the following: «R is isomorphic to a direct product of finitely many simple right Noetherian right V-rings».

A special class of right GV-rings is that of right SI-rings introduced by Goodearl in [8, Ch. III]; these are characterized by the fact that every singular right module is injective.

LEMMA 2.6. *A ring R is a right SI-ring with essential right socle if and only if* $\operatorname{Soc} R_R$ *is projective and* $R/\operatorname{Soc} R_R$ *is semisimple.*

PROOF. The «only if» part is a straightforward consequence of the definition of a right SI-ring. Conversely, assume that $\operatorname{Soc} R_R$ is projective and $R/\operatorname{Soc} R_R$ is semisimple. If M_R is singular, then $M(\operatorname{Soc} R_R) = 0$, so M is an injective right $R/\operatorname{Soc} R_R$-module. Being ${}_R(R/\operatorname{Soc} R_R)$ flat (see Prop. 1.1), it follows that M_R is injective. ■

Goodearl proved the following decomposition theorem for right SI-rings (see [8, Theor. 3.11]):

THEOREM 2.7. *R is a right SI-ring if and only if R decomposes as:*

$$R = S \times R_1 \times \ldots \times R_n, \tag{2}$$

where $\operatorname{Soc} S_S \trianglelefteq S_S$ (see above Lemma 2.6) *and each R_i is a simple right SI-ring which is Morita equivalent to a domain.*

The most difficult (and longest) steep in Goodearl's proof of this theorem is to find the decomposition (2), where S has essential right socle and each R_i is simple. We wish to exhibit here a simpler and shorter proof of this fact.

PROOF OF THE DECOMPOSITION (2). Assume that R is a right SI-ring and set $\mathfrak{k} = \operatorname{Soc} R_R$. According to [8, Prop. 3.6], $R/\mathfrak{k}$ is a right Noetherian, right V-ring and then, being FRI, from [13, Lemma 3.1] $R/\mathfrak{k} = R_1 \times \ldots \times R_m$, where each R_i is a simple right Noetherian, right V-ring. Let us write $R_i = \mathfrak{h}_i/\mathfrak{k}$, for a suitable two-sided ideal $\mathfrak{h}_i$ of R, and set $\mathfrak{l}_i = l_R(\mathfrak{k}) \cap \mathfrak{h}_i$ $(1 \leqslant i \leqslant m)$. By Prop. 1.1 we have $\mathfrak{l}_i \cap \mathfrak{k} = \mathfrak{l}_i\mathfrak{k} = 0$ and hence $\mathfrak{k}\mathfrak{l}_i = 0$. There are elements $a_1, \ldots, a_m \in R$ with the following properties:

(a) $a_i \in \mathfrak{h}_i$,

(b) $a_i a_i - a_i \in \mathfrak{k}$,

(c) $a_i a_j \in \mathfrak{k}$ whenever $i \neq j$,

(d) $a_i r - r a_i \in \mathfrak{k}$ for each $r \in R$,

(e) $(a_i + \mathfrak{k})(R/\mathfrak{k}) = R_i$.

Given $i \in \{1, \ldots, m\}$, if $\mathfrak{l}_i = 0$, then $\mathfrak{k} \trianglelefteq (\mathfrak{h}_i)_R$ and conversely. Assume that $\mathfrak{l}_i \neq 0$. Inasmuch ax $\mathfrak{h}_i/\mathfrak{k} = R_i$ is a simple ring and $\mathfrak{l}_i \cap \mathfrak{k} = 0$, then $\mathfrak{h}_i = \mathfrak{l}_i \oplus \mathfrak{k}$; hence there are unique $e_i \in \mathfrak{l}_i$, $f_i \in \mathfrak{k}$ such that $a_i = e_i + f_i$.

We claim that e_i is a central idempotent and $e_i R = \mathfrak{l}_i$. Indeed, it is clear that $a_i^2 = e_i^2 + f_i^2$, so that $\mathfrak{k} \ni a_i^2 - a_i = e_i^2 - e_i + f_i^2 - f_i$, whence $e_i^2 = e_i$. Given $r \in R$, we have $\mathfrak{k} \ni a_i r - r a_i = e_i r - r e_i + f_i r - r f_i$, therefore $e_i r - r e_i = 0$, being $\mathfrak{l}_i$, $\mathfrak{k}$ two-sided ideals and $\mathfrak{l}_i \cap \mathfrak{k} = 0$. Thus e_i is a central idempotent. Finally, since $0 \neq e_i R \subset \mathfrak{l}_i$ and $\mathfrak{l}_i \simeq \mathfrak{l}_i \oplus \mathfrak{k}/\mathfrak{k} = \mathfrak{h}_i/\mathfrak{k} = R_i$ is a simple ring, we conclude that $e_i R = \mathfrak{l}_i$ and the canonical map $R \to R/\mathfrak{k}$ induces an isomorphism $e_i R \simeq R_i$.

We may now assume that there is an integer n, with $0 \leqslant n \leqslant m$, such that $\mathfrak{l}_i \neq 0$ iff $1 \leqslant i \leqslant n$ and we have the central idempotents $e_1, \ldots, e_n$ such that every $e_i R$ is a simple ring. Taking into account of (c), if $1 \leqslant i \leqslant j \leqslant n$, then $\mathfrak{k} \ni a_i a_j = e_i e_j + e_i f_j + f_i e_j + f_i f_j$, therefore $e_i e_j = 0$ (remember that $\mathfrak{l}_i \mathfrak{k} = 0 = \mathfrak{k}\mathfrak{l}_i$). Thus the e_i's are orthogonal. If we set $f = 1 - e_1 - \ldots - e_n$, then f is a central idempotent and, by the above, $fR = \mathfrak{h}_{n+1} + \ldots + \mathfrak{h}_m$ and $\mathfrak{k} \trianglelefteq fR_R$. We conclude that R decomposes as in (2), where $S = fR$ is a right SI-ring with essential right socle and each $R_i = e_i R$ is a simple right Noetherian, right SI-ring. ■

Since a right SI-ring has projective right socle and a ring with essential, projective right socle is right non-singular, we deduce from Theor. 2.7 the following corollary:

COROLLARY 2.8. *A right SI-ring is right non-singular.* ■

The following result slightly improves [8, Coroll. 3.7].

PROPOSITION 2.9. *Given a ring R, the following conditions are equivalent*:

(1) *R is a right SI-ring,* $\operatorname{Soc} R_R \trianglelefteq R_R$ *and* $J(R) = 0$.

(2) *R is semiprime and $R/\operatorname{Soc} R_R$ is semisimple.*

(3) *R is a regular left SI-ring.*

PROOF. (1) $\Leftrightarrow$ (2). It is a consequence of Lemma 2.6.

(2) $\Rightarrow$ (3). Inasmuch as R is semiprime, then $\operatorname{Soc} R_R = (\operatorname{Soc} R_R)^2 = \operatorname{Soc} {}_R R$ and Coroll. 1.8 implies that R is regular. Combining Prop. 1.1 with [2, Prop. 1.4], we see that $\operatorname{Soc} R \trianglelefteq R_R$ implies $\operatorname{Soc} R \trianglelefteq {}_R R$. Thus R is a left SI-ring by Lemma 2.6.

(3) $\Rightarrow$ (2). If R is a regular left SI-ring, then $R/\operatorname{Soc} R$ is a left Noetherian regular ring by [8, Prop. 3.6]; hence $R/\operatorname{Soc} R$ is semisimple. ∎

3. Regular rings which are strongly regular modulo their socle.

Let R be a given ring. We call R an *ASR-ring* (almost strongly regular ring) if R is regular and $R/\operatorname{Soc} R$ is strongly regular. In the following theorem we characterize the ASR-rings among those rings for which all essential maximal right ideals are two-sided (it is clear that an ASR-ring has this property).

THEOREM 3.1. *Assume that every essential maximal right ideal of the ring R is two-sided. Then the following conditions are equivalent to the condition that R is ASR*:

(1) *R is right w.s. and all simple left R-modules are flat.*

(2) *R is right w.s. and $R/(\operatorname{Soc} R_R)^2$ is strongly regular.*

(3) *R is right w.s. and is a right p-V'-ring.*

(4) *R is right w.s. and is a right GV-ring.*

(5) *R is regular.*

(6) *R is FRI.*

If, in addition, every projective simple right R-module P is $R/r_R(P)$-injective, then the above conditions are equivalent to:

(7) *R is a right V-ring.*

Finally, if all maximal right ideals of R are two-sided, then all the above conditions are equivalent to the following one:

(8) *R is strongly regular.*

PROOF. (1) $\Rightarrow$ (2). This is a consequence of [2, Theor. 3.7], since the hypothesis of the theorem implies that all maximal right ideals of $R/(\mathrm{Soc}\, R_R)^2$ are two-sided.

(2) $\Rightarrow$ (5). It is a consequence of Coroll. 1.8.

(5) $\Rightarrow$ (6). It is clear.

(6) $\Rightarrow$ (4). If R is *FRI*, then R is semiprime and hence $\mathrm{Soc}\, R = (\mathrm{Soc}\, R_R)^2$; thus $R/\mathrm{Soc}\, R_R$ is a right V-ring by [2, Theor. 3.7] and R is a right GV-ring by Theor. 2.2.

(4) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (2). According to Prop. 1.6, if R is a right p-V'-ring, then $\mathrm{Soc}\, R_R = (\mathrm{Soc}\, R_R)^2$ and $R/\mathrm{Soc}\, R_R$ is a right p-V-ring. Thus $R/\mathrm{Soc}\, R_R$ is *FRI* by [16, Lemma 1] and (2) follows from [2, Theor. 3.7].

(5) $\Rightarrow$ (1). It is clear.

Now, by virtue of the equivalence between (5) and (2), we see that (5) is equivalent to the fact that R is *ASR*.

Assume that every simple projective right R-module P is $R/r_R(P)$-injective and that conditions (1) through (6) are fulfilled. Then ${}_R(R/r_R(P))$ is flat by (5) and hence P_R is injective. Thus (4) implies (7). Conversely (7) implies (4), because a right V-ring is semiprime.

The last part of the theorem is again a consequence of [2, Theor. 3.7]. ■

REMARK 3.2. Rings with the property that all essential maximal left ideals are two-sided were named *MELT* rings by Yue Chi Ming in [19, 20]; he also considered semiprime *MELT* left p-V'-rings, semiprime *MELT* left p-V-rings, *MELT* regular rings and semiprime *MELT* rings whose simple right modules are flat. In view of the foregoing theorem, we see that all the above concepts coincide with the concept of an ASR ring. Consequently, Theor. 3.1 improves [18 Theor. 2] and [19, Prop. 9 and 10].

REMARK 3.3. If R is an ASR prime ring, then $\operatorname{Soc} R \neq 0$; otherwise R would be a prime strongly regular ring, namely a division ring, and hence $\operatorname{Soc} R = R$: a contradiction. Thus $\operatorname{Soc} R \trianglelefteq R$ and, in particular, R is primitive.

Recall that R is a *right full linear ring* if R is isomorphic to the endomorphism ring of a right vector space over some division ring. The following lemma provides a generalization of [19, Theor. 11 equivalence of conditions (1) and (4)].

LEMMA 3.4. *Let R be a prime regular ring with maximal right quotient ring Q. If Q is ASR, then R is Artinian.*

PROOF. By the above Remark 3.3 $\operatorname{Soc} Q \trianglelefteq Q$; this implies that Q is a right full linear ring. It follows from the hypothesis that Q is Artinian, so R has finite right Goldie dimension. We conclude that R is simple Artinian, being regular. ■

For ASR prime right V-rings we have the following result:

PROPOSITION 3.5. *Suppose that R is a prime ASR ring. Then R is a right V-ring if and only if there is a right full linear ring Q such that* $\operatorname{Soc} Q \subset R \subset Q$. *If it is the case, then R is also a left V-ring if and only if R is Artinian.*

PROOF. Let us write $\mathfrak{k} = \operatorname{Soc} R$. According to Remark 3.3, $\mathfrak{k}$ is essential in R; moreover there is an idempotent $e \in R$ such that $\mathfrak{k} = ReR$ and eR is the unique (up to an isomorphism) simple projective right R-module. It is well known that $Q = \operatorname{End} \mathfrak{k}_R$, therefore $Q\mathfrak{k} = \mathfrak{k}$. Since $E(eR_R) = eQ$, it is clear that eR_R is injective iff $\mathfrak{k} = \mathfrak{k}Q$, namely iff $\operatorname{Soc} Q \subset R$. The first part of our proposition is now a consequence of Theor. 3.1.

If R is a right and left V-ring, then R has faithful injective, projective simple right and left module, hence R is Artinian by [12, Coroll. 2.2]. ■

If R is an ASR ring, then every essential right ideal is two-sided. Right selfinjective rings with the latter property were studied by K. Byrd in [5], under the name of right q-rings. We are going to prove a structure theorem for right selfinjective ASR rings which is a decisive improvement of [20, Theor. 27]. Since these rings are merely right q-rings with zero Jacobson radical, our theorem could be proved by using Byrd's structure theorem [5, Theor. 6]. However, we exhibit here a more direct proof.

THEOREM 3.6. *Let R be a right selfinjective ASR ring. Then R decomposes as $R = S' \times S'' \times T$, where S' is a semisimple ring, S'' is a direct product of division rings and T is a right (and hence left) selfinjective, strongly regular ring with zero socle. Thus R is a right and left V-ring with bounded index of nilpotence* (see [10, page 71]).

PROOF. Let us write $\mathfrak{k} = \operatorname{Soc} R$. Being R right selfinjective, there is an idempotent $e \in R$ such that $eR = E(\mathfrak{k}_R)$. We claim that e is central. Indeed, since $R/\mathfrak{k}$ is strongly regular, for each $r \in R$ there is $x \in \mathfrak{k}$ such that $er = re + x$, hence $re \in eR$. Thus $Re \subset eR$ and therefore $ReR = eR$. Being R semiprime, it follows that e is central. It is now clear that $T = (1-e)R$ is a right (and hence left) selfinjective strongly regular ring with zero socle.

The ring $S = eR$ is regular, right selfinjective and $\operatorname{Soc} S = \mathfrak{k}$ is essential in S. As it is well known (see e.g. [9, Theor. 3.28, page 92]), there is a family $(S_i)_{i\in I}$ of right full linear rings such that $S = \prod_{i\in I} S_i$. From the hypothesis, each S_i is ASR, so it is of the form $M_{n_i}(D_i)$, where D_i is a division ring and n_i a positive integer. We claim that $n_i = 1$ for all but a finite number of indices $i \in I$. Assume, on the contrary, that there is an infinite subset $J \subset I$ with $n_i > 1$ for all $i \in J$. For every $i \in J$, choose two non zero idempotents u_i, $v_i \in S_i$ such that $u_i v_i = v_i$ and $v_i u_i = 0$ and consider a, $b \in S$ with

$$a_i = \begin{cases} u_i & \text{if } i \in J \\ 0 & \text{if } i \in I - J \end{cases} \qquad b_i = \begin{cases} v_i & \text{if } i \in J \\ 0 & \text{if } i \in I - J . \end{cases}$$

Then a, b are idempotents and, taking into account that all idempotents in $S/\mathfrak{k} \subset R/\mathfrak{k}$ are central, we get $\mathfrak{k} = \bigoplus_{i\in I} S_i \not\ni b = ab - ba \in \mathfrak{k}$: a contradiction. We conclude that $S = S' \times S''$, where S' is a semisimple ring and S'' is a direct product of division rings. ■

4. Examples.

a) Let D be a division ring, let V_D be an infinite dimensional vector space over D, set $Q = \operatorname{End} V_D$ and $\mathfrak{k} = \operatorname{Soc} Q$. Let S be any strongly regular subring of Q and set $A = S + \mathfrak{k}$, the subring of Q generated by $\mathfrak{k}$ and S. It is clear that $\operatorname{Soc} A = \mathfrak{k} \trianglelefteq A$ and A is prime. Since $A/\mathfrak{k} = S + \mathfrak{k}/\mathfrak{k} \cong S/S \cap \mathfrak{k}$ is strongly regular and $\mathfrak{k}^2 = \mathfrak{k}$, it follows from Theor. 3.1 that A is ASR and is a right and left GV-ring. According to Prop. 3.5, A is a right V-ring which is not a left V-ring. In particular, if T is the subring of Q of all scalar transformations of V_D, then $B = T + \mathfrak{k}$ is the ring described in [10, example 6.19, page 68]. Note that B is a right (and left) SI-ring, since $B/\mathfrak{k} = D$.

The following examples envolve formal triangular matrix rings. For the basic properties of these rings see [9, Chap. 4].

b) Let S, T be two semisimple rings and let M be a faithful (T, S)-bimodule. Then $R = \begin{pmatrix} S & 0 \\ M & T \end{pmatrix}$ is a right and left SI-ring which is neither a right nor a left V-ring, since $J(R) = \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} \neq 0$.

c) Let A and $\mathfrak{k}$ be as in example *a*) and consider the ring $R = \begin{pmatrix} A & 0 \\ \mathfrak{k} & A \end{pmatrix}$. Then $\operatorname{Soc} R_R = \begin{pmatrix} \mathfrak{k} & 0 \\ \mathfrak{k} & 0 \end{pmatrix}$ and $\operatorname{Soc}{}_R R = \begin{pmatrix} 0 & 0 \\ \mathfrak{k} & \mathfrak{k} \end{pmatrix}$ are both projective. Since $R/\operatorname{Soc} R_R \cong R/\operatorname{Soc}_R R \cong A \times A/\mathfrak{k}$ and the latter is a right V-ring (recall that $A/\mathfrak{k}$ is strongly regular) which is not a left V-ring, it follows from Theor. 2.2 that R is a right GV-ring and is not a left GV-ring. Since $J(R) = \begin{pmatrix} 0 & 0 \\ \mathfrak{k} & 0 \end{pmatrix} \neq 0$, R is not a right V-ring. Observe that R is right and left nonsingular and is neither a right nor a left SI-ring.

d) Let R_M be the domain described in [6, page 78]. R_M is a simple non Artinian right and left V-ring which is a $\mathbb{Z}_2$-algebra. If we consider the ring $R = \begin{pmatrix} R_M & 0 \\ R_M & \mathbb{Z}_2 \end{pmatrix}$, then $\operatorname{Soc} R_R = 0$ and $\operatorname{Soc}_R R = \begin{pmatrix} 0 & 0 \\ R_M & \mathbb{Z}_2 \end{pmatrix}$ is idempotent. Since $R/\operatorname{Soc}_R R \cong R_M$ is a V-ring, it follows from Theor. 2.2 that R is a left GV-ring. It cannot be a right GV-ring, otherwise it would be a right V-ring, in contradiction with the fact that $J(R) \neq 0$.

e) Let B and $\mathfrak{k}$ be as in example *a*) and consider the ring $R = \begin{pmatrix} B & 0 \\ B/\mathfrak{k} & D \end{pmatrix}$. Then $\operatorname{Soc} R_R = \begin{pmatrix} \mathfrak{k} & 0 \\ B/\mathfrak{k} & 0 \end{pmatrix}$ and $\operatorname{Soc}{}_R R = \begin{pmatrix} \mathfrak{k} & 0 \\ B/\mathfrak{k} & D \end{pmatrix}$. Since $(\operatorname{Soc} R_R)^2 = \begin{pmatrix} \mathfrak{k} & 0 \\ 0 & 0 \end{pmatrix} \neq \operatorname{Soc} R_R$, R is not a right GV-ring, although

$R/\mathrm{Soc}\, R_R \simeq D \times D$ is a V-ring. On the other hand we have

$$\mathrm{Soc}_R R = \begin{pmatrix} \mathfrak{k} & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R = (\mathrm{Soc}_R R)^2 \quad \text{and} \quad R/\mathrm{Soc}_R R \simeq D,$$

thus R is a left SI-ring by Lemma 2.6. Note that $Z(R_R) = \begin{pmatrix} 0 & 0 \\ B/\mathfrak{k} & 0 \end{pmatrix} \neq 0.$

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