RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 71 (1984), p. 73-102

<http://www.numdam.org/item?id=RSMUP_1984__71__73_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 70 (1983)

Existence Theorems for Compressible Viscous Fluids Having Zero Shear Viscosity.

PAOLO SECCHI (*)

Compressible viscous fluids have been studied by several authors in the last twenty years. Existence theorems (local in time) for the Cauchy problem in $\mathbb{R}^3$ were proved in 1962 by Nash [13], in 1971 by Itaya [7] and in 1972 by Vol'pert-Hudjaev [23]. More recently, in 1980, Matsumara-Nishida [12] proved that the solution exists for all time, provided the initial velocity is small, and the initial density and temperature are close to constants. With regard to the initial-boundary value problem, Solonnikov [17] in 1976 proved an existence theorem for barotropic fluids (i.e. fluids for which the pressure depends only on the density) with constant viscosities. In 1977, Tani [19] proved the existence of a unique solution for the general case in bounded or unbounded domains. (A complete survey of these papers can be found in Solonnikov-Kazhikhov [18]). An existence theorem for the initial-boundary value problem was proved also by Valli [20], using an approach which is somewhat similar to ours.

In all these papers, the authors assume the shear viscosity μ (or « first coefficient of viscosity ») strictly positive and the dilatational viscosity μ' (or « second coefficient of viscosity ») to be such that $2\mu + 3\mu' \geqslant 0$. In the present work, we assume $\mu = 0$ and $\mu' > 0$. For the sake of convenience, we introduce the bulk viscosity $\zeta \equiv (2\mu + 3\mu')/3$, so that we have $\zeta = \mu' > 0$. A thorough discussion

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Work partially supported by G.N.A.F.A. of C.N.R.

of the meaning of these viscosity coefficients can be found in [14]. In [14], it is shown that there exist fluids for which $\mu' > 100\mu$ and for which the effects due to the dilatational viscosity are much greater than the ones due to the shear viscosity. Concerning these matters see also Chapman-Cowling [5].

The equations of motion that we study are obtained for instance in Serrin [15]. In particular, observe that the principal part of the system $(1.1)_1$, or $(1.6)_1$ (neglecting for simplicity the coefficient ζ) is the operator $L \equiv -\nabla$ div, which is *not* elliptic in the sense of Agmon-Douglis-Nirenberg [1], since det $L(\Xi) = 0$ for *any* real vector Ξ (see [1]). This implies that $(1.1)_1$ is *not* parabolic in the sense of the definitions given in Ladyženskaja-Solonnikov-Ural'ceva [9]). On the contrary, it is parabolic in the sense of the generalized definition of Kato [8], as L is generator of an analytic semigroup in $L^2(\Omega)$. Hence, equation $(1.1)_1$ is interesting also from a mathematical point of view, since it belongs to the class of parabolic equations only in a generalized sense and not in the sense of classical definitions. Also, observe that the equation $(1.1)_2$ for the density is hyperbolic, while the equation for the temperature is parabolic in case $(1.1)_3$ and hyperbolic in case $(1.6)_3$. Concerning the boundary conditions and their physical meaning, see for instance Serrin [16].

In this paper, we prove two existence theorems (local in time) for two cases of the initial-boundary value problems, written in general form. In the first case we take the coefficient of heat conductivity positive, and in the second we take it equal to zero. The solution is found in Sobolev spaces of Hilbert type.

A uniqueness theorem for this type of problems was first obtained by Serrin [16]. Another uniqueness result, in a slightly more general context (which, in particular, covers our cases), is proved by Valli in [22].

Finally, we conclude our brief survey on compressible fluids, by recalling the paper of Beirão da Veiga [2], where an existence theorem for barotropic *non*-viscous fluids is proved, making use of a fixed point argument, somewhat related to our method.

1. Statement of the problems and main results.

Let Ω be a bounded connected open subset of $\mathbf{R}^3$. Assume that the boundary Γ is a compact manifold of dimension 2, without boundary and that Ω is locally situated on one side of Γ. Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$ such that Γ_i ($i = 1, \dots, m$) are inside of Γ_0 and outside of one another. Set $Q_T \equiv]0, T[\times\Omega$, $\Sigma_T \equiv]0, T[\times\Gamma$. Let n be the unit outward normal vector to Γ.

Let us denote by $v = v(t, x)$, $\varrho = \varrho(t, x)$, $\theta = \theta(t, x)$ the velocity field, the density and the absolute temperature, respectively. The equations that we want to study are

$$
(1.1)\quad \begin{cases}
\varrho[\dot{v} + (v\cdot\nabla)v - b] = -\nabla p + \nabla(\zeta \operatorname{div} v) & \text{in } Q_T\,, \\
\dot{\varrho} + v\cdot\nabla\varrho + \varrho \operatorname{div} v = 0 & \text{in } Q_T\,, \\
c_V\varrho[\dot{\theta} + v\cdot\nabla\theta] = -\theta\dfrac{\partial p}{\partial\theta}\operatorname{div} v + \operatorname{div}(\chi\nabla\theta) + \varrho r + \zeta(\operatorname{div} v)^2 & \\
 & \text{in } Q_T\,, \\
v\cdot n = 0 & \text{on } \Sigma_T\,, \\
\theta = \theta_1 & \text{on } \Sigma_T\,, \\
v(0, x) = v_0(x) & \text{in } \Omega\,, \\
\varrho(0, x) = \varrho_0(x) & \text{in } \Omega\,, \\
\theta(0, x) = \theta_0(x) & \text{in } \Omega\,.
\end{cases}
$$

We shall also consider problem (1.6) (obtained from (1.1) setting $\chi = 0$ in $(1.1)_3$ and neglecting the boundary condition $(1.1)_5$). We denote by $\dot{v}$ the time derivative, $b = b(t, x)$ the external force field per unit mass and $r = r(t, x)$ the heat supply per unit mass per unit time. The pressure $p = p(\varrho, \theta)$ and the specific heat at constant volume $c_V = c_V(\varrho, \theta)$ are known functions of ϱ and θ, the coefficient of bulk viscosity $\zeta = \zeta(\varrho, \theta, v)$ and the coefficient of heat conductivity $\chi = \chi(\varrho, \theta, v)$ are given functions of ϱ, θ and v; $\theta_1 = \theta_1(t, y)$ ($y \in \Gamma$) is the assigned temperature on Σ_T. Finally $v_0 = v_0(x)$, $\varrho_0 = \varrho_0(x)$ and $\theta_0 = \theta_0(x)$ are the initial data.

Let us denote by $C^0(\bar{\Omega})$ the space of continuous (and bounded) functions on $\bar{\Omega}$ and by $C^k(\bar{\Omega})$ (k positive integer) the space of functions with derivatives up to order k in $C^0(\bar{\Omega})$. If m is a positive integer, $H^m(\Omega)$ is the Sobolev space of functions with m derivatives in $L^2(\Omega)$; we shall denote its norm by $\|\cdot\|_m$ and by $\|\cdot\|$ the norm of $H^0(\Omega) \equiv L^2(\Omega)$. For the definition of $H^s(\Gamma)$ (s not integral) see for instance Lions-Magenes [10]; we shall denote its norm by $\|\cdot\|_{s,\Gamma}$. If X is a Banach space, $L^2(0, T; X)$, $L^\infty(0, T; X)$, $H^m(0, T; X)$, $H^s(0, T; X)$ are the spaces of X-valued functions in L^2, L^∞, H^m and H^s, respectively.

We shall denote by $H^1_\infty(0, T; X)$ the space $\{u \in L^\infty(0, T; X)$ with $\dot{u} \in L^\infty(0, T; X)\}$. $C^0([0, T]; X)$ and $C^\alpha([0, T]; X)$ are the spaces of X-valued continuous and Hölder-continuous (with exponent α) functions, respectively. We shall denote by $|\cdot|_{p,m,T}$ the norm of $L^p(0, T; H^m(\Omega))$, $1 \leqslant p \leqslant +\infty$, by $\|\cdot\|_{m,m/2,T}$ the norm of the space $H^{m,m/2}(Q_T) \equiv \equiv L^2(0, T; H^m(\Omega)) \cap H^{m/2}(0, T; L^2(\Omega))$, and by $\|\cdot\|_{s,s/2,\Sigma_T}$ the norm of the space $H^{s,s/2}(\Sigma_T) \equiv L^2(0, T; H^s(\Gamma)) \cap H^{s/2}(0, T; L^2(\Gamma))$.

We prove the following results.

THEOREM A. Let Γ be of class C^4. Suppose $b \in H^{2,1}(Q_{T_0})$ with $\operatorname{rot} b \in L^2(0, T_0; H^2(\Omega))$, $r \in H^{2,1}(Q_{T_0})$, $p \in C^3$, $c_V \in C^3$, $c_V > 0$, $\zeta \in C^3$, $\zeta > 0$, $\chi \in C^3$, $\chi > 0$. Suppose $\theta_1 \in H^{7/2,7/4}(\Sigma_{T_0})$, $v_0 \in H^3(\Omega)$, $\varrho_0 \in H^3(\Omega)$ with $\min\limits_{x \in \bar{\Omega}} \varrho_0(x) \equiv \hat{\varrho}_0 > 0$, $\theta_0 \in H^3(\Omega)$.

Assume that the (necessary) compatibility conditions

$$(1.2) \qquad v_0 \cdot n = 0 \qquad \text{on } \Gamma,$$

$$(1.3) \qquad (v_0 \cdot \nabla) v_0 \cdot n - b(0) \cdot n + \frac{1}{\varrho_0} \frac{\partial}{\partial n} p(\varrho_0, \theta_0) - \frac{1}{\varrho_0} \frac{\partial}{\partial n} [\zeta(\varrho_0, \theta_0, v_0) \operatorname{div} v_0] = 0 \qquad \text{on } \Gamma,$$

$$(1.4) \qquad \theta_0 = \theta_1(0) \qquad \text{on } \Gamma,$$

$$(1.5) \qquad c_V(\varrho_0, \theta_0)\varrho_0[\dot{\theta}_1(0) + v_0 \cdot \nabla\theta_0] = -\theta_0 \frac{\partial p}{\partial \theta}(\varrho_0, \theta_0) \operatorname{div} v_0 + \operatorname{div}(\chi(\varrho_0, \theta_0, v_0)\nabla\theta_0) + \varrho_0 r(0) + \zeta(\varrho_0, \theta_0, v_0)(\operatorname{div} v_0)^2 \qquad \text{on } \Gamma$$

are satisfied.

Then there exist $T' \in]0, T_0]$,

$$v \in C^0([0, T']; H^3(\Omega)) \cap H^1(0, T'; H^2(\Omega)) \cap H^2(0, T'; L^2(\Omega))$$

with $\operatorname{div} v \in L^2(0, T'; H^3(\Omega))$, $\varrho \in C^0([0, T']; H^3(\Omega)) \cap H^2(0, T'; H^1(\Omega))$ with $\dot{\varrho} \in L^\infty(0, T'; H^2(\Omega))$ such that $\varrho > 0$ in $\bar{Q}_{T'}$ and $\theta \in H^{4,2}(Q_{T'})$ such that (v, ϱ, θ) is a solution of (1.1) in $Q_{T'}$.

THEOREM B. Let Γ be of class C^3. Suppose $b \in H^{2,1}(Q_{T_0})$ with $\operatorname{rot} b \in L^2(0, T_0; H^2(\Omega))$, $r \in L^2(0, T_0; H^3(\Omega)) \cap L^\infty(0, T_0; H^1(\Omega))$, $p \in C^3$ with $\partial p/\partial\theta \in C^3$, $c_V \in C^3$, $c_V > 0$, $\zeta \in C^3$, $\zeta > 0$. Assume $v_0 \in H^3(\Omega)$, $\varrho_0 \in H^3(\Omega)$ with $\min\limits_{x \in \bar{\Omega}} \varrho_0(x) \equiv \hat{\varrho}_0 > 0$, $\theta_0 \in H^3(\Omega)$.

Assume that the (necessary) compatibility conditions (1.2), (1.3) hold. Then there exist $T'' \in]0, T_0]$, $v \in C^0([0, T'']; H^3(\Omega)) \cap H^1(0, T''; H^2(\Omega)) \cap H^2(0, T''; L^2(\Omega))$ with $\operatorname{div} v \in L^2(0, T''; H^3(\Omega))$, $\varrho \in C^0([0, T'']; H^3(\Omega)) \cap H^2(0, T''; H^1(\Omega))$ with $\dot\varrho \in L^\infty(0, T''; H^2(\Omega))$ such that $\varrho > 0$ in $\bar{Q}_{T''}$, $\theta \in C^0([0, T'']; H^3(\Omega)) \cap H^1(0, T''; H^2(\Omega))$ with $\dot\theta \in L^\infty(0, T''; H^1(\Omega))$ such that (v, ϱ, θ) is a solution of problem

$$
(1.6)\qquad
\begin{cases}
\varrho[\dot v + (v\cdot\nabla)v - b] = -\nabla p + \nabla(\zeta \operatorname{div} v) & \text{in } Q_{T''}, \\
\dot\varrho + v\cdot\nabla\varrho + \varrho \operatorname{div} v = 0 & \text{in } Q_{T''}, \\
c_V\varrho[\dot\theta + v\cdot\nabla\theta] = -\theta \dfrac{\partial p}{\partial\theta} \operatorname{div} v + \varrho r + \zeta(\operatorname{div} v)^2 & \text{in } Q_{T''}, \\
v\cdot n = 0 & \text{on } \Sigma_{T''}, \\
v(0, x) = v_0(x) & \text{in } \Omega, \\
\varrho(0, x) = \varrho_0(x) & \text{in } \Omega, \\
\theta(0, x) = \theta_0(x) & \text{in } \Omega.
\end{cases}
$$

2. - Proof of Theorem A.

We suppose that Ω is simply-connected. For the case Ω not simply-connected see Remark 2.1. We shall prove Theorem A by the construction of three successive fixed points.

Let $T \in]0, T_0]$.

(H_1) Let $\psi \in L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega))$ with $\dot\psi \in L^\infty(0, T; L^2(\Omega))$ such that $\int_\Omega \psi\, dx = 0$ for each $t \in [0, T]$, $\psi(0) = \operatorname{div} v_0$ in Ω and

$$|\psi|_{2,3,T} + |\dot\psi|_{2,1,T} + |\dot\psi|_{\infty,0,T} \leqslant A\,,$$

where A is a positive constant, which will be specified in (2.41).

(H_2) Let $\tilde\theta \in H^{4,2}(Q_T)$ such that $\tilde\theta = \theta_1$ on Σ_T, $\tilde\theta(0) = \theta_0$ in Ω and $\|\tilde\theta\|_{4,2,T} \leqslant B$ (B will be specified in (2.25)).

(H_3) Let $\varphi \in C^0([0, T]; H^2(\Omega))$ such that for each $t \in [0, T]$ $\operatorname{div} \varphi = 0$ a.e. in Ω, $\int_{\Gamma_i} \varphi\cdot n\, d\sigma = 0$, $i = 1, \ldots, m$, $\varphi(0) = \operatorname{rot} v_0$ in Ω and $|\varphi|_{\infty,2,T} \leqslant D$ (D will be specified in (2.8)).

Then there exists a unique solution v of the elliptic system

$$\begin{cases} \operatorname{rot} v = \varphi & \text{in } Q_T\,, \\ \operatorname{div} v = \psi & \text{in } Q_T\,, \\ v\cdot n \;\; = 0 & \text{on } \Sigma_T\,, \end{cases} \tag{2.1}$$

such that $v \in C^0([0,T]; H^3(\Omega))$ and $|v|_{\infty,3,T} \leqslant C(A+D)$. Here and in the sequel, C will denote every generic constant; we shall denote by $\bar{c}(\cdot)$ a non-decreasing function of all its arguments, depending also at most on the data of the problem $\Omega, T_0, b, r, p, c_V, \zeta, \chi, \theta_1, \dot{v}_0, \varrho_0, \theta_0$.

Now consider the problem

$$\begin{cases} \dot{\varrho} + v\cdot\nabla\varrho + \varrho \operatorname{div} v = 0 & \text{in } Q_T\,, \\ \varrho(0) = \varrho_0 & \text{in } \Omega\,. \end{cases} \tag{2.2}$$

LEMMA 2.1. There exists a unique solution ϱ of problem (2.2), such that $\varrho \in C^0([0,T]; H^3(\Omega))$, $\dot{\varrho} \in L^\infty(0,T; H^2(\Omega))$ and $\varrho(t,x) > 0$ $\forall (t,x) \in \bar{Q}_T$.

PROOF. As $v \in C^0([0,T]; C^1(\bar{\Omega}))$, we can construct the solution ϱ by using the method of characteristics. Set $z(\sigma,t,x) \equiv z(\sigma)$, $\sigma, t \in [0,T]$, $x \in \bar{\Omega}$, where $z(\sigma)$ is the solution of

$$\begin{cases} \dfrac{dz}{d\sigma} = v(\sigma, z(\sigma)) & \text{in } [0,T]\,, \\ z(t) = x\,; \end{cases}$$

(such a solution is global since $v\cdot n = 0$ in Σ_T). Then the solution $h \equiv \log\varrho$ of

$$\dot{h} + v\cdot\nabla h = -\operatorname{div} v \qquad \text{in } Q_T\,,$$

$$h(0) = h_0 \equiv \log\varrho_0 \qquad \text{in } \Omega\,,$$

is given by

$$h(t,x) = h_0(z(0,t,x)) - \int_0^t \operatorname{div} v(s, z(s,t,x))\, ds\,.$$

From this expression one obtains $\varrho(t,x)$.

Applying the operator D^γ to $(2.2)_1$, where γ is a multi-index with $|\gamma|\leqslant 3$, multiplying for $D^\gamma \varrho$ and integrating over Ω, one easily gets

$$\frac{d}{dt}\|\varrho\|_3 \leqslant C\{\|v\|_3 + \|\psi\|_3\}\|\varrho\|_3 ,$$

and so $\varrho \in L^\infty(0, T; H^3(\Omega))$. Since $z(\sigma, t, x) \in C^0([0, T]\times[0, T]; H^3(\Omega))$ and, for each $\sigma, t \in [0, T]$, $z(\sigma, t, x)$ is a C^1 diffeomorphism from $\bar{\Omega}$ onto $\bar{\Omega}$, with $z(\partial\Omega) \subseteq \partial\Omega$, one obtains $\varrho \in C^0([0, T]; H^3(\Omega))$ (see Bourguignon-Brezis [4], Lemmas A3, A5, A6). Directly from the equation we have $\dot{\varrho} \in L^\infty(0, T; H^2(\Omega))$. Finally, one has

$$\begin{aligned} h(t, x) &\geqslant h_0(z(0, t, x)) - \int_0^t |\operatorname{div} v(s, z(s, t, x))|\, ds \\ &\geqslant \log \hat{\varrho}_0 - C|\psi|_{2,3,T} T_0^{\frac12} \geqslant \log \hat{\varrho}_0 - CAT_0^{\frac12} , \end{aligned}$$

where $\hat{\varrho}_0 = \min\limits_{x\in\bar{\Omega}} \varrho_0(x) > 0$; then

$$\varrho(t, x) = \exp[h(t, x)] \geqslant \hat{\varrho}_0 \exp[-CAT_0^{\frac12}] > 0 \qquad \forall (t, x) \in \bar{Q}_T . \quad \square$$

Now consider the following equation (formally obtained by taking the curl of $\dot{v} + (v\cdot\nabla)v - b = (1/\varrho)[-\nabla p + \nabla(\zeta \operatorname{div} v)]$ and writing $\operatorname{rot} v = \xi$):

$$(2.3) \qquad \begin{cases} \dot{\xi} + (v\cdot\nabla)\xi - (\xi\cdot\nabla)v + \xi \operatorname{div} v = \\ \qquad\qquad = \operatorname{rot} b + \dfrac{\nabla\varrho}{\varrho^2} \wedge \nabla\bar{p} - \dfrac{\nabla\varrho}{\varrho^2} \wedge \nabla(\bar{\zeta} \operatorname{div} v) & \text{in } Q_T , \\ \xi(0) = \operatorname{rot} v_0 & \text{in } \Omega , \end{cases}$$

where $\bar{p} = p(\varrho, \bar{\theta})$, $\bar{\zeta} = \zeta(\varrho, \bar{\theta}, v)$. As for equation (2.2), we can construct the solution ξ by using the method of characteristics. Moreover one has the following results:

LEMMA 2.2. Let ξ be the solution of (2.3). Then $\xi \in C^0([0, T]; H^2(\Omega))$ with $\dot{\xi} \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ and

$$(2.4) \qquad |\xi|_{\infty,2,T} \leqslant \exp[\bar{c}(A, D)T]\{\|\operatorname{rot} v_0\|_2 + \bar{c}(A, B, D)T^{\frac12}\} .$$

Moreover, for each $t \in [0, T]$,

$$\operatorname{div} \xi = 0 \quad \text{a.e. in } \Omega , \tag{2.5}$$

$$\int_{\Gamma_i} \xi \cdot n = 0 \quad i = 1, \ldots, m . \tag{2.6}$$

PROOF. Applying the operator D^γ to $(2.3)_1$, where γ is a multi-index with $|\gamma| \leqslant 2$, multiplying for $D^\gamma \xi$ and integrating over Ω, one gets

$$\frac{1}{2} \frac{d}{dt} \|\xi\|_2^2 \leqslant C \left\{ \|v\|_3 \|\xi\|_2 + \|\operatorname{rot} b\|_2 + \left\| \frac{\nabla \varrho}{\varrho^2} \right\|_2 \|\bar{p}\|_3 + \left\| \frac{\nabla \varrho}{\varrho^2} \right\|_2 \|\bar{\zeta}\|_3 \|\psi\|_3 \right\} \|\xi\|_2 ,$$

from which one has

$$\frac{d}{dt} \|\xi\|_2 \leqslant \bar{c}(A, D) \|\xi\|_2 + \bar{c}(A, B, D) + C [\, \|\operatorname{rot} b\|_2 + \bar{c}(A, B, D) \|\psi\|_3] .$$

From Gronwall's lemma one has (2.4). By the same arguments used to prove $\varrho \in C^0([0, T]; H^3(\Omega))$, one obtains $\xi \in C^0([0, T]; H^2(\Omega))$.

From $(2.3)_1$ one obtains

$$|\dot{\xi}|_{2,1,T} + |\dot{\xi}|_{\infty,0,T} \leqslant \bar{c}(A, B, D) .$$

Observe that, by interpolation, from $\bar{\theta} \in H^{4,2}(Q_T)$ one obtains $\bar{\theta} \in C^0([0, T]; H^3(\Omega))$ with

$$|\bar{\theta}|_{\infty,3,T} \leqslant C \{ \|\bar{\theta}\|_{4,2,T} + \|\bar{\theta}(0)\|_3 \}$$

where C is independent of T (see Lions-Magenes [11]); hence one has $\bar{p}, \bar{\zeta} \in L^\infty(0, T; H^3(\Omega))$ with $|\bar{p}|_{\infty,3,T} \leqslant \bar{c}(A, B, D)$, $|\bar{\zeta}|_{\infty,3,T} \leqslant \bar{c}(A, B, D)$. Analogously from $b \in H^{2,1}(Q_{T_0})$ one obtains $b \in C^0([0, T_0]; H^1(\Omega))$.

In order to prove (2.5), observe that, by the general formula

$$(v \cdot \nabla) \xi - (\xi \cdot \nabla) v + \xi \operatorname{div} v = v \operatorname{div} \xi - \operatorname{rot} (v \wedge \xi) ,$$

and since

$$\operatorname{rot} b + \frac{\nabla \varrho}{\varrho^2} \wedge \nabla \bar{p} - \frac{\nabla \varrho}{\varrho^2} \wedge \nabla (\bar{\zeta} \operatorname{div} v) = \operatorname{rot} \left(b - \frac{\nabla \bar{p}}{\varrho} + \frac{\nabla (\bar{\zeta} \operatorname{div} v)}{\varrho} \right),$$

one can write (2.3) as

$$\text{(2.7)} \qquad \begin{cases} \dot{\xi} + v \operatorname{div} \xi - \operatorname{rot}(v \wedge \xi) = \operatorname{rot}\left(b - \dfrac{\nabla \bar{p}}{\varrho} + \dfrac{\nabla(\xi \operatorname{div} v)}{\varrho}\right) \\ \xi(0) = \operatorname{rot} v_0 \,. \end{cases}$$

Applying the operator div to both side of (2.7) one gets

$$\begin{cases} \dfrac{\partial}{\partial t} \operatorname{div} \xi + v \cdot \nabla \operatorname{div} \xi + \operatorname{div} v \operatorname{div} \xi = 0 \\ \operatorname{div} \xi(0) = 0 \,, \end{cases}$$

from which (2.5) follows.
Finally, by $(2.7)_1$ we have

$$\frac{d}{dt} \int_{\Gamma_i} \xi \cdot n \, d\sigma = \int_{\Gamma_i} \dot{\xi} \cdot n \, d\sigma = 0 \qquad \forall i = 1, \dots, m \,,$$

since $\int_{\Gamma_i} \operatorname{rot} G \cdot n \, d\sigma = 0$ for each G. Hence, for each $t \in [0, T]$,

$$\int_{\Gamma_i} \xi \cdot n \, d\sigma = \int_{\Gamma_i} \operatorname{rot} v_0 \cdot n \, d\sigma = 0 \qquad \forall i = 1, \dots, m \,. \qquad \square$$

We can now construct a fixed point of the map $\Phi_1 \colon \varphi \to \xi$. In fact, choose

$$\text{(2.8)} \qquad D > \|\operatorname{rot} v_0\|_2 \,.$$

Then from estimate (2.4) one sees that there exists $T_1 \in \,]0, T_0]$ such that the set

$$S_1 = \Big\{ \varphi \in C^0([0, T_1]; H^2(\Omega)) \colon |\varphi|_{\infty,2,T_1} \leqslant D, \ \operatorname{div} \varphi = 0 \text{ a.e. in } \Omega, \int_{\Gamma_i} \varphi \cdot n \, d\sigma = 0 \ \forall i = 1, \dots, m, \quad \varphi(0) = \operatorname{rot} v_0 \Big\}$$

satisfies $\Phi_1[S_1] \subseteq S_1$.
S_1 is obviously convex, bounded and closed in $X_1 \equiv C^0([0, T_1]; H^1(\Omega))$.

As $\Phi_1[S_1]$ is bounded in $H^1(0, T_1; H^1(\Omega))$, $\Phi_1[S_1]$ is a bounded subset of $C^{\frac{1}{2}}([0, T_1]; H^1(\Omega)) \cap C^0([0, T_1]; H^2(\Omega))$. From the Ascoli-Arzelà theorem, $\Phi_1[S_1]$ is relatively compact in X_1.

Let now φ_n, $\varphi \in S_1$, $\varphi_n \to \varphi$ in X_1. Then the solutions v_n of the elliptic system (2.1) (where $\operatorname{div} v_n = \operatorname{div} v = \psi$) converge to v in $C^0([0, T_1]; H^2(\Omega))$. From (2.2) one obtains

$$\frac{1}{2}\frac{d}{dt}\|\varrho_n - \varrho\|_2^2 \leqslant C\{\|v_n - v\|_2\|\varrho_n\|_3\|\varrho_n - \varrho\|_2 + \\ + \|v\|_3\|\varrho_n - \varrho\|_2^2 + \|\psi\|_2\|\varrho_n - \varrho\|_2^2\},$$

$$\frac{d}{dt}\|\varrho_n - \varrho\|_2 \leqslant \bar{c}(A, D)\{\|v_n - v\|_2 + \|\varrho_n - \varrho\|_2\}.$$

Hence from the Gronwall's lemma one has $\varrho_n \to \varrho$ in $C^0([0, T_1]; H^2(\Omega))$. Finally, by evaluating $d/dt\|\xi_n - \xi\|_1^2$ in a standard way, one has

$$\frac{d}{dt}\|\xi_n - \xi\|_1 \leqslant \bar{c}(A, B, D)\{\|\xi_n - \xi\|_1 + (1 + \|\psi\|_3)(\|v_n - v\|_2 + \|\varrho_n - \varrho\|_2)\},$$

and consequently $\xi_n \to \xi$ in X_1.

Then Φ_1 is continuous in the topology of X_1. From the Schauder fixed point theorem, there exists $\varphi = \xi$. Let v be the corresponding solution of (2.1). As $\dot{\xi} \in L^2(0, T_1; H^1(\Omega)) \cap L^\infty(0, T_1; L^2(\Omega))$, by differentiating in t system (2.1), we get $\dot{v} \in L^2(0, T_1; H^2(\Omega)) \cap L^\infty(0, T_1; H^1(\Omega))$. From (2.2) we obtain $\dot{\varrho} \in H^1(0, T_1; H^1(\Omega))$. Moreover, from

$$\begin{cases} \operatorname{rot}[v(0) - v_0] = \xi(0) - \operatorname{rot} v_0 = 0 & \text{in } \Omega, \\ \operatorname{div}[v(0) - v_0] = \psi(0) - \operatorname{div} v_0 = 0 & \text{in } \Omega, \\ [v(0) - v_0]\cdot n = 0 & \text{on } \Gamma, \end{cases}$$

one has $v(0) = v_0$ in Ω.

Then we have the following result:

THEOREM 2.1. For each $\psi, \bar{\theta}$ satisfying hypotheses (H_1), (H_2) respectively, there exist $T_1 \in]0, T_0]$, $v \in C^0([0, T_1]; H^3(\Omega)) \cap H^1(0, T_1; H^2(\Omega))$ with $\dot{v} \in L^\infty(0, T_1; H^1(\Omega))$, $\varrho \in C^0([0, T_1]; H^3(\Omega)) \cap H^2(0, T_1; H^1(\Omega))$ with $\dot{\varrho} \in L^\infty(0, T_1; H^2(\Omega))$ such that $\varrho(t, x) > 0$ $\forall (t, x) \in \bar{Q}_{T_1}$, such that (v, ϱ) is the unique solution of

$$
(2.9)\qquad \begin{cases}
\dfrac{\partial}{\partial t}(\operatorname{rot} v) + (v\cdot\nabla)\operatorname{rot} v - \big((\operatorname{rot} v)\cdot\nabla\big)v + \psi \operatorname{rot} v = \\
\qquad = \operatorname{rot} b + \dfrac{\nabla\varrho}{\varrho^2}\wedge\nabla p(\varrho, \bar\theta) - \dfrac{\nabla\varrho}{\varrho^2}\wedge\nabla\big(\zeta(\varrho, \bar\theta, v)\psi\big) & \text{in } Q_{T_1}, \\
\dot\varrho + v\cdot\nabla\varrho + \varrho \operatorname{div} v = 0 & \text{in } Q_{T_1}, \\
\operatorname{div} v = \psi & \text{in } Q_{T_1}, \\
v\cdot n = 0 & \text{on } \Sigma_{T_1}, \\
v(0) = v_0 & \text{in } \Omega, \\
\varrho(0) = \varrho_0 & \text{in } \Omega.
\end{cases}
$$

Proof. We have only to prove the uniqueness of the solution. Let (v', ϱ') be another solution of (2.9); set $\xi' \equiv \operatorname{rot} v'$. From

$$
(2.10)\qquad \begin{cases}
\operatorname{rot}(v - v') = \xi - \xi' & \text{in } Q_{T_1}, \\
\operatorname{div}(v - v') = 0 & \text{in } Q_{T_1}, \\
(v - v')\cdot n = 0 & \text{on } \Sigma_{T_1},
\end{cases}
$$

we have $\|(v - v')(t)\|_1 \leqslant C\|(\xi - \xi')(t)\|$ for each $t \in [0, T_1]$. From the equation (2.9) for ϱ we get

$$
\begin{aligned}
\frac{d}{dt}\|\varrho - \varrho'\|_1 &\leqslant C\{\|v - v'\|_1\|\varrho\|_3 + \|v'\|_3\|\varrho - \varrho'\|_1 + \|\psi\|_2\|\varrho - \varrho'\|_1\} \\
&\leqslant \bar c(A, D)\{\|v - v'\|_1 + \|\varrho - \varrho'\|_1\} \\
&\leqslant \bar c(A, D)\{\|\xi - \xi'\| + \|\varrho - \varrho'\|_1\}.
\end{aligned}
$$

From equation $(2.9)_1$, one obtains

$$
\begin{aligned}
\frac{d}{dt}\|\xi - \xi'\| &\leqslant \bar c(A, B, D)\,\{\|\xi - \xi'\| + (1 + \|\psi\|_3)[\|\varrho - \varrho'\|_1 + \|v - v'\|_1]\} \\
&\leqslant \bar c(A, B, D)(1 + \|\psi\|_3)\{\|\xi - \xi'\| + \|\varrho - \varrho'\|_1\}.
\end{aligned}
$$

Then

$$
\frac{d}{dt}[\|\varrho - \varrho'\|_1 + \|\xi - \xi'\|] \leqslant \bar c(A, B, D)(1 + \|\psi\|_3)\{\|\varrho - \varrho'\|_1 + \|\xi - \xi'\|\};
$$

hence, from Gronwall's lemma, as $(\varrho - \varrho')(0) = 0$, $(\xi - \xi')(0) = 0$, one obtains $\varrho = \varrho'$, $\xi = \xi'$ in Q_{T_1}. From (2.10) one has $v = v'$ in Q_{T_1}. $\square$

Construction of the second fixed point.

Let $T \in]0, T_1]$; consider the problem

$$(2.11)\qquad \begin{cases} \dot\theta - \dfrac{1}{c_0\varrho_0}\operatorname{div}(\chi_0\nabla\theta) = -v\cdot\nabla\bar\theta - \dfrac{1}{\bar c_V\varrho}\,\bar\theta\,\dfrac{\partial\bar p}{\partial\theta}\operatorname{div} v + \\ \qquad + \dfrac{1}{\bar c_V\varrho}\operatorname{div}(\bar\chi\nabla\bar\theta) - \dfrac{1}{c_0\varrho_0}\operatorname{div}(\chi_0\nabla\bar\theta) + \dfrac{r}{\bar c_V} + \dfrac{\bar\zeta}{\bar c_V\varrho}(\operatorname{div} v)^2 & \text{in } \Omega_T, \\ \theta = \theta_1 & \text{on } \Sigma_T, \\ \theta(0) = \theta_0 & \text{in } \Omega, \end{cases}$$

where $c_0 \equiv c_V(\varrho_0, \theta_0)$, $\chi_0 \equiv \chi(\varrho_0, \theta_0, v_0)$, $\bar c_V \equiv c_V(\varrho, \bar\theta)$, $\bar\chi \equiv \chi(\varrho, \bar\theta, v)$, $\bar p \equiv p(\varrho, \bar\theta)$ and $\bar\zeta \equiv \zeta(\varrho, \bar\theta, v)$. Observe that $c_0 > 0$ in $\bar\Omega$, $\chi_0 > 0$ in $\bar\Omega$, $\bar c_V > 0$ in $\bar Q_T$.

First we define the operator A_1 in $H^2(\Omega)$ setting

$$D(A_1) = \{\theta \in H^4(\Omega) \,|\, \theta = 0 \text{ on } \Gamma\},$$

$$A_1\theta = -\frac{1}{c_0\varrho_0}\operatorname{div}(\chi_0\nabla\theta) \qquad \forall\theta \in D(A_1);$$

(we have $c_0 \in H^3(\Omega)$, $\chi_0 \in H^3(\Omega)$).

The following result holds.

LEMMA 2.3. For any $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \geqslant 0$, $\lambda + A_1$ is an isomorphism from $D(A_1)$ (endowed with the graph norm) into $H^2(\Omega)$.

Moreover one has, for each $\theta \in D(A_1)$,

$$(2.12)\qquad \|\theta\|_{D(A_1)} + |\lambda|^2\|\theta\| \leqslant C\{\|(\lambda + A_1)\theta\|_2 + (1 + |\lambda|)\|(\lambda + A_1)\theta\|\},$$

where the constant C does not depend on λ.

PROOF. Consider the problem

$$(2.13)\qquad \lambda\theta + A_1\theta = f,$$

where $f \in H^2(\Omega)$, $\lambda \in \mathbf{C}$. By well-known results on elliptic problems, one gets a unique solution $\theta \in D(A_1)$, for any $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda \geqslant 0$. Multiplying (2.13) by $c_0 \varrho_0 \theta$ and integrating over Ω, one easily obtains

$$|\lambda| \|\theta\| \leqslant C \|f\| = C \|(\lambda + A_1)\theta\| . \tag{2.14}$$

By this inequality one has

$$\|\theta\|_2 \leqslant C \|A_1 \theta\| \leqslant C \{\|(\lambda + A_1)\theta\| + |\lambda| \|\theta\|\} \leqslant C \|(\lambda + A_1)\theta\| .$$

Then

$$\begin{aligned} \|\theta\|_{D(A_1)} = \|\theta\|_2 + \|A_1\theta\|_2 \leqslant \|\theta\|_2 + \|(\lambda + A_1)\theta\|_2 + |\lambda| \|\theta\|_2 \\ \leqslant C(1 + |\lambda|) \|(\lambda + A_1)\theta\| + \|(\lambda + A_1)\theta\|_2 , \end{aligned}$$

from which (2.12) immediately follows. □

Then one has:

LEMMA 2.4. *Let $F \in H^{2,1}(Q_T)$, $\theta_1 \in H^{7/2,7/4}(\Sigma_T)$, $\theta_0 \in H^3(\Omega)$. Assume that the (necessary) compatibility conditions*

$$\begin{cases} \theta_0 = \theta_1(0) & \text{on } \Gamma , \\ \dot{\theta}_1(0) + A_1 \theta_0 = F(0) & \text{on } \Gamma , \end{cases} \tag{2.15}$$

are satisfied. Then there exists a unique solution $\theta \in H^{4,2}(Q_T)$ of problem

$$\begin{cases} \dot{\theta} + A_1 \theta = F & \text{in } Q_T , \\ \theta = \theta_1 & \text{on } \Sigma_T , \\ \theta(0) = \theta_0 & \text{in } \Omega . \end{cases} \tag{2.16}$$

Moreover,

$$\begin{aligned} \|\theta\|_{4,2,T} \leqslant C \{ \|F\|_{2,1,T} + \|\theta_1\|_{7/2,7/4,\Sigma_T} + \\ + \|\theta_0\|_3 + \|\theta_1(0)\|_{5/2,\Gamma} + \|F(0)\|_1 \} \end{aligned} \tag{2.17}$$

where C does not depend on T.

PROOF. The traces $\theta_1(0)$ and $\dot\theta_1(0)$ on Γ belong $H^{5/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ respectively to. Hence one can find a function $\Psi \in H^{7/2,7/4}(]0, +\infty[\times\Gamma)$ such that $\Psi(0) = \theta_1(0)$, $\dot\Psi(0) = \dot\theta_1(0)$, and

$$\|\Psi\|_{7/2,7/4,\Sigma_\infty} \leqslant C\{\|\theta_1(0)\|_{5/2,\Gamma} + \|\dot\theta_1(0)\|_{1/2,\Gamma}\},$$

where the constant C does not depend on T.
Now, we can extend $\theta_1 - \Psi$ from $[0, T]\times\Gamma$ to $\mathbf{R}\times\Gamma$ in such a way that the extension $P(\theta_1 - \Psi) \in H^{7/2,7/4}(\mathbf{R}\times\Gamma)$, $P(\theta_1 - \Psi) = 0$ for $t < 0$ and

$$\|P(\theta_1 - \Psi)\|_{7/2,7/4,\Sigma_\infty} \leqslant C\|\theta_1 - \Psi\|_{7/2,7/4,\Sigma_T},$$

where the constant C does not depend on T (extension by reflection around $t = T$: see Lions-Magenes [10], Theorem 2.2, chap. 1, and Theorem 11.2, chap. 1). Hence we have extended θ_1 to $\tilde\theta_1 \equiv P(\theta_1 - \Psi) + \Psi$, and $\tilde\theta_1 \in H^{7/2,7/4}(]0, +\infty[\times\Gamma)$ with

$$\|\tilde\theta_1\|_{7/2,7/4,\Sigma_\infty} \leqslant C\{\|\theta_1\|_{7/2,7/4,\Sigma_T} + \|\theta_1(0)\|_{5/2,\Gamma} + \|\dot\theta_1(0)\|_{1/2,\Gamma}\}. \tag{2.18}$$

Now, the compatibility conditions (2.15) are necessary and sufficient to find a function $H \in H^{4,2}(]0, +\infty[\times\Omega)$ such that

$$\begin{cases} H = \tilde\theta_1 & \text{on }]0, +\infty[\times\Gamma, \\ H(0) = \theta_0 & \text{on } \Omega, \\ \dot H(0) = -A_1\theta_0 + F(0) & \text{on } \Omega, \end{cases}$$

and the following estimate holds:

$$\|H\|_{4,2,+\infty} \leqslant C\{\|\tilde\theta_1\|_{7/2,7/4,\Sigma_\infty} + \|\theta_0\|_3 + \|F(0)\|_1\}, \tag{2.19}$$

where the constant C does not depend on T.
Now we can find the solution of

$$\begin{cases} \dot\Theta + A_1\Theta = F - (\dot H + A_1 H) & \text{in } Q_T, \\ \Theta = 0 & \text{on } \Sigma_T, \\ \Theta(0) = 0 & \text{in } \Omega. \end{cases}$$

Since Lemma 2.3 holds and the condition $F(0) - \dot H(0) - A_1 H(0) = 0$

is satisfied, we can apply Theorem 5.2, chap. 4 of Lions-Magenes [11], and we find a solution $\Theta \in H^{4,2}(Q_T)$, verifying

$$(2.20) \qquad \|\Theta\|_{4,2,T} \leqslant C\{\|F\|_{2,1,T} + \|\dot{H} + A_1 H\|_{2,1,T}\} \leqslant C\{\|F\|_{2,1,T} + \|H\|_{4,2,T}\},$$

where C does not depend on T.

The function $\theta = H + \Theta$ is the solution of (2.16) and by (2.18), (2.19), (2.20) we get (2.17). □

Thanks to Lemma 2.4, we can now solve problem (2.11).

LEMMA 2.5. There exists a unique solution $\theta \in H^{4,2}(Q_T)$ of problem (2.11). Moreover the following estimate holds:

$$(2.21) \qquad \|\theta\|_{4,2,T} \leqslant C_1 \Big\{ \bar{c}(A, B)T^{\frac{1}{2}} + \bar{c}(A, B)\|r\|_{2,1,T} + \|\theta_1\|_{\frac{7}{2},\frac{7}{4},\Sigma_{T_0}} + \|\theta_0\|_3 + \Big\| - v_0 \cdot \nabla\theta_0 - \frac{\theta_0}{c_0\varrho_0}\frac{\partial p_0}{\partial\theta}\operatorname{div} v_0 + \frac{r(0)}{c_0} + \frac{\zeta_0}{c_0\varrho_0}(\operatorname{div} v_0)^2 \Big\|_1 \Big\},$$

where $p_0 \equiv p(\varrho_0, \theta_0)$, $\zeta_0 \equiv \zeta(\varrho_0, \theta_0, v_0)$ and the constant C_1 does not depend on T.

PROOF. As one can easily verify, we have

$$\bar{c}_V, \bar{p}, \bar{\chi}, \bar{\zeta} \in C^0([0, T]; H^3(\Omega)) \cap H^1(0, T; H^2(\Omega)) \cap H^1_\infty(0, T; H^1(\Omega)).$$

Recall that, if $f \in L^\infty(0, T; X)$, where X is a Banach space, then

$$|f|_{2,X,T} \leqslant |f|_{\infty,X,T} T^{\frac{1}{2}},$$

and, if $f \in H^1(0, T; X)$, then

$$|f(t) - f(0)|_X \leqslant |\dot{f}|_{2,X,T} t^{\frac{1}{2}}.$$

Then we have

$$(2.22) \qquad \Big\| - v \cdot \nabla\bar{\theta} - \frac{\bar{\theta}}{\bar{c}_V\varrho}\frac{\partial\bar{p}}{\partial\theta}\operatorname{div} v + \frac{\bar{\zeta}}{\bar{c}_V\varrho}(\operatorname{div} v)^2 \Big\|_{2,1,T} \leqslant \bar{c}(A, B)T^{\frac{1}{2}},$$

$$(2.23)\qquad \left\|\frac{1}{\bar{c}_V\varrho}\operatorname{div}(\bar{\chi}\nabla\bar{\theta})-\frac{1}{c_0\varrho_0}\operatorname{div}(\chi_0\nabla\bar{\theta})\right\|_{2,1,T}\leqslant$$

$$\leqslant\left|\left(\frac{1}{\bar{c}_V\varrho}-\frac{1}{c_0\varrho_0}\right)\operatorname{div}(\bar{\chi}\nabla\bar{\theta})\right|_{2,2,T}+\left|\frac{1}{c_0\varrho_0}\operatorname{div}((\bar{\chi}-\chi_0)\nabla\bar{\theta})\right|_{2,2,T}+$$

$$+\left|\frac{\partial}{\partial t}\left(\frac{1}{\bar{c}_V\varrho}\right)\operatorname{div}(\bar{\chi}\nabla\bar{\theta})\right|_{2,0,T}+\left|\left(\frac{1}{\bar{c}_V\varrho}-\frac{1}{c_0\varrho_0}\right)\frac{\partial}{\partial t}\operatorname{div}(\bar{\chi}\nabla\bar{\theta})\right|_{2,0,T}+$$

$$+\left|\frac{1}{c_0\varrho_0}\frac{\partial}{\partial t}\operatorname{div}((\bar{\chi}-\chi_0)\nabla\bar{\theta})\right|_{2,0,T}\leqslant\bar{c}(A,B)T^{\frac{1}{4}}.$$

Moreover

$$(2.24)\qquad \left\|\frac{r}{\bar{c}_V}\right\|_{2,1,T}\leqslant\bar{c}(A,B)\|r\|_{2,1,T}.$$

The compatibility conditions (2.15) are satified since (1.4), (1.5) hold. Then we can apply Lemma 2.4 and find a unique solution $\theta\in H^{4,2}(Q_T)$ of problem (2.11). From (2.17), (2.22), (2.23), (2.24) we get (2.21), observing that

$$\|\theta_1(0)\|_{5/2,\Gamma}+\|\dot{\theta}_1(0)\|_{1/2,\Gamma}\leqslant C\|\theta_1\|_{7/2,7/4,\Sigma_{T_0}},$$

where C does not depend on T. $\square$

Now, we can construct a fixed point of the map $\Phi_2\colon\bar{\theta}\to\theta$. Choose

$$(2.25)\qquad B>C_1\left\{\|\theta_1\|_{\frac{7}{2},\frac{7}{4},\Sigma_{T_0}}+\|\theta_0\|_3+\right.$$
$$\left.+\left\|-v_0\nabla\theta_0-\frac{\theta_0}{c_0\varrho_0}\frac{\partial p_0}{\partial\theta}\operatorname{div}v_0+\frac{r(0)}{c_0}+\frac{\zeta_0}{c_0\varrho_0}(\operatorname{div}v_0)^2\right\|_1\right\}.$$

Then, by estimate (2.21), there exists $T_2\in\,]0,T_1]$ such that the convex set

$$S_2=\left\{\bar{\theta}\in H^{4,2}(Q_{T_2})\colon\bar{\theta}=\theta_1\text{ on }\Sigma_{T_2},\ \bar{\theta}(0)=\theta_0\text{ in }\Omega,\ \|\bar{\theta}\|_{4,2,T_2}\leqslant B\right\}$$

satisfies $\Phi_2[S_2]\subseteq S_2$.

S_2 is bounded in $C^0([0, T_2]; H^3(\Omega)) \cap C^1([0, T_2]; H^1(\Omega))$; hence, by the Ascoli-Arzelà theorem, it is compact in $X_2 \equiv C^0([0, T_2]; H^1(\Omega))$. Let $\bar{\theta}_n$, $\bar{\theta} \in S_2$, $\bar{\theta}_n \to \bar{\theta}$ in X_2. Let (v_n, ϱ_n), (v, ϱ) be the corresponding solutions of (2.9) (where $\operatorname{div} v_n = \operatorname{div} v = \psi$); set $\xi_n \equiv \operatorname{rot} v_n$, $\xi \equiv \operatorname{rot} v$. Then, from (2.1) and $(2.9)_2$, we obtain

$$\|v_n - v\|_1 \leqslant C\|\xi_n - \xi\|$$

$$\frac{d}{dt}\|\varrho_n - \varrho\|_1 \leqslant C\{\|\varrho_n - \varrho\|_1 + \|v_n - v\|_1\} \leqslant \leqslant C\{\|\varrho_n - \varrho\|_1 + \|\xi_n - \xi\|\}. \tag{2.26}$$

From $(2.9)_1$, one gets

$$\frac{d}{dt}\|\xi_n - \xi\| \leqslant C(1 + \|\psi\|_3)\{\|\varrho_n - \varrho\|_1 + \|\xi_n - \xi\| + \|\bar{\theta}_n - \bar{\theta}\|_1\}. \tag{2.27}$$

Then, adding (2.26) to (2.27), one has $\varrho_n \to \varrho$ in X_2, $\xi_n \to \xi$ in $C^0([0, T_2]; L^2(\Omega))$; hence $v_n \to v$ in X_2.

From $(2.11)_1$ one gets, after some calculations,

$$\frac{1}{2}\frac{d}{dt}\|\theta_n - \theta\|^2 + \int_\Omega \frac{\chi_0}{c_0\varrho_0}|\nabla(\theta_n - \theta)|^2 \leqslant \varepsilon\int_\Omega |\nabla(\theta_n - \theta)|^2 +$$
$$+ C\{\|\theta_n - \theta\| + \|\bar{\theta}_n - \bar{\theta}\|_1 + \|v_n - v\|_1 + \|\varrho_n - \varrho\|_1\}\|\theta_n - \theta\|$$
$$\left(\text{where } \frac{\chi_0}{c_0\varrho_0} \geqslant \min_{\bar{\Omega}} \frac{\chi_0}{c_0\varrho_0} > 0\right),$$

hence $\theta_n \to \theta$ in $C^0([0, T_2]; L^2(\Omega))$. Since θ_n, $\theta \in S_2$, which is compact in X_2, we have $\theta_n \to \theta$ in X_2. Then Φ_2 is continuous in the X_2 topology. From the Schauder theorem there exists a fixed point $\bar{\theta} = \theta$. Let (v, ϱ) be the solution of (2.9), given by Theorem 2.1, corresponding to the fixed point θ. Then we have

THEOREM 2.2. *For each ψ satisfying hypothesis (H_1), there exist $T_2 \in]0, T_1]$, $v \in C^0([0, T_2]; H^3(\Omega)) \cap H^1(0, T_2; H^2(\Omega))$ with $\dot{v} \in L^\infty(0, T_2; H^1(\Omega))$, $\varrho \in C^0([0, T_2]; H^3(\Omega)) \cap H^2(0, T_2; H^1(\Omega))$ with $\dot{\varrho} \in L^\infty(0, T_2; H^2(\Omega))$ such that $\varrho(t, x) > 0$ in $\bar{Q}_{T_2}$, $\theta \in H^{4,2}(Q_{T_2})$, such that (v, ϱ, θ)*

is the unique solution of

$$(2.28)\quad \begin{cases} \dfrac{\partial}{\partial t}(\operatorname{rot} v) + (v\cdot\nabla)\operatorname{rot} v - \big((\operatorname{rot} v)\cdot\nabla\big)v + \psi \operatorname{rot} v = \operatorname{rot} b + \\ \qquad + \dfrac{\nabla\varrho}{\varrho^2}\wedge\nabla p(\varrho,\theta) - \dfrac{\nabla\varrho}{\varrho^2}\wedge\nabla\big(\zeta(\varrho,\theta,v)\psi\big) & \text{in } Q_{T_2}, \\ \dot\varrho + v\cdot\nabla\varrho + \varrho \operatorname{div} v = 0 & \text{in } Q_{T_2}, \\ c_V(\varrho,\theta)\varrho[\dot\theta + v\cdot\nabla\theta] = -\theta\dfrac{\partial p}{\partial\theta}\operatorname{div} v + \operatorname{div}\big(\chi(\varrho,\theta,v)\nabla\theta\big) + \\ \qquad + \varrho r + \zeta(\varrho,\theta,v)(\operatorname{div} v)^2 & \text{in } Q_{T_2}, \\ \operatorname{div} v = \psi \quad \text{in } Q_{T_2}, \\ v\cdot n = 0 \quad \text{on } \Sigma_{T_2}, \\ \theta = \theta_1 \quad \text{on } \Sigma_{T_2}, \\ v(0) = v_0 \quad \text{in } \Omega, \\ \varrho(0) = \varrho_0 \quad \text{in } \Omega, \\ \theta(0) = \theta_0 \quad \text{in } \Omega. \end{cases}$$

PROOF. We have only to prove that the solution is unique.

Let (v', ϱ', θ') be another solution of (2.28). Set $\xi \equiv \operatorname{rot} v$, $\xi' \equiv \operatorname{rot} v'$. As in the proof of the continuity of Φ_2, one obtains, from (2.1), $(2.28)_2$ and $(2.28)_1$, respectively,

$$\|v' - v\|_1 \leqslant C\|\xi' - \xi\|,$$

$$\frac{1}{2}\frac{d}{dt}\|\varrho' - \varrho\|_1^2 \leqslant C\{\|\varrho' - \varrho\|_1 + \|\xi' - \xi\|\}\|\varrho' - \varrho\|_1,$$

$$\frac{1}{2}\frac{d}{dt}\|\xi' - \xi\|^2 \leqslant C(1 + \|\psi\|_3)\{\|\varrho' - \varrho\|_1 + \|\xi' - \xi\| + \|\theta' - \theta\|_1\}\|\xi' - \xi\|.$$

From $(2.28)_3$, one has

$$\frac{1}{2}\frac{d}{dt}\|\theta' - \theta\|^2 + \int_\Omega \frac{\chi(\varrho,\theta,v)}{c_V(\varrho,\theta)\varrho}|\nabla(\theta' - \theta)|^2 \leqslant \varepsilon\int_\Omega |\nabla(\theta' - \theta)|^2 + \\ + C\{\|\theta' - \theta\| + \|\varrho' - \varrho\|_1 + \|v' - v\|_1\}\|\theta' - \theta\|$$

$\left(\text{where } \dfrac{\chi(\varrho,\theta,v)}{c_V(\varrho,\theta)\varrho} \geqslant \min\limits_{\bar Q_T} \dfrac{\chi(\varrho,\theta,v)}{c_V(\varrho,\theta)\varrho} > 0\right)$. Then we obtain

$$\frac{1}{2}\frac{d}{dt}[\|\varrho'-\varrho\|_1^2+\|\xi'-\xi\|^2+\|\theta'-\theta\|^2]+\int_\Omega \frac{\chi(\varrho,\theta,v)}{c_V(\varrho,\theta)\varrho}|\nabla(\theta'-\theta)|^2\leqslant$$

$$\leqslant\varepsilon\int_\Omega|\nabla(\theta'-\theta)|^2+C(1+\|\psi\|_3^2)[\|\varrho'-\varrho\|_1^2+\|\xi'-\xi\|^2+\|\theta'-\theta\|^2]\,,$$

from which $\varrho=\varrho'$, $v=v'$, $\theta=\theta'$ in Q_{T_2}. □

Construction of the third fixed point.

Finally, let us consider the problem (formally obtained by taking the divergence of $\dot v+(v\cdot\nabla)v-b=-(1/\varrho)\nabla p+(1/\varrho)\nabla(\zeta\operatorname{div}v)$ in Ω and writing $\operatorname{div}v=w$ or ψ, and by taking the scalar product of the same equation with the outward normal n on Γ)

$$(2.29)\quad\begin{cases}\dot w-\operatorname{div}\left(\dfrac{\zeta_0}{\varrho_0}\nabla w\right)=-\sum\limits_{i,j}D_iv_jD_jv_i-v\cdot\nabla\psi+\operatorname{div}b+\\[2mm]\qquad-\operatorname{div}\dfrac{\nabla p}{\varrho}+\operatorname{div}\left(\left(\dfrac{\zeta}{\varrho}-\dfrac{\zeta_0}{\varrho_0}\right)\nabla\psi\right)+\operatorname{div}\left(\dfrac{\psi}{\varrho}\nabla\zeta\right)\equiv G\quad\text{in } Q_T\,,\\[2mm]\dfrac{\partial w}{\partial n}=\dfrac{\varrho_0}{\zeta_0}\left[(v\cdot\nabla)v\cdot n-b\cdot n+\dfrac{1}{\varrho}\dfrac{\partial p}{\partial n}-\left(\dfrac{\zeta}{\varrho}-\dfrac{\zeta_0}{\varrho_0}\right)\dfrac{\partial\psi}{\partial n}-\dfrac{\psi}{\varrho}\dfrac{\partial\zeta}{\partial n}\right]\equiv E\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{on } \Sigma_T\,,\\ w(0)=\operatorname{div}v_0\qquad\qquad\qquad\qquad\qquad\qquad\text{in } \Omega\,,\end{cases}$$

where $T\in\,]0,T_2]$, $\zeta_0\equiv\zeta(\varrho_0,\theta_0,v_0)>0$ in $\bar\Omega$.
Let us define the operator A_2 in the space $H^1(\Omega)$, setting

$$D(A_2)=\left\{w\in H^3(\Omega):\frac{\partial\omega}{\partial n}=0\quad\text{on }\Gamma\right\},$$

$$A_2w=-\operatorname{div}\left(\frac{\zeta_0}{\varrho_0}\nabla w\right)\qquad\forall w\in D(A_2)\,,$$

(we have $\zeta_0\in H^3(\Omega)$).
Then we have

LEMMA 2.6. For any $\lambda\in\mathbf{C}$ with $\operatorname{Re}\lambda\geqslant\lambda_0>0$, $\lambda+A_2$ is an isomorphism from $D(A_2)$ (endowed with the graph norm) into $H^1(\Omega)$.

Moreover, for each $w \in D(A_2)$, $\operatorname{Re}\lambda \geqslant \lambda_0$, one has

$$\|w\|_1 \leqslant \frac{C}{1+|\lambda|} \|(\lambda + A_2)w\|_1 , \tag{2.30}$$

where the constant C does not depend on λ.

PROOF. Consider the elliptic problem

$$\lambda w + A_2 w = g \tag{2.31}$$

where $g \in H^1(\Omega)$, $\lambda \in \mathbf{C}$. Then for any $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \geqslant \lambda_0 > 0$ there exists a unique solution $w \in D(A_2)$. Multiplying (2.31) by $\overline{w}$ and integration over Ω, one obtains

$$\lambda \int_\Omega |w|^2 + \int_\Omega \frac{\zeta_0}{\varrho_0} |\nabla w|^2 = \int_\Omega g\overline{w} ,$$

from which one gets

$$\|w\| \leqslant \frac{C}{|\lambda|} \|g\| , \qquad \|w\|_1 \leqslant C\|g\| ,$$

where C does not depend on λ. Multiplying (2.31) by $\operatorname{div}\left((\zeta_0/\varrho_0)\nabla\overline{w}\right)$ and integrating over Ω, one has

$$\lambda \int_\Omega \frac{\zeta_0}{\varrho_0} |\nabla w|^2 + \int_\Omega \left| \operatorname{div}\left(\frac{\zeta_0}{\varrho_0}\nabla\omega\right)\right|^2 = \int_\Omega \frac{\zeta_0}{\varrho_0} \nabla g \cdot \nabla\overline{w} ,$$

from which one obtains

$$\|\nabla w\| \leqslant \frac{C}{|\lambda|} \|\nabla g\| .$$

Then one has

$$|\lambda| \int_\Omega |w|^2 + |\lambda| \int_\Omega |\nabla w|^2 \leqslant \left| \int_\Omega g\overline{w} - \int_\Omega \frac{\zeta_0}{\varrho_0} |\nabla w|^2 \right| + c\|\nabla g\| \, \|\nabla\omega\| \leqslant$$

$$\leqslant \frac{C}{|\lambda|} \|g\|^2 + \frac{C}{|\lambda|} \|\nabla g\| \, \|g\| + \frac{C}{|\lambda|} \|\nabla g\|^2 \leqslant \frac{C}{|\lambda|} \|g\|_1^2 .$$

Hence

$$\|w\|_1 \leq \frac{C}{|\lambda|} \|g\|_1 ,$$

from which, as $\|w\|_1 \leq C\|g\|_1$, one has (2.30). $\square$

LEMMA 2.7. Let $G \in L^2(0, T; H^1(\Omega))$, $E \in H^{3/2,3/4}(\Sigma_T)$, $\operatorname{div} v_0 \in H^2(\Omega)$. Assume that the (necessary) compatibility condition

$$\frac{\partial}{\partial n} \operatorname{div} v_0 = E(0) \quad \text{on } \Gamma , \tag{2.32}$$

holds. Then there exists a unique solution $w \in L^2(0, T; H^3(\Omega)) \cap \cap H^1(0, T; H^1(\Omega))$ of problem

$$\begin{cases} \dot{w} + A_2 w = G & \text{in } Q_T , \\ \dfrac{\partial w}{\partial n} = E & \text{on } \Sigma_T , \\ w(0) = \operatorname{div} v_0 & \text{in } \Omega . \end{cases} \tag{2.33}$$

Moreover

$$\begin{aligned} |w|_{2,3,T} + |\dot{w}|_{2,1,T} &\leq \\ &\leq C\{|G|_{2,1,T} + \|E\|_{3/2,3/4,\Sigma_T} + \|\operatorname{div} v_0\|_2 + \|E(0)\|_{1/2,\Gamma}\} , \end{aligned} \tag{2.34}$$

where C does not depend on T.

PROOF. As in the proof of Lemma 2.4, one can extend E from $[0, T] \times \Gamma$ to $[0, +\infty[\times \Gamma$ in such a way that the extension $\tilde{E} \in H^{3/2,3/4}(]0, +\infty[\times \Gamma)$, with

$$\|\tilde{E}\|_{3/2,3/4,\Sigma_\infty} \leq C\{\|E\|_{3/2,3/4,\Sigma_T} + \|E(0)\|_{1/2,\Gamma}\} , \tag{2.35}$$

where C does not depend on T.
Now, compatibility condition (2.32) is necessary and sufficient to find a function $U \in H^{3,3/2}(]0, +\infty[\times \Omega)$ such that

$$\begin{cases} U(0) = \operatorname{div} v_0 & \text{in } \Omega , \\ \dfrac{\partial U}{\partial n} = \tilde{E} & \text{in }]0, +\infty[\times \Gamma , \end{cases}$$

and the following estimate holds:

$$\|U\|_{3,3/2,+\infty} \leqslant C\{\|\tilde{E}\|_{3/2,3/4,\Sigma_\infty} + \|\operatorname{div} v_0\|_2\}, \tag{2.36}$$

where C does not depend on T.

Let us consider the problem

$$\begin{cases} \dot{W} + A_2 W = G - (\dot{U} + A_2 U) & \text{in } Q_T, \\ \dfrac{\partial W}{\partial n} = 0 & \text{on } \Sigma_T, \\ W(0) = 0 & \text{in } \Omega. \end{cases}$$

Since Lemma 2.6 holds, we can apply Theorem 3.2, chap. 4 of Lions-Magenes [11], and find a solution $W \in L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega))$ such that

$$|W|_{2,3,T} + |\dot{W}|_{2,1,T} \leqslant C\{|G|_{2,1,T} + \|U\|_{3,3/2,T}\}, \tag{2.37}$$

where C does not depend on T.

The function $w = W + U$ is the solution of (2.33). By (2.35), (2.36), (2.37) we get (2.34). □

We can now solve the problem (2.29).

LEMMA 2.8. There exists a unique solution $w \in L^2(0, T; H^3(\Omega)) \cap \cap H^1(0, T; H^1(\Omega)) \cap H^1_\infty(0, T; L^2(\Omega))$ of problem (2.29). Moreover w satisfies

$$|w|_{2,3,T} + |\dot{w}|_{2,1,T} + |\dot{w}|_{\infty,0,T} \leqslant C_2 \Big\{\bar{c}(A) T^{\frac{1}{4}} + \|b\|_{2,1,T} + \\ + \|\operatorname{div} v_0\|_2 + \left(1 + \left\|\frac{\varrho_0}{\zeta_0}\right\|_2\right) \left\|(v_0 \cdot \nabla) v_0 - b(0) + \frac{1}{\varrho_0} \nabla p_0 - \frac{\operatorname{div} v_0}{\varrho_0} \nabla \zeta_0 \right\|_1 \Big\}, \tag{2.38}$$

where C_2 does not depend on T, and

$$\int_\Omega w\, dx = 0 \qquad \forall t \in [0, T].$$

PROOF. First, one easily verifies that $p, \zeta \in C^0([0, T]; H^3(\Omega)) \cap \cap H^1(0, T; H^2(\Omega)) \cap H^1_\infty(0, T; H^1(\Omega))$. Then, proceeding as in the proof of Lemma 2.5, we obtain

$$
(2.39)\qquad \left\{
\begin{aligned}
&|G|_{2,1,T} \leqslant \bar{c}(A)\, T^{\frac{1}{2}} + |\operatorname{div} b|_{2,1,T}\,,\\
&\|E\|_{\frac{3}{2},\frac{3}{4},\Sigma_T} \leqslant C_3 \Bigg\{ \bar{c}(A)\, T^{\frac{1}{2}} + C\|b\|_{2,1,T} + \\
&\qquad + \left\|\frac{\varrho_0}{\zeta_0}\right\|_2 \left\| (v_0\cdot\nabla)v_0 - b(0) + \frac{\nabla p_0}{\varrho_0} - \frac{\operatorname{div} v_0}{\varrho_0}\nabla\zeta_0 \right\|_1 \Bigg\},
\end{aligned}
\right.
$$

where C_3 does not depend on T.

Compatibility condition (2.32) is satisfied since (1.3) holds. Then, by Lemma 2.7, we have a unique solution $w \in L^2(0, T; H^3(\Omega)) \cap \cap H^1(0, T; H^1(\Omega))$ of problem (2.29). By interpolation $w \in C^0([0, T]; H^2\, \Omega))$, hence $\operatorname{div}\big((\zeta_0/\varrho_0)\nabla\omega\big) \in C^0([0, T]; L^2(\Omega))$ with

$$
\left|\operatorname{div}\left(\frac{\zeta_0}{\varrho_0}\nabla w\right)\right|_{\infty,0,T} \leqslant C\left\{|w|_{2,3,T} + |\dot{w}|_{2,1,T} + \|\operatorname{div} v_0\|_2\right\},
$$

where C does not depend on T.

Observing that, if $f \in H^1(0, T; L^2(\Omega))$, then

$$
|f|_{\infty,0,T} \leqslant |\dot{f}|_{2,0,T}\, T^{\frac{1}{2}} + \|f(0)\|\,,
$$

by $(2.29)_1$, we get

$$
(2.40)\qquad |\dot{w}|_{\infty,0,T} \leqslant \left|\operatorname{div}\left(\frac{\zeta_0}{\varrho_0}\nabla w\right)\right|_{\infty,0,T} + |G|_{\infty,0,T} \leqslant
$$
$$
\leqslant C\left\{|w|_{2,3,T} + |\dot{w}|_{2,1,T} + \|\operatorname{div} v_0\|_2 + \bar{c}(A)\, T^{\frac{1}{2}} + \|G(0)\|\right\}.
$$

From (2.34), (2.39), (2.40) we get (2.38).

Finally, we have $\Big($since $-\sum\limits_{i,j} D_i v_j D_j v_i - v\cdot\nabla\psi = -\operatorname{div}\big((v\cdot\nabla)v\big)\Big)$

$$
\frac{d}{dt}\int_\Omega w\, dx = \int_\Omega \dot{w}\, dx = \int_\Omega \operatorname{div}\left[\frac{\zeta_0}{\varrho_0}\nabla\omega - (v\cdot\nabla)v + b - \frac{\nabla p}{\varrho} + \right.
$$
$$
\left. + \left(\frac{\zeta}{\varrho} - \frac{\zeta_0}{\varrho_0}\right)\nabla\psi + \frac{\psi}{\varrho}\nabla\zeta\right] dx
$$
$$
= \int_\Gamma \left[\frac{\zeta_0}{\varrho_0}\frac{\partial w}{\partial n} + \left(-(v\cdot\nabla)v + b - \frac{\nabla p}{\varrho} + \left(\frac{\zeta}{\varrho} - \frac{\zeta_0}{\varrho_0}\right)\nabla\psi + \frac{\psi}{\varrho}\nabla\zeta\right)\cdot n\right] d\sigma = 0\,,
$$

by the boundary condition $(2.29)_2$.

As

$$\int_{\Omega} w(0)\,dx = \int_{\Omega} \operatorname{div} v_0\,dx = \int_{\Gamma} v_0\cdot n\,d\sigma = 0\,,$$

by (1.2), we have

$$\int_{\Omega} w\,dx = 0 \qquad \forall t\in[0,T]\,. \qquad \square$$

We now prove that there exists a fixed point of the map $\Phi_3\colon \psi\to w$. In fact, choose

$$(2.41)\qquad A > C_2\left\{\|\operatorname{div} v_0\|_2 + \left(1+\left\|\frac{\varrho_0}{\zeta_0}\right\|_2\right)\left\|(v_0\cdot\nabla)v_0 - b(0) + \frac{1}{\varrho_0}\nabla p_0 - \frac{\operatorname{div} v_0}{\varrho_0}\nabla\zeta_0\right\|_1\right\}.$$

Then there exists $T'\in\,]0,T_2]$ such that, by estimate (2.38), the convex set

$$S_3 = \Big\{\psi\in L^2(0,T';H^3(\Omega))\cap H^1(0,T';H^1(\Omega)) \text{ with } \dot\psi\in L^\infty(0,T';L^2(\Omega))\colon \int_{\Omega}\psi\,dx = 0 \text{ for each } t\in[0,T'],$$
$$\psi(0)=\operatorname{div} v_0,\ |\psi|_{2,3,T'} + |\dot\psi|_{2,1,T'} + |\dot\psi|_{\infty,0,T'}\leqslant A\Big\}$$

satisfies $\Phi_3[S_3]\subseteq S_3$.

By interpolation and a Sobolev imbedding, S_3 is bounded in $C^0([0,T'];H^2(\Omega))\cap C^{1/2}([0,T'];H^1(\Omega))$; hence by the Ascoli-Arzelà theorem, it is compact in $X_3\equiv C^0([0,T'];H^1(\Omega))$. Let $\psi_n,\psi\in S_3$, $\psi_n\to\psi$ in X_3. Let (v_n,ϱ_n,θ_n), (v,ϱ,θ) be the corresponding solutions of (2.28); set $\xi_n\equiv\operatorname{rot} v_n$, $\xi\equiv\operatorname{rot} v$.

From system (2.1) for $v_n - v$, and $(2.28)_2$ respectively, one has

$$(2.42)\qquad \|v_n - v\|_1\leqslant C\{\|\xi_n-\xi\| + \|\psi_n-\psi\|\}\,,$$

$$(2.43)\qquad \frac{1}{2}\frac{d}{dt}\|\varrho_n-\varrho\|_1^2\leqslant C\{\|\varrho_n-\varrho\|_1 + \|v_n-v\|_1 + \|\psi_n-\psi\|_1\}\,\|\varrho_n-\varrho\|_1\,.$$

From $(2.28)_1$ and $(2.28)_3$, one obtains (by (2.42))

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_n-\xi\|^2\leqslant C(1+\|\psi\|_3)\{\|\varrho_n-\varrho\|_1+\|\xi_n-\xi\|+ \\ +\|\theta_n-\theta\|_1+\|\psi_n-\psi\|_1\}\|\xi_n-\xi\|, \tag{2.44}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_n-\theta\|^2+\int_\Omega\frac{\chi(\varrho,\theta,v)}{c_V(\varrho,\theta)\varrho}|\nabla(\theta_n-\theta)|^2\leqslant\varepsilon\int_\Omega|\nabla(\theta_n-\theta)|^2+ \\ +C\{\|\theta_n-\theta\|+\|\varrho_n-\varrho\|_1+\|v_n-v\|_1+\|\psi_n-\psi\|\}\|\theta_n-\theta\|. \tag{2.45}$$

Then, adding (2.43), (2.44) and (2.45), one has $v_n\to v$ in X_3, $\varrho_n\to\varrho$ in X_3, $\theta_n\to\theta$ in $C_0([0,T'];L^2(\Omega))$ and in $L^2(0,T';H^1(\Omega))$. Finally, from $(2.29)_1$, taking account of the boundary condition $(2.29)_2$, we get

$$\frac{1}{2}\frac{d}{dt}\|w_n-w\|^2+\int_\Omega\frac{\zeta_0}{\varrho_0}|\nabla(w_n-w)|^2\leqslant\varepsilon\int_\Omega|\nabla(w_n-w)|^2+ \\ +C\{\|v_n-v\|_1^2+\|\varrho_n-\varrho\|_1^2+\|\theta_n-\theta\|_1^2+\|\psi_n-\psi\|_1^2\},$$

from which one has $w_n\to w$ in $C^0([0,T'];L^2(\Omega))$. Since $w_n, w\in S_3$ which is compact in X_3, we have $w_n\to w$ in X_3. Then Φ_3 is continuous in the X_3 topology. From the Schauder theorem there exists a fixed point $\psi=w$. Let (v,ϱ,θ) be the corresponding solution of (2.28). Set $V\equiv\dot v+(v\cdot\nabla)v-b+\nabla p/\varrho-\nabla(\zeta\operatorname{div}v)/\varrho$. Then, from $(2.28)_1$, $(2.29)_1$ and $(2.29)_2$ ($v\cdot n|_\Gamma=0$ implies $\dot v\cdot n|_\Gamma=0$), we have

$$\begin{cases}\operatorname{rot}V=0 & \text{in } Q_{T'},\\ \operatorname{div}V=0 & \text{in } Q_{T'},\\ V\cdot n=0 & \text{on } \Sigma_{T'};\end{cases}$$

hence $V=0$ in $Q_{T'}$, i.e. the equation $(1.1)_1$ is satisfies in $Q_{T'}$. Finally, from $(1.1)_1$, one has $v\in H^2(0,T';L^2(\Omega))$.

REMARK 2.1. Assume that Ω is not simply-connected. Then (see Foias-Temam [6]) there exist N vector fields $u^{(l)}(x)$, $l=1,\dots,N$, defined in Ω, which are a basis for the linear space of the solutions of the system $\operatorname{rot}u=0$, $\operatorname{div}u=0$ in Ω, $u\cdot n=0$ on Γ and such that $(u^{(l)},u^{(j)})=\delta_{lj}$ ($(\ ,\ )$ denotes the scalar product in $L^2(\Omega)$). N is the number of cuts needed to make Ω simply-connected.

Given $\psi, \bar{\theta}, \varphi$ as in (H_1)-(H_3), instead of solving successively (2.1), (2.2), one simultaneously finds (by a fixed point argument, see for instance Beirão da Veiga-Valli [3]) a solution (v, ϱ) of

$$\text{(2.46)} \quad \begin{cases} \operatorname{rot} v = \varphi & \text{in } Q_T\,, \\ \operatorname{div} v = \psi & \text{in } Q_T\,, \\ v\cdot n = 0 & \text{on } \Sigma_T\,, \\ \big(v(0) - v_0, u^{(l)}\big) = 0 \quad l = 1, \dots, N\,, \\ \left(\dot{v} + (v\cdot\nabla)v - b + \dfrac{\nabla p(\varrho, \bar{\theta})}{\varrho} - \dfrac{\nabla(\zeta(\varrho, \bar{\theta}, v)\psi)}{\varrho}, u^{(l)}\right) = 0 \\ \qquad\qquad\qquad l = 1, \dots, N\,, \quad \forall t \in [0, T] \\ \dot{\varrho} + v\cdot\nabla\varrho + \varrho \operatorname{div} v = 0 & \text{in } Q_T\,, \\ \varrho(0) = \varrho_0\,. & \text{in } \Omega\,. \end{cases}$$

Then one proceeds as in our proof, in the same way.

One has only to observe that we shall have $v(0) = v_0$ in Ω from the system

$$\begin{cases} \operatorname{rot}[v(0) - v_0] = \xi(0) - \operatorname{rot} v_0 = 0 & \text{in } \Omega\,, \\ \operatorname{div}[v(0) - v_0] = \psi(0) - \operatorname{div} v_0 = 0 & \text{in } \Omega\,, \\ [v(0) - v_0]\cdot n = 0 & \text{on } \Gamma\,, \\ \big(v(0) - v_0, u^{(l)}\big) = 0 & l = 1, \dots, N\,. \end{cases}$$

Moreover, equation $(1.1)_1$ is satisfied, since we have

$$\begin{cases} \operatorname{rot} V = 0 & \text{in } Q_{T'}\,, \\ \operatorname{div} V = 0 & \text{in } Q_{T'}\,, \\ V\cdot n = 0 & \text{on } \Sigma_{T'}\,, \\ (V, u^{(l)}) = 0 & \forall t \in [0, T']\,, \quad l = 1, \dots, N\,, \end{cases}$$

where the last equation follows from $(2.46)_5$.

REMARK 2.2. One can solve our problem also for the following boundary conditions for the temperature θ (with the obvious modifications in the compatibility conditions):

$$\frac{\partial\theta}{\partial n}=\theta_2 \quad \text{or} \quad \frac{\partial\theta}{\partial n}=-k(\theta-\theta_2) \quad \text{on } \Sigma_T,$$

where k is a given positive constant and $\theta_2\in H^{5/2,5/4}(\Sigma_{T_0})$.

REMARK 2.3. Instead of (2.29), one can solve the problem (directly obtained by (1.1))

$$(2.47)\quad \begin{cases} \dot u-\dfrac{\zeta_0}{\varrho_0}\nabla\operatorname{div}u=-(v\cdot\nabla)v+b-\dfrac{\nabla p}{\varrho}+\dfrac{\nabla\zeta}{\varrho}\psi+\left(\dfrac{\zeta}{\varrho}-\dfrac{\zeta_0}{\varrho_0}\right)\nabla\psi & \\ & \text{in } Q_T,\\ u\cdot n=0 & \text{on } \Sigma_T,\\ u(0)=v_0 & \text{in } \Omega. \end{cases}$$

The operator $A_3\equiv-(\zeta_0/\varrho_0)\nabla\operatorname{div}$ is not elliptic (see Agmon-Douglis-Nirenberg [1]). First one solves the problem $\lambda u+A_3u=f\in L^2(\Omega)$, finding by the Lax-Milgram lemma the solution in the space $\{u\in L^2(\Omega), \operatorname{div}u\in L^2(\Omega), u\cdot n=0 \text{ on } \Gamma\}$; then, if $f\in H^2(\Omega)$, one obtains $u\in D(A_3)\equiv\{u\in H^2(\Omega), \nabla\operatorname{div}u\in H^2(\Omega), u\cdot n=0 \text{ on } \Gamma\}$. A_3 satisfies the inequality (2.12), so that one can solve (2.47) by means of Theorem 5.2, chap. 1 of Lions-Magenes [11], as for the problem (2.11).

Then one proves, if T is small enough, that there exists a fixed point $\psi=\operatorname{div}u$. By $(2.47)_1$ and $(2.46)_5$, one has $\operatorname{rot}(\dot u-\dot v)=0$ in Q_T and $(\dot u-\dot v, u^{(l)})=0$ in $[0,T]$, $l=1,\dots,N$. Hence

$$\begin{cases} \operatorname{rot}(u-v)=0 & \text{in } Q_T,\\ \operatorname{div}(u-v)=0 & \text{in } Q_T,\\ (u-v)\cdot n=0 & \text{on } \Sigma_T,\\ (u-v,u^{(l)})=0 & \forall t\in[0,T],\quad l=1,\dots,N, \end{cases}$$

from which $u=v$ in Q_T.

3. Proof of Theorem B.

The proof is similar to the one of Theorem A. We start from ψ as in (H_1) and φ as in (H_3), while we take $\bar\theta\in C^0([0,T];H^3(\Omega))$ such that $\bar\theta(0) = \theta_0$ in Ω and $|\bar\theta|_{\infty,3,T} \leqslant B'$ (that we specify later). One finds the first fixed point $\varphi = \xi$ as in the proof of Theorem A. Then, insstead of (2.11), one considers the problem

$$\begin{cases} \dot\theta + v\cdot\nabla\theta = -\dfrac{\bar\theta}{\bar c_V \varrho}\dfrac{\partial\bar p}{\partial\theta}\psi + \dfrac{r}{\bar c_V} + \dfrac{\bar\zeta}{\bar c_V\varrho}\psi^2 \equiv F' & \text{in } Q_T\,, \\ \theta(0) = \theta_0 & \text{in } \Omega\,. \end{cases}$$

We solve it by the method of characteristics, as the equation for the density ϱ. Observe now that $L^2(0, T; H^3(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$ is an algebra and consequently $\psi^2 \in L^2(0, T; H^3(\Omega))$; hence $F' \in L^2(0, T; H^3(\Omega))$, and then one obtains $\theta \in C^0([0, T]; H^3(\Omega))$; from the equation $\big(F'$ belongs also to $L^\infty(0, T; H^1(\Omega))\big)$ one has $\dot\theta \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$. Moreover

$$|\theta|_{\infty,3,T} \leqslant \exp[\bar c(A, B')T]\{\|\theta_0\|_3 + \bar c(A, B')T^{1/2}\}\,.$$

Then, if one takes $B' > \|\theta_0\|_3$, there exists T''' small enough such that θ belongs to the convex set $\{\bar\theta \in C^0([0, T''']; H^3(\Omega)):\ \bar\theta(0) = \theta_0$ in Ω, $|\bar\theta|_{\infty,3,T'''} \leqslant B'\}$. It is not hard to show that the map $\bar\theta \to \theta$ has a fixed point. The remainder of the proof is as the third part of the proof of Theorem A. □

REMARK 3.1. Proceeding as in Valli [20] (see also [21]), we can obtain analogous results also for the problem

$$\begin{cases} \varrho[\dot v + (v\cdot\nabla)v - b] = -\nabla p + \sum\limits_k [D_k(\mu D_k v) + \\ \qquad\qquad + D_k(\mu\nabla v^k)] + \nabla[(\zeta - \frac{2}{3}\mu)\operatorname{div} v] & \text{in } Q_T\,, \\ \dot\varrho + v\cdot\nabla\varrho + \varrho\operatorname{div} v = 0 & \text{in } Q_T\,, \\ c_V\varrho[\dot\theta + v\cdot\nabla\theta] = -\theta\dfrac{\partial p}{\partial\theta}\operatorname{div} v + \varrho r + \dfrac{\mu}{2}\sum\limits_{i,k}(D_k v^i + D_i v^k)^2 + \\ \qquad\qquad + \left(\zeta - \dfrac{2}{3}\mu\right)(\operatorname{div} v)^2 & \text{in } Q_T\,, \\ v\cdot n = 0 & \text{on } \Sigma_T\,, \\ v(0, x) = v_0(x) & \text{in } \Omega\,, \\ \varrho(0, x) = \varrho_0(x) & \text{in } \Omega\,, \\ \theta(0, x) = \theta_0(x) & \text{in } \Omega\,, \end{cases}$$

where $\mu = \mu(\varrho, \theta, v) > 0$, $\zeta = \zeta(\varrho, \theta, v) \geqslant 0$ and χ is equal to zero. The third equation is solved by the method of characteristics as in the proof of Theorem B.

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Manoscritto pervenuto in redazione il 7 giugno 1982.