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Asymptotic Stability Theorems for Viscous Fluid Motions in Exterior Domains (*).

PAOLO MAREMONTI (**)

Introduction.

Let $\mathcal{F}$ be a viscous incompressible fluid, filling the domain Ω exterior to ν compact subregions of the euclidean three-dimensional space R^3. In this paper we shall study the attractivity of a given motion m_0 of $\mathcal{F}$. As is well known, such a problem in the case Ω *bounded* has been investigated by several authors [1-3] and they proved that, provided a suitable condition on the Reynolds number Re associated to m_0 is fulfilled, all perturbations satisfying the « energy inequality » fall off as $t \to +\infty$ in the L^2 norm with an exponential decay order. The key tool in proving this result is furnished by the validity of the Poincarè inequality. It is therefore quite natural to expect that when one considers the case of an *exterior* domain, where this inequality fails, the problem becomes much more involved and, further, in general the above results no longer hold [4]. To solving this problem the efforts of several writers have been directed [4-13]. In particular, we quote the results of [9, 11, 13] because of the strict connection between them and the main theorems proved in the present paper. In [9] the author shows that, provided the unperturbed mo-

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tion m_0 satisfies some regularity assumptions (1) and the number Re is not « too large », all perturbations verifying the « energy inequality » smooth out as the time increases and finally decay to zero in suitable norms and with a suitable order of decay. In [13], among other things, the result of [9] are improved as far as the order of decay is concerned, but the unperturbed motion reduces to the rest. Finally, in [11] following analogous formulations for bounded domain (cf. [1-2]) a variational approach to stability is proved under assumptions on the unperturbed motion m_0 larger than those considered in [9, 13]. In particular, in the case when m_0 is unsteady, no « infinitesimality » for large t on the kinetic field $\boldsymbol{v}$ associated to m_0 is needed. However, results proved in [11] hold for small initial data and no decay order is given as $t \to +\infty$.

In this paper, following the approach given in [11], we give a variational formulation of stability of motions in exterior domains. To better explain our results, let us begin to denote by R^{-1} the maximum of a suitable quadratic functional depending on perturbation through the rate stress tensor of m_0. We also assume that the kinetic field associated to m_0 has a « nice » behaviour at large spatial distances and, in the case m_0 unsteady, for large time as well. We start with perturbations $\boldsymbol{u}$ to the kinetic field associated to m_0 satisfying the « energy inequality » and prove that if $Re < R$ there exists an instant T, such that for $t \geq T$, $\boldsymbol{u}$ smooths out and ultimately decays to zero with suitable order. In particular we have for m_0 *steady*

$$\text{(I)} \qquad \begin{aligned} \|\boldsymbol{u}_t\| &= O(t^{-1}) , \\ \|\nabla \boldsymbol{u}\| &= O(t^{-1/2}) , \\ \sup_{\Omega} |\boldsymbol{u}(x, t)| &= O(t^{-1/2}) , \end{aligned}$$

(1) As pointed out by the author to Professor K. Masuda, assumption 2′ made in [9] p. 298 should be strengthened. Professor Masuda kindly replied that the right assumption to be made is the following

$$t^{1/2}\left(\frac{\partial}{\partial t}\right) \boldsymbol{w}(x, t) \in L^{\infty}((0, +\infty); L^3(\Omega))$$

where $\boldsymbol{w}$ is the kinetic field associated to the unsteady unperturbed motion.

and for m_0 *unsteady*

$$\text{(II)} \qquad \begin{aligned} \|\boldsymbol{u}_t\| &= O(t^{-1/2}), \\ \|\nabla \boldsymbol{u}\| &= O(t^{-1/2}), \\ \sup_{\Omega} |\boldsymbol{u}(x,t)| &= O(t^{-1/2}). \end{aligned}$$

In the above, $\|\cdot\|$ represents the L^2-norm and the subscript t denotes differentiation with respect to time. The above decay estimates markedly improve those given in [9], and coincide with those given in [13] which are proved, however, *only when* m_0 *is the rest.* In this connection, it is worth remarking that, as noticed in [13] p. 674, the methods employed in [13] are not able to give any behaviour for large t when m_0 is different from the rest. Moreover, when m_0 is *steady* a suitable coupling of methods of [13] and [9] would give an asymptotic behaviour which, however, is worse than that provided in (I).
Finally regarding $(\mathrm{I})_1$ it seem interesting to remark that it is a consequence of a sort Poincarè inequality which we prove to hold for $\boldsymbol{u}_t$. Obviously, this inequality is true a *priori* only along the solutions and, in fact, the constant appearing in it depends on m_0, on the initial data of $\boldsymbol{u}$, on Re and R.

The paper is subdivided into three main sections. The first one is devoted to some mathematical preliminaries concerning embedding theorems, the Stokes problem in exterior domains (subsection 1.1) and to the statement of the main theorems (subsection 1.2). The second section is devoted to the proof of stability of steady motions, i.e., to the proof of (I). To this end, in subsection 2.1, we begin to give some existence theorems and to prove (along the lines of [9, 13, 14]) that for t sufficiently large the perturbation smooths out in a suitable sense. In subsection 2.2 after several preliminary lemmas we give the proof of (I). Finally, section 3 is devoted to show the stability of unsteady motions, i.e., to show relations (II). This is accomplished by first proving existence theorems of the kind previously proved in the steady case (subsection 3.1). In subsection 3.2 we give the proof of (II).

Last but not least, the author wishes to express his deep gratitude to Professor G. P. Galdi for suggesting this research and for helpful comments and suggestions.

1. Preliminary results and statement of the main theorems.

1.1. *Preliminaries.*

Throughout this paper we indicate by Ω a C^2-smooth domain exterior to ν (≥ 0) compact regions of the euclidean three dimensional space R^3. For $T > 0$ we set $\Omega_T \equiv \Omega \times [0, T)$ and denote by (x, t) a given point in Ω_T.

We introduce some spaces whose members are vector functions $\boldsymbol{u}$: $\Omega \to R^3$. $L^p(\Omega)$ ($p \in [1, +\infty]$) is the usual Lebesgue space endowed with the norm

$$\|\boldsymbol{u}\|_p \equiv \Big(\int_\Omega |\boldsymbol{u}|^p\, dx\Big)^{1/p};$$

in the case $p = 2$ we put $\|\boldsymbol{u}\|_2 \equiv \|\boldsymbol{u}\|$. Moreover, $W_2^m(\Omega), m = 1, 2, \ldots,$ is the Sobolev space of functions $\boldsymbol{u}$ which are square summable over Ω together with their m-th (generalized) derivatives inclusive. As is well known, $W_2^m(\Omega)$ is a Banach space equipped with the norm

$$\|\boldsymbol{u}\|_{W_2^m} = \Big(\sum_{|\mu| \leq m} \|D^\mu \boldsymbol{u}\|^2\Big)^{1/2}$$

where

$$\mu = (\mu_1, \mu_2, \mu_3)\,,\ \mu_i \geq 0\,,\ |\mu| = \mu_1 + \mu_2 + \mu_3\,,\ D^\mu \boldsymbol{u} = \frac{\partial^\mu \boldsymbol{u}}{\partial x_1^{\mu_1}\, \partial x_2^{\mu_2}\, \partial x_3^{\mu_3}}\,.$$

We shall set $\mathring{W}_2^m$ the completion of $C_0^\infty(\Omega)$ in $W_2^m(\Omega)$. Furthermore we let

$$\mathcal{C}_0(\Omega) = \{\boldsymbol{u}\colon \boldsymbol{u} \in C_0^\infty(\Omega) \text{ and } \nabla \cdot \boldsymbol{u} = 0\}\,,$$

$$H(\Omega) = \text{completion of } \mathcal{C}_0(\Omega) \text{ in } L^2(\Omega)\,,$$

$$H^1(\Omega) = \text{completion of } \mathcal{C}_0(\Omega) \text{ in } \mathring{W}_2^1(\Omega)\,,$$

$$\tilde{H}(\Omega) = \text{completion of } \mathcal{C}_0(\Omega) \text{ with Dirichlet norm: } \|\nabla(\cdot)\|\,.$$

For $a, b > 0$ by $L^p(a, b; X)$, $p \in [1, +\infty]$, X Banach space we denote the set of functions $f:(a, b) \to X$ such that

$$\left(\int_a^b \|f\|_X^p\right)^{1/p} < +\infty \qquad (\|f\|_X \equiv X\text{-norm})\,.$$

Analogously, denoting by I an interval in R, by $C(I; X)$ we indicate the set of functions f: $I \to X$ which are continuous from I into X. A natural norm in this (Banach) space is

$$\|f\|_C \equiv \max_I \|f\|_X\,.$$

As is well known [15] $L^2 = G \oplus H$ where

$$G = \{\boldsymbol{u} \in L^2(\Omega):\ \boldsymbol{u} = \nabla p \text{ for some } p(x) \in W^1_{2\text{loc}}(\Omega)\}\,.$$

By $P_H \equiv P$ we denote the projection operator from L^2 into H. The following lemma can be proved (cf., e.g., [16])

LEMMA 1.1. *Let* $\boldsymbol{w} \in W_2^2 \cap H^1$ *be a solution to*

$$\begin{cases} \Delta \boldsymbol{w}(x) = -\nabla p(x) + \boldsymbol{f}(x)\,, & \text{with } \boldsymbol{f} \in L^2(\Omega)\,,\\ \nabla\cdot \boldsymbol{w}(x) = 0\,, & \\ \boldsymbol{w}(y)|_{\partial\Omega} = 0\,, & \end{cases}$$

then

$$\begin{cases} \|D^2\boldsymbol{w}\| \leqslant C_0(\|P\boldsymbol{f}\| + \|\nabla \boldsymbol{w}\|)\,,\\ \|\nabla \boldsymbol{w}\|_3 \leqslant C_0(\|P\boldsymbol{f}\|^{1/2}\,\|\nabla \boldsymbol{w}\|^{1/2} + \|\nabla \boldsymbol{w}\|)\,, \end{cases} \tag{1.1}$$

where C_0 is a constant depending on the geometry on Ω. In this connection we notice that, throughout the paper the symbol C_n ($n = 0, 1, ...$) will be used to denote a positive constant depending at most on the geometry of Ω and on the «size» of the motions whose stability is to be investigated. The precise value of C_n is unessential to our aims and therefore it will be omitted.

In the following lemmas $\boldsymbol{u}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ denote vector functions on Ω, $(\cdot, \cdot)$ denote the scalar product in $L^2(\Omega)$ and α is a positive real number.

LEMMA 1.2. *Let* $\boldsymbol{u} \in W_2^2 \cap H^1$, $\boldsymbol{b} \in L^2(\Omega)$, $\boldsymbol{a} \in L^6(\Omega)$ *and* $\nabla \boldsymbol{a} \in L^3(\Omega)$. *Then*, $\forall \eta, \varepsilon > 0$,

$$(1.2)\quad \begin{cases} |\alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant \dfrac{C_1}{\varepsilon^2\eta}\alpha^4\|\nabla\boldsymbol{u}\|^6 + \dfrac{C_2}{\varepsilon}\alpha^2\|\nabla\boldsymbol{u}\|^4 + 2\varepsilon\|\boldsymbol{b}\|^2 + \eta\|P\Delta\boldsymbol{u}\|^2, \\ |\alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{a}, \boldsymbol{b})| \leqslant \dfrac{C_3}{\varepsilon}\alpha^2\|\nabla\boldsymbol{a}\|_3^2\|\nabla\boldsymbol{u}\|^2 + \varepsilon\|\boldsymbol{b}\|^2, \\ |\alpha(\boldsymbol{a}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant \left(\dfrac{C_4}{\varepsilon}\alpha^2\|\boldsymbol{a}\|_6^2 + \dfrac{C_5}{\varepsilon^2\eta}\alpha^4\|\boldsymbol{a}\|_6^4\right)\cdot \\ \qquad\qquad \cdot\|\nabla\boldsymbol{u}\|^2 + 2\varepsilon\|\boldsymbol{b}\|^2 + \eta\|P\Delta\boldsymbol{u}\|^2. \end{cases}$$

PROOF. We employ the Holder inequality with exponents 6, 3, and 2 to obtain

$$(1.3)\qquad |(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant \|\boldsymbol{u}\|_6\|\nabla\boldsymbol{u}\|_3\|\boldsymbol{b}\|.$$

Since

$$(1.4)\qquad \|\varphi\|_6 \leqslant C_6\|\nabla\varphi\|, \qquad \forall\varphi \in \mathring{W}_2^1(\Omega),$$

taking into account $(1.1)_2$ from (1.3) we have

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant C_0C_6\|\nabla\boldsymbol{u}\|(\|\nabla\boldsymbol{u}\|^{1/2}\|P\Delta\boldsymbol{u}\|^{1/2} + \|\nabla\boldsymbol{u}\|)\|\boldsymbol{b}\|.$$

From this last relation we deduce $(1.2)_1$. Concerning $(1.2)_2$, we notice that from Holder inequality we obtain

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{a}, \boldsymbol{b})| \leqslant \|\boldsymbol{u}\|_6\|\nabla\boldsymbol{a}\|_3\|\boldsymbol{b}\|.$$

From (1.4) and Cauchy inequality we thus deduce $(1.3)_2$.

Applying again Holder inequality with exponents 6, 3 and 2, and taking into account $(1.1)_2$ we have

$$|(\boldsymbol{a}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant C\|\boldsymbol{a}\|_6(\|\nabla\boldsymbol{u}\|^{1/2}\|P\Delta\boldsymbol{u}\|^{1/2} + \|\nabla\boldsymbol{u}\|)\|\boldsymbol{b}\|.$$

Employing the Cauchy inequality we finally recover $(1.2)_3$.

LEMMA 1.3. *Let* $\boldsymbol{u} \in W_2^2 \cap H^1$, $\boldsymbol{a} \in L^6(\Omega)$ *and* $\nabla \boldsymbol{a} \in L^3(\Omega)$. *Then,* $\forall \eta > 0$

$$(1.5)\quad \begin{cases} \alpha|(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, P\Delta\boldsymbol{u})| \leqslant \dfrac{C_7}{\eta^3}\alpha^4\|\nabla\boldsymbol{u}\|^6 + \dfrac{C_8}{\eta}\alpha^2\|\nabla\boldsymbol{u}\|^4 + 2\eta\|P\Delta\boldsymbol{u}\|^2, \\ \alpha|(\boldsymbol{u}\cdot\nabla\boldsymbol{a}, P\Delta\boldsymbol{u})| \leqslant \dfrac{C_9}{\eta}\alpha^2\|\nabla\boldsymbol{a}\|_3^2\|\nabla\boldsymbol{u}\|^2 + \eta\|P\Delta\boldsymbol{u}\|^2, \\ \alpha|(\boldsymbol{a}\cdot\nabla\boldsymbol{u}, P\Delta\boldsymbol{u})| \leqslant \left[\dfrac{C_{10}}{\eta^3}\alpha^4\|\boldsymbol{a}\|_6^4 + \dfrac{C_{11}}{\eta}\alpha^2\|\boldsymbol{a}\|_6^2\right]\cdot \\ \qquad\qquad\qquad\qquad \cdot\|\nabla\boldsymbol{u}\|^2 + 2\eta\|P\Delta\boldsymbol{u}\|^2. \end{cases}$$

PROOF. The proof of this lemma is analogous to that of Lemma 1.2 with $\boldsymbol{b} = P\Delta\boldsymbol{u}$ and $2\varepsilon = \eta$.

LEMMA 1.4. *Let* $\boldsymbol{u}, \boldsymbol{b} \in H^1$ *and* $\boldsymbol{a} \in L^3(\Omega)$. *Then,* $\forall \varepsilon_1 > 0$

$$(1.6)\quad \begin{cases} |\alpha(\boldsymbol{b}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant \begin{cases} \dfrac{1}{4}\alpha\|\boldsymbol{b}\|^2\|\nabla\boldsymbol{u}\|^2 + \dfrac{3}{4}\alpha\|\nabla\boldsymbol{b}\|^2\|\nabla\boldsymbol{u}\|^{2/3}, \\ \dfrac{C}{4\varepsilon_1^3}\alpha^4\|\boldsymbol{b}\|^2\|\nabla\boldsymbol{u}\|^4 + \dfrac{3}{4}\varepsilon_1\|\nabla\boldsymbol{b}\|^2, \end{cases} \\ |\alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{a}, \boldsymbol{b})| \leqslant \dfrac{C_{12}}{\varepsilon_1}\alpha^2\|\boldsymbol{a}\|_3^2\|\nabla\boldsymbol{u}\|^2 + \varepsilon_1\|\nabla\boldsymbol{b}\|^2, \\ |\alpha(\boldsymbol{a}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant \dfrac{C_{12}}{\varepsilon_1}\alpha^2\|\boldsymbol{a}\|_3^2\|\nabla\boldsymbol{u}\|^2 + \varepsilon_1\|\nabla\boldsymbol{b}\|^2. \end{cases}$$

PROOF. Applying Schwartz inequality and the following

$$\|\varphi\|_4^4 \leqslant 4\|\varphi\|\,\|\nabla\varphi\|^3, \qquad \forall\varphi \in \mathring{W}_2^1(\Omega)$$

we obtain

$$(1.7)\qquad |(\boldsymbol{b}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant 2\|\boldsymbol{b}\|^{1/2}\|\nabla\boldsymbol{b}\|^{3/2}\|\nabla\boldsymbol{u}\|.$$

Therefore both relations $(1.6)_1$ are deduced from (1.7) after a suitable application of Cauchy inequality. On the other hand, taking into account

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{a}, \boldsymbol{b}) = -(\boldsymbol{u}\cdot\nabla\boldsymbol{b}, \boldsymbol{a})$$

and employing Holder inequality in $(1.6)_2$ and $(1.6)_3$ with exponents 6, 2 and 3 and 3, 2 and 6 respectively, we have

$$\begin{cases} |(\boldsymbol{u}\cdot\nabla\boldsymbol{b},\boldsymbol{a})|\leqslant C_6\|\nabla\boldsymbol{u}\|\,\|\nabla\boldsymbol{b}\|\,\|\boldsymbol{a}\|_3\,, \\ |(\boldsymbol{a}\cdot\nabla\boldsymbol{u},\boldsymbol{b})|\leqslant C_6\|\nabla\boldsymbol{u}\|\,\|\nabla\boldsymbol{b}\|\,\|\boldsymbol{a}\|_3\,. \end{cases}$$

From these last relations we deduce $(1.6)_2$ and $(1.6)_3$.

LEMMA 1.5. *Let* $\boldsymbol{u},\boldsymbol{b}\in H^1$, $\boldsymbol{a}\in L^6(\Omega)$ *and* $\nabla\boldsymbol{a}\in L^3(\Omega)$. *Then*, $\forall\eta_1>0$

$$(1.8)\quad \begin{cases} |\alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{b})|\leqslant\dfrac{C_{13}}{\eta_1}\alpha^2\,\|\boldsymbol{u}\|_4^2\,\|\nabla\boldsymbol{u}\|^2+\dfrac{\eta_1}{2}\,\|\boldsymbol{b}\|^2+\dfrac{3}{2}\eta_1\|\nabla\boldsymbol{b}\|^2\,, \\ |\alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{a},\boldsymbol{b})|\leqslant\dfrac{C_{14}}{\eta_1}\alpha^2\|\nabla\boldsymbol{a}\|_3^2\,\|\nabla\boldsymbol{u}\|^2+\dfrac{1}{2}\eta_1\|\boldsymbol{b}\|^2 \\ |\alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{b})|\leqslant\dfrac{C_{15}}{\eta_1}\alpha^2\,\|\boldsymbol{a}\|_6^2\|\nabla\boldsymbol{u}\|^2+\dfrac{\eta_1}{2}\,\|\boldsymbol{b}\|^2+\dfrac{\eta_1}{2}\,\|\nabla\boldsymbol{b}\|^2. \end{cases}$$

PROOF. To obtain $(1.8)_1$, we notice that

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{b})|\leqslant\sqrt{2}\,\|\boldsymbol{u}\|_4\,\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{b}\|^{1/4}\,\|\nabla\boldsymbol{b}\|^{3/4}$$

where the inequality

$$\|\varphi\|_4^4\leqslant 4\,\|\varphi\|\,\|\nabla\varphi\|^3\,,\qquad \forall\varphi\in\mathring{W}_2^1(\Omega)$$

has been employed. Therefore, $(1.8)_1$ follows from this last relation along with the use of Young and Cauchy inequality. Analogously, to prove $(1.8)_2$ we notice that it follows from Holder inequality with exponents 6, 3 and 2 and again Young inequality. Finally, it is

$$(1.9)\qquad |(\boldsymbol{a}\cdot\nabla\boldsymbol{u},\boldsymbol{b})|\leqslant(2)^{1/3}\,\|\boldsymbol{a}\|_6\,\|\nabla\boldsymbol{u}\|\,\|\boldsymbol{b}\|^{1/2}\,\|\nabla\boldsymbol{b}\|^{1/2}$$

where use has been made of [18]

$$\|\varphi\|_3\leqslant(2)^{1/3}\,\|\varphi\|^{1/2}\,\|\nabla\varphi\|^{1/2}\,,\qquad \forall\varphi\in\mathring{W}_2^1(\Omega)\,.$$

Using Young and Cauchy inequality in (1.9), $(1.8)_3$ follows.

LEMMA 1.6. *Let* $\boldsymbol{u}, \boldsymbol{b} \in H^1$, $\boldsymbol{a} \in L^3(\Omega)$. *Then*

$$
(1.10) \qquad \begin{cases} |\alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant C_{16}\,\alpha\|\boldsymbol{u}\|^{\frac{1}{2}}\ \|\nabla\boldsymbol{u}\|^{\frac{3}{2}}\|\ \|\nabla\boldsymbol{b}\|\,, \\ |\alpha(\boldsymbol{a}\cdot\nabla\boldsymbol{u}, \boldsymbol{b})| \leqslant C_6\alpha\|\boldsymbol{a}\|_3\ \|\nabla\boldsymbol{u}\|\ \|\nabla\boldsymbol{b}\|\,, \\ |\alpha(\boldsymbol{u}\cdot\nabla\boldsymbol{a}, \boldsymbol{b})| \leqslant C_6\alpha\|\boldsymbol{a}\|_3\ \|\nabla\boldsymbol{u}\|\ \|\nabla\boldsymbol{b}\|\,. \end{cases}
$$

PROOF. We have

$$
(1.11) \qquad |(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{b}) \leqslant \|\boldsymbol{u}\|_3\ \|\nabla\boldsymbol{u}\|\ \|b\|_6\,.
$$

Since

$$
(1.12) \qquad \|\varphi\|_3 \leqslant (2)^{1/3}\,\|\varphi\|^{1/2}\,\|\nabla\varphi\|^{1/2} \text{ and } \|\varphi\|_6 \leqslant C_6\|\nabla\varphi\|\,, \qquad \forall\varphi\in \mathring{W}_2^1(\Omega)\,,
$$

$(1.10)_1$ is a consequence of (1.11)-(1.12). Estimates $(1.10)_2$ and $(1.10)_3$ are easily obtainable by applying suitable Holder inequality and noticing that

$$
(\boldsymbol{u}\cdot\nabla\boldsymbol{a}, \boldsymbol{b}) = -(\boldsymbol{u}\cdot\nabla\boldsymbol{b}, \boldsymbol{a})\,.
$$

For the proof of the first part of Lemma 1.7 below cf. also [19].

LEMMA 1.7. *Let* $\varphi(t) \in C^1(t_0, +\infty)$ *such that* $\varphi(t) \geqslant 0$

$$
(1.13) \qquad \begin{cases} \dot\varphi(t) \leqslant a(t)\varphi(t) + K_2\varphi^2(t) + K_3\varphi^3(t)\,, \\ \varphi(t_0) = \varphi_0\,, \\ E = \displaystyle\int_{t_0}^{+\infty}\varphi(t)\,dt < +\infty\,, \end{cases}
$$

with $a(t) \leqslant K_1$, $\forall t \geqslant t_0$. *Assume, moreover, that for some* $\delta > 0$

$$
E < \frac{\delta}{2K(1+\delta)^2} \quad \text{with } K = \operatorname*{Max}_i K_i
$$

then

$$
(1.14) \qquad \varphi_0 < \frac{\delta}{2} \Rightarrow \varphi(t) < \delta\,, \quad \forall t \geqslant t_0\,.
$$

Moreover, if

$$M = \int_{t_0}^{+\infty} a(t)\, dt < +\infty$$

necessarily

$$\varphi(t) \leqslant \frac{[K_2\varphi_0(t_0+\beta)+1]\exp[(M+K_2E+K_3\delta E)-1}{K_2(t+\beta)}, \qquad \forall t \geqslant t_0$$

for each $\beta > -t_0$.

PROOF. We proceed per *absurdum*. Since $\varphi_0 < \delta/2$ $(<\delta)$, by continuity there exists $t^* > t_0$ such that $\varphi(t) < \delta$, $\forall t \in [t_0, t^*)$, and $\varphi(t^*) = \delta$. We shall show that this leads to a contradiction, thus proving $t = +\infty$. In fact, from $(1.13)_1$ we have $\forall t \in [t_0, t^*]$

$$\varphi(t) \leqslant \varphi_0 + K_1\int_{t_0}^{t^*}\varphi(s)\, ds + K_2\int_{t_0}^{t^*}\varphi^2(s)\, ds + K_3\int_{t_0}^{t^*}\varphi^3(s)\, ds$$

and hence

$$\varphi(t^*) < \frac{\delta}{2} + KE(1+\delta+\delta^2) < \frac{\delta}{2} + KE(1+\delta)^2 < \delta\,,$$

which shows (1.14). Assume now $M = \int_{t_0}^{+\infty} a(t)\, dt < +\infty$. Then multiplying $(1.13)_1$ by $(t+\beta)$ $(\beta > -t_0)$ we deduce

$$\frac{d}{dt}[(t+\beta)\varphi(t)] \leqslant a(t)(t+\beta)\varphi(t) + K_2(t+\beta)\varphi^2(t) + {} + K_3(t+\beta)\varphi^3(t) + \varphi(t)\,. \tag{1.15}$$

Setting $z(t) = (t+\beta)\varphi(t)$ from (1.15) we obtain

$$z'(t) \leqslant a(t)z(t) + K_2\varphi(t)z(t) + K_3\varphi^2(t)z(t) + \varphi(t)$$

whith is equivalent to

$$\frac{z'(t)}{K_2 z(t)+1} \leqslant \frac{a(t)z(t)}{K_2 z(t)+1} + \varphi(t) + \frac{K_3\varphi^2(t)z(t)}{K_2 z(t)+1}\,.$$

From this relation we deduce

$$\frac{K_2 z'(t)}{K_2 z(t)+1} \leqslant a(t) + K_2\varphi(t) + K_3\varphi^2(t)\,. \tag{1.16}$$

Integrating (1.16) over $[t_0, t]$ it follows

$$\log\frac{K_2 z(t)+1}{K_2 z(0)+1} \leqslant M + K_2 E + K_3 \delta E$$

whith gives

$$\varphi(t) \leqslant \frac{[K_2\varphi_0(t_0+\beta)+1]\exp(M+K_2E+K_3\delta E)-1}{K_2(t+\beta)}\,, \qquad \forall t \geqslant t_0\,,$$

thus proving the lemma.

We end this section by recalling a well known theorem. For $a, b > 0$ we set

$$W(a, b) = \left\{ f \in L^2(a, b;\, X) : \frac{d^h f}{dt^h} \in L^2(a, b;\, Y) \right\}$$

where the derivatives are taken in a distributional sense. It is known that $W(a, b)$ endowed with the norm

$$\|f\|_W = \left[\|f\|^2_{L^2(a,b;X)} + \|f^h\|^2_{L^2(a,b;Y)}\right]^{1/2}$$

becomes a Hilbert space. Denoting by $[X, Y]_\theta$, $0 \leqslant \theta \leqslant 1$, the intermediate space [20], the following lemma holds [20].

LEMMA 1.8. *If $f \in W(a, b)$, then*

$$\frac{df^j}{dt^j} \in C\big(a, b;\, [X, Y]_{(j+\frac12)/h}\big) \qquad 0 \leqslant j \leqslant h-1\,.$$

1.2. *Statement of stability results.*

Let $\mathcal{F}$ be an incompressible viscous fluid filling the region Ω. We assume that the motion of $\mathcal{F}$ is governed by the Navier-Stokes equations. In this section we state the problem of the attractivity of a motion m_0 of $\mathcal{F}$ with respect to perturbations to initial data. As is well known, indicating by $(\boldsymbol{u}, \pi)$ the perturbation to the kinetic and pressure fields associated to m_0, we have that $(\boldsymbol{u}, \pi)$ is a solution to the following initial boundary value problem:

$$(1.17)\qquad \begin{cases} \boldsymbol{u}_t + \mathrm{Re}\,(\boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{a}\cdot\nabla\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{a}) = \Delta\boldsymbol{u} - \nabla\pi\,, \\ \nabla\cdot\boldsymbol{u} = 0\,, \quad \forall(x,t)\in\Omega_s\,, \\ \boldsymbol{u}(y,t)_{\partial\Omega}\big| = 0\,, \quad \forall(y,t)\in\partial\Omega\times[0,T)\,. \end{cases}$$

In the above $\boldsymbol{u}_t = \partial\boldsymbol{u}(x,t)/\partial t$, $\boldsymbol{a}$ is the velocity field associated to m_0 and Re is the Reynolds number associated to $\boldsymbol{a}$. Throughout this paper, we shall replace the letter $\boldsymbol{a}$ in (1.17) with $\boldsymbol{w}$ or $\boldsymbol{v}$ according to whether the motion m_0 is steady or not. Concerning $\boldsymbol{a}$, we make the following assumptions:

i) There exist $M_i > 0$ $(i = 1, 2, 3)$, $q \in (3, +\infty]$ such that

$\|\nabla\boldsymbol{v}\|_q \leqslant M_1$ uniformly in $t \geqslant 0$, (resp. $\|\nabla\boldsymbol{w}\|_q < +\infty$) ;

$\|\boldsymbol{v}\|_6 \leqslant M_2$ and $\|\nabla\boldsymbol{v}\|_3 \leqslant M_3$ uniformly in $t \geqslant 0$,

$$(\text{resp. } \|\boldsymbol{w}\|_6 + \|\nabla\boldsymbol{w}\|_3 < +\infty)\ ;$$

ii) there exists $M_4 > 0$ such that

$$\int_0^{+\infty}(\|\boldsymbol{v}(t)\|_6^2 + \|\nabla\boldsymbol{v}(t)\|_3^2)\,dt \leqslant M_4\,;$$

iii) there exists (in a suitable function space) the maximum (²) of the following functional

$$\mathcal{F}(\boldsymbol{u}) = -\int_\Omega \frac{\boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u}}{\|\nabla\boldsymbol{u}\|^2} \qquad \left(\text{resp. } \mathcal{F}^*(\boldsymbol{u}) = -\int_\Omega \frac{\boldsymbol{u}\cdot\boldsymbol{D}^*\cdot\boldsymbol{u}}{\|\nabla\boldsymbol{u}\|^2}\right)^{(3)}\,.$$

(²) The existence of Max $\mathcal{F}(\boldsymbol{u})$ in $\tilde{H}$, under suitable hypotheses for unperturbed motion, has been proved by G. P. Galdi in [11]. Cf. also remark (2) below.

(³) With $\boldsymbol{D}$ we denote the rate stress tensor associated to $\boldsymbol{a}$.

Moreover, setting

$$\frac{1}{R(t)} = \operatorname*{Max}_{u} \mathcal{F}(\boldsymbol{u}),$$

it holds

$$\frac{1}{R} = \sup_{t\geqslant 0} \frac{1}{R(t)} < \frac{1}{Re} \tag{1.18}$$

where Re is the Reynolds number associated to $\boldsymbol{v}$ (resp. $\boldsymbol{w}$);

iv) we suppose that $\boldsymbol{v}$ is continuously differentiable function of x and t; moreover

$$\|\boldsymbol{v}_t\|_3 \leqslant M_5 \qquad \text{uniformly in } t \geqslant 0 .$$

Finally, we suppose that $\boldsymbol{w} \in L^3(\Omega)$.

Now we give the following definition of weak and strong solutions to equations (1.17).

DEDINITION 1.1. A field $\boldsymbol{u}$: $\Omega_T \to R^3$ ($T = +\infty$) is said to be a *weak solution* of (1.23) if and only if

h$_1$) $\boldsymbol{u} \in L^\infty(0, +\infty; H) \cap L^2(0, +\infty; \tilde{H})$;

h$_2$) for some given $\boldsymbol{u}_0 \in H \lim\limits_{t\to 0^+} \|\boldsymbol{u}(t) - \boldsymbol{u}_0\|^2 = 0$;

h$_3$) the equation

$$\int_s^t \{(\boldsymbol{u}, \boldsymbol{\varphi}_\tau) - (\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + Re[(\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{v}) + (\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}, \boldsymbol{u})]\}\, d\tau = \\ = \big(\boldsymbol{u}(x, t), \boldsymbol{\varphi}(x, t)\big) - \big(\boldsymbol{u}(x, s), \boldsymbol{\varphi}(x, s)\big)$$

is satisfied

$$\forall (s, t) \in [0, +\infty) \text{ and } \forall \boldsymbol{\varphi} \in L^\infty(0, +\infty; H^1) \cap H^1(0, +\infty; H);$$

h$_4$)

$$\|\boldsymbol{u}(t)\|^2 + 2\int_s^t \|\nabla \boldsymbol{u}\|^2\, d\tau + 2Re\int_s^t (\boldsymbol{u}\cdot D, \boldsymbol{u})\, d\tau \leqslant \|\boldsymbol{u}(s)\|^2$$

for almost all $s \geqslant 0$ and with $t > s$.

The existence of a weak solution for every perturbation $\boldsymbol{u}_0 \in H(\Omega)$ has been proved by G. P. Galdi in [10].

DEFINITION 1.2: A field $\boldsymbol{u}$: $\Omega_T \to R^3$ is said be a *strong* (*or classical*) *solution* if and only if

a) $\boldsymbol{u} \in L^2(0, T; W_2^2 \cap H) \cap L^\infty(0, T; H)$, $\boldsymbol{u}_t \in L^2(0, T; L^2(\Omega))$;

b) $\boldsymbol{u} \in C([0, T); W_2^1(\Omega))$;

c) $\boldsymbol{u}$ satisfies system (1.17) almost everywhere.

THEOREM 1.1 (attractivity of steady motions). *Let* m_0 *be a steady motion and let* $\boldsymbol{w}$ *satisfy assumptions* i)-iv). *Moreover, let* $\boldsymbol{u}_0 \in H$ *and* $\boldsymbol{u}$ *be a weak solution corresponding to* $\boldsymbol{u}_0$. *Then there exists* $T_0 \geqslant 0$ *such that* $\boldsymbol{u}$ *becomes strong* $\forall t \geqslant T_0$. *Moreover, there exist constants* A, B, A_i *and* B_i (> 0) *such that for* $T \in (T_0, T_0 + A\|\boldsymbol{u}_0\|^2 + B)$ *the following estimates hold*:

$$\|\boldsymbol{u}_t(t)\|^2 \leqslant \frac{A_1}{(t - T + B_1)^2} \qquad t \geqslant T\,, \tag{1.19}$$

$$\|\nabla \boldsymbol{u}(t)\|^2 \leqslant \frac{A_2}{(t - T + B_2)} \qquad t \geqslant T\,, \tag{1.20}$$

$$\|P\Delta \boldsymbol{u}(t)\|^2 \leqslant \frac{A_3}{(t - T + B_3)} \qquad t \geqslant T\,, \tag{1.21}$$

$$(\sup_\Omega |\boldsymbol{u}(x, t)|)^2 \leqslant \frac{A_4}{(t - T + B_4)} \qquad t \geqslant T\,. \tag{1.22}$$

THEOREM 1.2 (attractivity of unsteady motions). *Let* m_0 *be an unsteady motion and let* $\boldsymbol{v}$ *satisfy assumptions* i)-iv). *Moreover, let* $\boldsymbol{u}_0 \in H$ *and* $\boldsymbol{u}$ *be a weak solution corresponding to* $\boldsymbol{u}_0$. *Then there exists* $T_0 \in [0, A'\|\boldsymbol{u}_0\|^2]$ (A' *is a constant*) *such that* $\boldsymbol{u}$ *becomes strong* $\forall t \geqslant T_0$. *Moreover, there exist some constants* A'_i (> 0) *such that for* $T \in (T_0, T_0 + 1]$ *the following estimates hold*:

$$\|\nabla \boldsymbol{u}(t)\|^2 \leqslant \frac{A'_1}{t + 1} \qquad t \geqslant T_0\,, \tag{1.23}$$

$$\|\boldsymbol{u}_t(t)\|^2 \leqslant \frac{A'_2}{t - T + 1} \qquad t \geqslant T\,, \tag{1.24}$$

$$\|PΔ\boldsymbol{u}(t)\|^2 \leqslant \frac{A_3'}{t-T+1} \qquad t \geqslant T\,, \tag{1.25}$$

$$(\sup_{\Omega} |\boldsymbol{u}(x,t)|)^2 \leqslant \frac{A_4'}{t-T+1} \qquad t \geqslant T\,. \tag{1.26}$$

REMARKS. (1) We notice that apart from «regularity» assumptions (*i.e.* behaviour at large spatial distances and large times) on the unperturbed motion, the theorems we give are based upon a variational formulation of the same kind of that introduced in [1-2] for bounded domains and in [11] for exterior domains. In this regards, we recall the importance of such a formulation both for applications [3, 21-22] and the connection between linear and non linear stability [3, 19, 23].

(2) It is important to stress that *if* m_0 *is steady and* $\nabla \boldsymbol{w} \in L^2(\Omega)$ *all the assumptions* i)-iv) *are automatically satisfied* with the only exception, of course, of condition (1.18). This can be easily checked by suitable coupling the results of [24], Corollary 2 and [13], Theorem 1. On the other hand it is well known that the class of solutions verifying the above hypothesis, (the so-called D-solutions) is certainly non void (cf., e.g. [25]). Therefore, *in the class of D-solutions condition* (1.18) *is sufficient for stability in the sense of Theorem* 1.1 (4). This theorem therefore, improves on analogous results proved in [5, 9, 11, 13]. Moreover, we should also notice that *the orders as* $t \to +\infty$ *derived in Theorem* 1.1 are better than those proved by the Authors of [9, 13].

(3) When the unperturbed motion m_0 is *unsteady* Theorem 1.2 should be compared with analogous results of K. Masuda [9] and G.P. Galdi [11]. As pointed out by the author to Professor K. Masuda, the hypothesis on $\boldsymbol{v}_t$ in Assumption 2′ on p. 298 of [9] is to be strenghtened to the following

$$t^{1/2}\left(\frac{\partial}{\partial t}\right)\boldsymbol{v}(x,t) \in L^\infty((0,+\infty);\, L^3(\Omega))\,.$$

Therefore, it is worth remarking that our assumptions on the unper-

(4) It is needless to say that both strong and weak solutions to (1.17) belong, for fixed t, to the class $\tilde{H}$. In fact, from known results [29], it follows $W_2^2 \cap H = W_2^2 \cap H^1 \subset H^1 \subset \tilde{H}$.

turbed motion do not require any specific behaviour as $t \to +\infty$ (only ii) is needed here). In any case, the order of decay proved in this paper is better than that given in [9]. As far as paper [11] is concerned, the assumptions there made on m_0 are better than ours, (in particular, no «infinitesimality» at large t is needed). However, results proved in [11] hold small initial data and no decay is given as $t \to +\infty$.

2. Asymptotic stability of steady flows.

2.1. *Existence theorems with smooth initial data.*

Under suitable hypothesis on the unperturbed motion m_0 (steady or not) we have the following (local in time) existence theorem for strong solutions.

THEOREM 2.1. *Let assumption* i) *be satisfied and let* $\boldsymbol{u}_0 \in H^1$, *then there exists one and only one classical solution* $(\boldsymbol{u}, \pi)$ *to the system* (1.17). *in* $\Omega \times [0, T^*)$:

$$a)\quad \boldsymbol{u} \in L^2(0, T^*; W_2^2 \cap H) \cap L^\infty(0, T^*; H) ,$$

$$\boldsymbol{u}_t \in L^2(0, T^*; H) , \qquad \nabla\pi \in L^2(0, T^*; L^2) ;$$

$$b)\quad \boldsymbol{u} \in C([0, T); W_2^1) , \qquad T < T^* .$$

PROOF. The first statement is a particular case of a theorem proved by V.A. Solonnikov (cf. [26], Theorem 10.1). The second is an immediate consequence of Lemma 1.8 when we notice that for $\theta = 1/2$ the following relation holds true (cf. [17])

$$[W_2^2, W_2^0]_{1/2} = W_2^1 .$$

We notice now that in order to obtain a global existence theorem from the preceding one, it is sufficient to prove that the solution $\boldsymbol{u}$ of the Theorem 2.1 verifies an uniform estimate of the type

$$(A = \text{cost} > 0) \ \ \|\boldsymbol{u}\|_{H^1} \leqslant A , \qquad \forall t \in [0, T^*] .$$

We shall show that this is possible provided $\|u_0\|_{H^1}$ is sufficiently small.

To this end, we propose some lemmas.

LEMMA 2.1 (cf. [27]). *If* $u \in C([0, T); H^1) \cap L^2(0, T; W_2^2 \cap H)$, $u_t \in L^2(0, T; H)$ *then*

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 = -(u_t, P\Delta u), \qquad \forall t \in [0, T) \text{ a.e.} . \tag{2.1}$$

LEMMA 2.2. *Let* (u, π) *be a solution to system* (1.17), *and let* iii) *be satisfied. Then*

$$\|u(t)\|^2 \leqslant \|u_0\|^2, \qquad \forall t \in [0, T^*) \tag{2.2}$$

and

$$\frac{R - Re}{R}\int_0^t \|\nabla u(s)\|^2\, ds \leqslant \frac{1}{2}\|u_0\|^2, \qquad \forall t \in [0, T^*), \tag{2.3}$$

where $1/R = \sup_{t \geqslant 0} 1/R(t)$.

PROOF. It can be easily seen that multiplying, in $L^2(\Omega)$ both sides of $(1.17)_1$ by u we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = -Re(u \cdot D, u). \tag{2.4}$$

Hence, integrating (2.4) over $[0, t]$, $\forall t < T^*$, and taking into account iii) we deduce both (2.2) and (2.3).

We are now in a position to prove the following global existence theorem:

THEOREM 2.2. *Let* i) *and* iii) *be satisfied. Set*

$$K_1 = 10Re^2 C_9 \|\nabla w\|_3^2 + 250 Re^4 C_{10} \|w\|_6^4 + 10 Re^2 C_{11} \|w\|_6^2,$$

$$K_2 = 10 R_e^2 C_8^1 \text{ and } K_3 = 250 R_e^4 C_7$$

and $K = \operatorname{Max}_i K_i$. *Suppose for some* $\delta > 0$

$$\|\nabla u_0\|^2 < \frac{\delta}{2}, \qquad \|u_0\|^2 < \frac{1}{K}\frac{\delta}{(1+\delta)^2}\frac{R - Re}{R},$$

then there exists a classical solution (u, π), $\forall T^* > 0$.

PROOF. Multiplying in $L^2(\Omega)$, both sipes in $(1.17)_1$ by $PΔu$ and taking into account (2.1) we have

$$(2.5) \qquad \frac{1d}{2dt}\|\nabla u\|^2 + \|P\Delta u\|^2 =$$
$$= Re((u\cdot\nabla u, P\Delta u) + (u\cdot\nabla w, P\Delta u) + (w\cdot\nabla u, P\Delta u)) .$$

According to Lemma 1.3 it is possible to increase (2.5) to obtain

$$(2.6) \qquad \frac{1d}{2dt}\|\nabla u\|^2 + (1 - 5\eta)\|P\Delta u\|^2 \leqslant C_7\frac{Re^4}{\eta^3}\|\nabla u\|^6 + C_8\frac{Re^2}{\eta}\|\nabla u\|^4 +$$
$$+ \left[C_9\frac{Re^2}{\eta}\|\nabla w\|_3^2 + C_{10}\frac{Re^4}{\eta^3}\|w\|_6^4 + C_{11}\frac{Re^2}{\eta}\|w\|_6^2\right]\|\nabla u\|^2 .$$

Choosing $\eta = 1/5$ into (2.6) and taking into account lemma (1.7) we prove the theorem.

REMARK. 1 We explicitly observe that the condition

$$\|u_0\|^2 < \frac{\delta}{(1+\delta)^2}\frac{1}{K}\frac{R - Re}{R}$$

can be omitted whenever

$$\int_0^{t^*}\|\nabla u(s)\|^2\,ds < \frac{1}{2}\frac{\delta}{(1+\delta)^2}\frac{1}{K}$$

Such a condition, for example, will be employed in the sequel for an analogous result (cf. Lemma 2.4).

We now want to prove that for every initial perturbation $u_0 \in H(\Omega)$ we can be determine a $T > 0$ such that the weak solution corresponding to u_0 becomes classical $\forall t \geqslant T$ for this kind of problem cf. also [9, 13-14].

To this end, we begin to recall the following uniqueness theorem essentially due to J. Sather and J. Serrin [28].

THEOREM 2.3. *Let* $\boldsymbol{u}$ *and* $\boldsymbol{u}'$ *be two weak solutions to system* (1.23). *Let* iii) *be satisfied and let* $\boldsymbol{u}' \in L^s(0, T; L^{s'}(\Omega))$ *with* s *and* s' *such that*

$$\frac{3}{s} + \frac{2}{s'} = 1 \quad \text{and} \quad 3 < s < +\infty .$$

If $\boldsymbol{u}_0 \equiv \boldsymbol{u}'_0$, *then* $\boldsymbol{u} \equiv \boldsymbol{u}'$.

From the Definition 1.1 of weak solution, it follows obviously, that $(0, +\infty)$ can be considered as the union of two disjoint open sets θ and θ' such that

$$\boldsymbol{u}(x, t) \in W_2^1 \cap H , \qquad \forall t \in \theta$$

θ' is a set Lebesgue measure zero.

Taking into account Theorems 2.1 and 2.3 is easy to deduce the following lemma.

LEMMA 2.3. *Let* $t_0 \in \theta$. *Setting* $\boldsymbol{u}_0 = \boldsymbol{u}(x, t_0)$, *then there exists an interval* $[t_0, t_0 + T)$ *where the weak solution becomes classical* (*i.e. enjoys properties* a) *and* b) *of Theorem* 2.1).

LEMMA 2.4. *Let* $\boldsymbol{u}$ *be a weak solution to system* (1.17). *Then there exists an instant* $T_0 \geqslant 0$ *such that* $\boldsymbol{u}$ *becomes a classical solution* $\forall t \geqslant T_0$.

PROOF. From the energy inequality we deduce the existence of $T_1 > 0$ such that for some $\delta > 0$

$$\int_{T_1}^{+\infty} \|\nabla \boldsymbol{u}(t)\|^2 \, dt < \frac{\delta}{2K(1+\delta)^2} . \tag{2.7}$$

Setting

$$T_2 = T_1 + \|\boldsymbol{u}_0\|^2 2 \left(\frac{R - Re}{R} \delta\right)^{-1} ,$$

there exists $T_0 \in [T_1, T_2] \cap \theta$ such that

$$\|\nabla \boldsymbol{u}(T_0)\|^2 < \frac{\delta}{2} . \tag{2.8}$$

In fact, if $\|\nabla u(t)\|^2 \geq \delta/2$, $\forall t \in [T_1, T_2] \cap \theta$, since $[T_1, T_2] \cap \theta'$ is still a set of Lebesgue measure zero, we have

$$\int_{T_1}^{T_2} \|\nabla u(t)\|^2 \, dt \geq \int_{T_1}^{T_2} \frac{\delta}{2} = \|u_0\|^2 \frac{R}{R - Re}$$

contradicting h_4).

Thus from (2.7) and (2.8) we have

$$\int_{T_0}^{+\infty} \|\nabla u(t)\|^2 \, dt < \frac{\delta}{2K(1+\delta)^2}, \qquad \|\nabla u(T_0)\|^2 < \frac{\delta}{2}.$$

Now, consider a solution having $u_0 = u(T_0) \in H^1$ as initial data. According to Theorem 2.2 such a solution exists $\forall t \geq T_0$, and the lemma follows as a consequence of Lemma 2.3.

In order to obtain the time estimate appearing in Theorem 1.1, we shall first construct suitable solutions to problem (1.17). To this end we may employ, for example, a variant of the usual Faedo-Galerkin method in the way suggested in [13] to which the reader is referred for details. Let $\{\Omega_k\}_{k \in N}$ be an increasing sequence of compact subdomains of R^3 invading Ω and let $\{a^n\}^{n \in N}$ and $\{\lambda_n\}_{n \in N}$ be eigenfunctions and eigenvalues respectively of the Stokes operator $-P\Delta$ in $H(\Omega_k)$, i.e.

$$P\Delta a^n = \lambda_n a^n \quad \text{with } a^n \in H(\Omega_k), \ \|a^n\|_{H(\Omega_k)} = 1, \qquad \forall n \in N.$$

We set

$$u^m(x, t) = \sum_{n=1}^{m} c_{nm}(t) \, a^n(x)$$

and require that the coefficients $c_{nm}(t)$ are solutions of the following (ordinary) initial value problem

$$\begin{cases} (u_t^m, a^n) - (\Delta u^m, a^n) = -Re[(u^m \cdot \nabla u^m, a^n) + \\ \qquad\qquad + (u^m \cdot \nabla w, a^n) + (w \cdot \nabla u^m, a^n)], & n = 1, 2, \ldots, m, \\ c_{nm}(0) = (u(T), a^n), & n = 1, 2, \ldots, m. \end{cases}$$

It is easy to check that $\{\boldsymbol{u}^m\}_{m\in N}$ verify the following relations

$$(2.9)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2+\|P\Delta \boldsymbol{u}\|^2= \\ = Re[(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, P\Delta\boldsymbol{u})+(\boldsymbol{u}\cdot\nabla\boldsymbol{w}, P\Delta\boldsymbol{u})+(\boldsymbol{w}\cdot\nabla\boldsymbol{u}, P\Delta\boldsymbol{u})]\,,$$

$$(2.10)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2+\|\boldsymbol{u}_t\|^2= \\ = -Re[(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{u}_t)+(\boldsymbol{u}\cdot\nabla\boldsymbol{w}, \boldsymbol{u}_t)+(\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{u}_t)]\,,$$

$$(2.11)\qquad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2+\|\nabla\boldsymbol{u}\|^2=-Re(\boldsymbol{u}\cdot\nabla\boldsymbol{w}, \boldsymbol{u})\,,$$

$$(2.12)\qquad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2+\|\nabla\boldsymbol{u}_t\|^2=-Re[(\boldsymbol{u}_t\cdot\nabla\boldsymbol{u}, \boldsymbol{u}_t)+(\boldsymbol{u}_t\cdot\nabla\boldsymbol{w}, \boldsymbol{u}_t)]\,.$$

In the above, for the sake of simplicity, the superscript m has been omitted. The right hand side of (2.9) can be increased by employing inequalities (1.5), thus obtaining

$$(2.13)\qquad \frac{1}{2}\frac{d}{ds}\|\nabla \boldsymbol{u}\|^2+(1-5\eta)\|P\Delta\boldsymbol{u}\|^2\leqslant C_7\frac{Re^4}{\eta^3}\|\nabla\boldsymbol{u}\|^6+C_8\frac{Re^2}{\eta}\|\nabla\boldsymbol{u}\|^4+ \\ +\left[C_9\frac{Re^2}{\eta}\|\nabla\boldsymbol{w}\|_3^2+C_{10}\frac{Re^4}{\eta^3}\|\boldsymbol{w}\|_6^4+C_{11}\frac{Re^2}{\eta}\|\boldsymbol{w}\|_6^2\right]\|\nabla\boldsymbol{u}\|^2\,.$$

On the other hand, since

$$\|\nabla\boldsymbol{u}^m(0)\|\leqslant\|\nabla\boldsymbol{u}(T)\|\,,\qquad \forall m\in N$$

and choosing $\eta=1/6$ in (2.13), by a well known lemma we deduce

$$(2.14)\qquad \|\nabla\boldsymbol{u}(s)\|^2\leqslant F(s)\ \text{ and } \int_0^s\|P\,\Delta\boldsymbol{u}(\tau)\|^2d\tau\leqslant\tilde{F}(s)\,,\qquad \forall s\in[0, T^*)$$

where $F(s)$ and $\tilde{F}(s)$ are two continuous functions in $[0, T^*)$ with $F(0)=\tilde{F}(0)=\|\nabla\boldsymbol{u}(T)\|^2$ and where T^* depends on $\|\nabla\boldsymbol{u}(T)\|$ and Re. From (1.1) we have

$$\|D^2\boldsymbol{u}(s)\|^2\leqslant C_0(\|P\Delta\boldsymbol{u}(s)\|^2+\|\nabla\boldsymbol{u}(s)\|^2)\,,$$

which along with (2.14) implies

$$\int_0^s \|D^2 \boldsymbol{u}(\tau)\|^2 \, d\tau \leqslant \overline{F}(s) , \qquad \forall s \in [0, T^*) , \tag{2.15}$$

where $\overline{F}(s)$ is a suitable continuous function of s. Let us now consider (2.10). Applying (1.2) we deduce

$$\|\boldsymbol{u}_s\|^2 < C_{17}\|\nabla \boldsymbol{u}\|^6 + C_{17}\|\nabla \boldsymbol{u}\|^4 + C_{17}\|\nabla \boldsymbol{u}\|^2 - \frac{1d}{2ds}\|\nabla \boldsymbol{u}\|^2 . \tag{2.16}$$

Taking into account

$$-\frac{1d}{2ds}\|\nabla \boldsymbol{u}\|^2 = (\boldsymbol{u}_s, P\Delta \boldsymbol{u}) \leqslant \varepsilon \|\boldsymbol{u}_s\|^2 + \frac{1}{4\varepsilon}\|P\Delta \boldsymbol{u}\|^2 ,$$

from (2.16) we also have

$$\int_0^s \|\boldsymbol{u}_\tau\|^2 \, d\tau \leqslant F(s) , \qquad \forall s \in [0, T^*) . \tag{2.17}$$

Since estimates (2.14)-(2.17) hold uniformly with respect to m we can select a subsequence which by standard argument can be proved to converge to a solution to (1.17) in $\Omega_k \times [0, T^*)$. To obtain a solution on the *whole* of Ω we can employ the methods suggested in [13] and since the proof is routine it will be omitted.

2.2. *Stability of steady motions: proof of Theorem* 1.1.

First of all we notice that owing to the uniqueness Theorem 2.3 the solution constructed by the Galerkin method and assuming $\boldsymbol{u}(x, T)$ as initial data coincides with the solution derived in Lemma 3.1 and assuming in T the value $\boldsymbol{u}(x, T)$ ([5]).

Therefore, we have

$$\boldsymbol{u}(x, T) \equiv \boldsymbol{U}(x, 0) \Rightarrow \boldsymbol{u}(x, t) \equiv \boldsymbol{U}(x, s) \qquad \forall t \geqslant T \text{ and } s \geqslant 0 \text{ such that } t = T + s ,$$

([5]) Obviously, also the coefficients of equation (1.17) which in this case depend on t must be evaluated for $t \geqslant T$.

where by $\boldsymbol{U}$ we have denoted the solution constructed by the Galerkin method. In this subsection, as usually done previously, we shall continue to denote by $\boldsymbol{u}$ the Galerkin approximations and their limit as well. Moreover we shall still adopt for the time variable the symbol t (instead of s).

We give some preliminary lemmas.

LEMMA 2.5. *Let $\boldsymbol{u}$ be a weak solution and let T_0 denote the instant of time after which the solution becomes strong. Setting:*

$$\delta_1 = \min\{m_1, m_2\}$$

with

$$m_1 = \frac{1}{\left(2\left(\frac{R}{R-Re}\right)^7 Re^4\|\boldsymbol{u}_0\|^3\left[ReC_{16}\|\boldsymbol{u}_0\|^{1/2}\left(\frac{2}{3}\frac{R-Re}{R\cdot Re}\right)^{3/4} + 2ReC_6\|\boldsymbol{w}\|_3+1\right]^2\exp\left(\frac{\|\boldsymbol{u}_0\|^2 ReR}{8R-Re}\right)\right)^2}$$

$$m_2 = \left(\frac{R-Re}{R\|\boldsymbol{u}_0\|}\left(\frac{2}{3}\frac{R-Re}{R\cdot Re}\right)^3\right)^2\left(\exp\left(-\frac{1}{4}\|\boldsymbol{u}_0\|^2\frac{R\cdot Re}{R-Re}\right)\right),$$

$$C_{18} = 648\cdot C_7 Re^4\delta^2\frac{R}{R-Re} + 18C_8Re^2\delta\frac{R}{R-Re} + \\ + 18C_9Re^2\|\nabla\boldsymbol{w}\|_3^2 + 648C_{10}Re^4\|\boldsymbol{w}\|_6^4 + 18C_{11}Re^2\|\boldsymbol{w}\|_6^2\frac{R}{R-Re},$$

$$C_{19} = 343C_1Re^4\delta^2 + 49C_2Re^2\delta + \\ + [49C_3Re^2\|\nabla\boldsymbol{w}\|_3^2 + 49C_4Re^2\|\boldsymbol{w}\|_6^2 + 343C_5Re^4\|\boldsymbol{w}\|_6^4],$$

there exists

$$T\in\left[T_0,\ T_0 + \frac{105}{8}C_{18}\|\boldsymbol{u}_0\|^2 + \frac{105}{4}\delta + \frac{C_{19}\delta}{2(1+\delta)^2K}\right]$$

such that

$$C_{19}\|\nabla\boldsymbol{u}(T)\|^2 + \frac{105}{4}\|P\Delta\boldsymbol{u}(T)\|^2 \leqslant \delta_1 .$$

PROOF. We commerce by considering inequality (2.6). Choosing $\eta = 1/6$ we deduce

$$\int_{T_0}^{+\infty}\|P\Delta\boldsymbol{u}\|^2\,dt \leqslant C_{18}\|\boldsymbol{u}_0\|^2 + 3\delta .$$

Then, taking into account (2.7) it follows

$$\int_{T_0}^{+\infty}\left(\frac{105}{4}\|PΔ\boldsymbol{u}\|^2 + C_{19}\|\nabla \boldsymbol{u}\|^2\right)dt \leq \tag{2.20}$$

$$\leq \frac{105}{4} C_{18}\|\boldsymbol{u}_0\|^2 + \frac{315}{4}\delta + \frac{C_{19}\delta}{2K(1+\delta)^2}.$$

We claim that putting

$$T_1 = T_0 + \left(\frac{105}{4} C_{18}\|\boldsymbol{u}_0\|^2 + \frac{315}{4}\delta + \frac{C_{19}\delta}{2K(1+\delta)^2}\right)\frac{2}{\delta_1}$$

there exists $T \in [T_0, T_1]$ such that

$$\frac{105}{4}\|PΔ\boldsymbol{u}(T)\|^2 + C_{19}\|\nabla \boldsymbol{u}(T)\|^2 .$$

In fact, assuming per absurdum that

$$\frac{105}{4}\|PΔ\boldsymbol{u}(t)\|^2 + C_{19}\|\nabla \boldsymbol{u}(t)\|^2 > \delta_1 , \qquad \forall t \in [T_0, T_1]$$

we deduce, in particular,

$$\int_{T_0}^{T_1}\left(\frac{105}{4}\|PΔ\boldsymbol{u}(t)\|^2 + C_{19}\|\nabla \boldsymbol{u}(t)\|^2\right)dt > \delta_1(T - T_0) =$$

$$= 2\left(\frac{105}{4} C_{18}\|\boldsymbol{u}_0\|^2 + \frac{315}{4}\delta + \frac{C_{19}\delta}{2K(1+\delta)^2}\right)$$

contradicting (2.20).

LEMMA 2.6 [6]. *Let $\boldsymbol{u}$ be a weak solution and T_0 be the instant of time after which the solution becomes strong. Denote by T $(\geq T_0)$ the instant of time such that*

$$C_{19}\|\nabla \boldsymbol{u}(T)\|^2 + \frac{105}{4}\|PΔ\boldsymbol{u}(T)\|^2 \leq \delta_1 .$$

[6] For the proof cf. also [11].

Then

$$(2.21)\qquad \begin{cases} \|\boldsymbol{u}_t(t)\|^2 \leqslant \delta_1 \exp\left(\dfrac{1}{4}\|\boldsymbol{u}_0\|^2 \dfrac{ReR}{R-Re}\right), & \forall t > T\,, \\ \|\nabla \boldsymbol{u}(t)\|^{2/3} \leqslant \dfrac{2}{3}\dfrac{R-Re}{RRe}\,, & \forall t \geqslant T\,. \end{cases}$$

PROOF. Let $\boldsymbol{u}^m$ be the m-th Galerkin approximation assuming in $t=0$ the data $\boldsymbol{u}(T)$. It is easily verified that

$$(2.22)\qquad C_{19}\|\nabla \boldsymbol{u}^m(0)\|^2 + \frac{105}{4}\|P\Delta \boldsymbol{u}^m(0)\|^2 \leqslant \delta_1\,.$$

Let us now consider relation (2.10). Since

$$(2.22)\qquad (\boldsymbol{u}_t, P\Delta \boldsymbol{u}) \leqslant \varepsilon\|\boldsymbol{u}_t\|^2 + \frac{1}{4\varepsilon}\|P\,\Delta \boldsymbol{u}\|^2\,,$$

increasing the right hand side of (2.10) with the aid of (1.2) we obtain

$$(2.23)\qquad (1-6\varepsilon)\|\boldsymbol{u}_t\|^2 \leqslant C_1 Re^4 \frac{1}{\varepsilon^2\eta}\|\nabla \boldsymbol{u}\|^6 + \frac{C_2 Re^2}{\varepsilon}\|\nabla \boldsymbol{u}\|^4 + \\ + \left[\frac{C_3 Re^2}{\varepsilon}\|\nabla \boldsymbol{w}\|_3^2 + \frac{C_4 Re^2}{\varepsilon}\|\boldsymbol{w}\|_6^2 + \frac{C_5 Re^4}{\varepsilon^2\eta}\|\boldsymbol{w}\|_6^4\right]\|\nabla \boldsymbol{u}\|^2 + \\ + \left(2\eta + \frac{1}{4\eta}\right)\|P\Delta \boldsymbol{u}\|^2.$$

Setting $\eta = 1$, $\varepsilon = 1/7$ and $t = 0$ in (2.23) from (2.22) we have

$$(2.24)\qquad \|\boldsymbol{u}_t^m(0)\|^2 \leqslant \delta_1\,.$$

Let us now consider relation (2.12) which for the reader sake we rewrite in the next line

$$(2.25)\qquad \frac{1d}{2dt}\|\boldsymbol{u}_t\|^2 + \|\nabla \boldsymbol{u}_t\|^2 = -Re[(\boldsymbol{u}_t\cdot\nabla \boldsymbol{u}, \boldsymbol{u}_t) + (\boldsymbol{u}_t\cdot\nabla \boldsymbol{w}, \boldsymbol{u}_t)]\,.$$

As usual, the superscript m has been dropped in (2.23)-(2.25). From

(1.6) and (2.25) we thus recover

$$(2.26)\qquad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2+\left(\frac{R-Re}{R}-\frac{3}{4}Re\,\|\nabla\boldsymbol{u}\|^{2/3}\right)\|\nabla\boldsymbol{u}_t\|^2\leqslant\frac{1}{4}Re\|\boldsymbol{u}_t\|^2\,\|\nabla\boldsymbol{u}\|^2 .$$

Moreover, from (2.11) it easily follows

$$(2.27)\qquad \|\nabla\boldsymbol{u}(t)\|^2\leqslant\|\boldsymbol{u}_0\|\,\|\boldsymbol{u}_t(t)\|\,\frac{R}{R-Re}$$

which we claim to imply

$$\|\nabla\boldsymbol{u}\|^{2/3}\leqslant\frac{2}{3}\,\frac{R-Re}{RRe} .$$

To prove this last assertion it suffices to notice that from (2.24) we obtain

$$\|\nabla\boldsymbol{u}(0)\|^{2/3}\leqslant\frac{2}{3}\,\frac{R-Re}{RRe} .$$

By continuity it is possible to determine a right neighborhood N_0 of $t=0$ such that

$$\frac{R-Re}{R}-\frac{3}{4}Re\|\nabla\boldsymbol{u}\|^{2/3}>0 .$$

Integrating (2.26) over N_0 we deduce

$$(2.29)\qquad \|\boldsymbol{u}_t(t)\|^2\leqslant\delta_1\exp\left(\frac{1}{4}\,\|\boldsymbol{u}_0\|^2\,\frac{ReR}{R-Re}\right),\qquad \forall t\in N_0 .$$

Relations (2.27), (2.29) and (2.18) imply

$$\|\nabla\boldsymbol{u}(t)\|^{2/3}\leqslant\frac{2}{3}\,\frac{R-Re}{RRe},\qquad \forall t\geqslant 0 ,$$

thus proving (2.28). Finally, once (2.28) has been obtained, (2.29) holds uniformly in $t\geqslant 0$. Therefore, the lemma is completely proved.

The following lemma, which is crucial to prove the time behaviour quoted in Theorem 1.1, shows the validity of a sort of Poincarè inequality holding for the L^2-norm of $\boldsymbol{u}_t$. In this connection it is worth remarking that the main difficulty for obtaining time decay for solutions in unbounded domains is conneted to the fact that in this case the Poincarè inequality *a priori* fails along solutions ([8, 4]). However, in the following lemma we shall prove that it is always possible to find a constant $\hat{K}$ (>0) depending on the data of the problem such that L^2-norm of $\boldsymbol{u}_t$ is bounded by $\hat{K}$ times the L^2-norm of $\nabla\boldsymbol{u}_t$ rised to a suitable power.

LEMMA 2.7. *Let* $\boldsymbol{u}$, T_0 *and* T *be as in Lemma* 2.6 *above. Then the following inequality holds*

$$\|\boldsymbol{u}_t(t)\| \leqslant \hat{K}\|\nabla\boldsymbol{u}_t(t)\|^{2/3}, \qquad \forall t \geqslant T \tag{2.30}$$

where

$$\hat{K} = \left\{\left[ReC_{16}\|\boldsymbol{u}_0\|^{1/2}\left(\frac{2}{3}\frac{R-Re}{RRe}\right)^{3/4} + 2Re\, C_6\|\boldsymbol{w}\|_3 + 1\right]^2 \|\boldsymbol{u}_0\|\frac{R}{R-Re}\right\}^{1/3}.$$

PROOF. Let us consider relation (2.10). Since

$$\left|\frac{1d}{2dt}\|\nabla\boldsymbol{u}\|^2\right| \leqslant \|\nabla\boldsymbol{u}\|\,\|\nabla\boldsymbol{u}_t\|$$

taking into account (1.10) we have

$$\|\boldsymbol{u}_t\|^2 \leqslant (ReC_{16}\|\boldsymbol{u}\|^{1/2}\|\nabla\boldsymbol{u}\|^{1/2} + 2EReC_6\|\boldsymbol{w}\|_3 + 1)\|\nabla\boldsymbol{u}\|\,\|\nabla\boldsymbol{u}_t\|. \tag{2.31}$$

Therefore, after a simple manipulation (2.30) follows from $(2.21)_2$ and (2.31).

From the above lemma we have

COROLLARY 2.1. *Let* $\boldsymbol{u}$, T *and* T_0 *be as in Lemma* 2.7. *Then*

$$\|\boldsymbol{u}_t(t)\|^2 \leqslant \frac{\delta_1}{(\hat{K}_1\delta_1^{1/2}(t-T)+1)^2}, \qquad \forall t \geqslant T, \tag{2.32}$$

$$\|\nabla\boldsymbol{u}(t)\|^2 \leqslant \frac{\|\boldsymbol{u}_0\|\delta_1^{1/2}}{\hat{K}_1\delta_1^{1/2}(t-T)+1}\frac{R}{R-Re}, \qquad \forall t \geqslant T. \tag{2.33}$$

PROOF. From $(1.6)_2$ and (2.7) we deduce

$$(2.34)\qquad |Re(\boldsymbol{u}_t\cdot\nabla\boldsymbol{u},\boldsymbol{u}_t)|\leqslant\frac{1}{4}Re^4\left(\frac{R}{R-Re}\right)^5\|\boldsymbol{u}_0\|^2\|\boldsymbol{u}_t\|^4+\frac{3}{4}\frac{R-Re}{R}\|\nabla\boldsymbol{u}_t\|^2.$$

Now, we increase the right hand side of (2.12) by (2.34) to obtain

$$(2.35)\qquad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2+\frac{1}{4}\frac{R-Re}{R}\|\nabla\boldsymbol{u}_t\|^2\leqslant\frac{1}{4}Re^4\left(\frac{R}{R-Re}\right)^5\|\boldsymbol{u}_t\|^4\|\boldsymbol{u}_0\|^2.$$

Since (2.30) implies

$$-\frac{1}{4}\frac{R-Re}{R}\|\nabla\boldsymbol{u}_t\|^2\leqslant-\frac{1}{4}\frac{R-Re}{R\hat{K}^3}\|\boldsymbol{u}_t\|^3$$

from (2.35) we deduce

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2\leqslant-\|\boldsymbol{u}_t\|^3\left(\frac{R-Re}{4R}\frac{1}{\hat{K}^3}-\frac{1}{4}Re^4\|\boldsymbol{u}_0\|^2\left(\frac{R}{R-Re}\right)^5\|\boldsymbol{u}_t\|\right),$$

and hence

$$(2.36)\qquad \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2\leqslant-\|\boldsymbol{u}_t\|^3\left[\left(\frac{R-Re}{4R}\frac{1}{\hat{K}^3}\right)-\frac{1}{4}Re^4\|\boldsymbol{u}_0\|^2\left(\frac{R}{R-Re}\right)^5\right]\cdot$$
$$\cdot\,\delta_1^{1/2}\left(\exp\left(\frac{1}{4}\|\boldsymbol{u}_0\|^2\frac{ReR}{R-Re}\right)\right)^{1/2}\leqslant-\|\boldsymbol{u}_t\|^3\left(\frac{R-Re}{8R}\frac{1}{\hat{K}^3}\right).$$

In the last inequality we have taken into account $(2.21)_1$. Setting in (2.36)

$$\hat{K}_1=\left(\frac{R-Re}{8R}\frac{1}{\hat{K}^3}\right)$$

we have

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_t\|^2\leqslant-\hat{K}_1u_t^3\Leftrightarrow\frac{d}{dt}\|\boldsymbol{u}_t\|\leqslant-\hat{K}_1\|\boldsymbol{u}_t\|^2,$$

from which (2.32) follows.

The preceding Corollary proves (1.19) and (1.20) of Theorem 1.1.

It remains to show (1.21) and (1.22). To this end, multiply both sides of $(1.17)_1$ by $P\Delta\boldsymbol{u}$ in $L^2(\Omega)$ to obtain

$$\|P\Delta\boldsymbol{u}\|^2 \leqslant Re[(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, P\Delta\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla\boldsymbol{w}, P\Delta\boldsymbol{u}) + (\boldsymbol{w}\cdot\nabla\boldsymbol{u}, P\Delta\boldsymbol{u})] + (\boldsymbol{u}_t, P\Delta\boldsymbol{u}).$$

Increasing the right hand side in the above through (1.5) and taking into account

$$|(\boldsymbol{u}_t, P\Delta\boldsymbol{u})| \leqslant \frac{1}{4\eta}\|\boldsymbol{u}_t\|^2 + \eta\|P\Delta\boldsymbol{u}\|^2$$

we obtain

$$(1-6\eta)\|P\Delta\boldsymbol{u}\|^2 \leqslant C_7 Re^4 \frac{1}{\eta^3}\|\nabla\boldsymbol{u}\|^6 + C_8\frac{Re^2}{\eta}\|\nabla\boldsymbol{u}\|^4 + \left[\frac{C_9 Re^2}{\eta}\|\nabla\boldsymbol{w}\|_3^2 + C_{10} Re^4\frac{1}{\eta^3}\|\boldsymbol{w}\|_6^4 + C_{11} Re^2\frac{1}{\eta}\|\boldsymbol{w}\|_6^2\right]\|\nabla\boldsymbol{u}\|^2 + \frac{1}{4\eta}\|\boldsymbol{u}_t\|^2.$$

Choosing $\eta = 1/7$ in the last relation proves (1.22). To prove (1.22) we recall the following inequality [17]:

$$\sup_{\Omega}|\varphi(x)| \leqslant C_{20}\|\varphi\|_{m,p} \qquad \text{with } mp > 3. \tag{2.37}$$

Choosing $m = 1$ and $p = 6$ (2.37) and $(1.1)_1$ we thus deduce

$$\sup_{\Omega}|\boldsymbol{u}(x,t)| \leqslant C_{21}\|\nabla\boldsymbol{u}(t)\| + C_{22}\|P\Delta\boldsymbol{u}(t)\|$$

which shows (1.22).

3. Asymptotic stability of unsteady flows.

3.1. *Existence theorems with smooth initial data.*

In the case when the unperturbed motion m_0 is not steady, the results established in section 2.1 continue to hold as in the steady case. Moreover, under the assumptions made, these results can be further improved. In fact, it is possible to determine explicitly the value of T_0 and to determine for the solution of Theorem 2.2 an asymptotic behaviour of $\boldsymbol{u}$ in the Dirichlet norm. The above results are proved in the next lemma.

LEMMA 3.1. *Let* $\boldsymbol{u}$ *be a weak solution. Then there exists* $T_0 \in \in (0, \|\boldsymbol{u}_0\|^2(R/(R-Re))\,\varepsilon^{-1}]$ *such that*

$$\|\nabla \boldsymbol{u}(T_0)\|^2 \leqslant \varepsilon$$

where

$$\varepsilon = \frac{1}{2}\,h\left(\exp\left(2M_6 + (C_{23} + C_{24}h)\|\boldsymbol{u}_0\|^2 \frac{R}{R-Re}\right)\right)^{-1}$$

with M_6, C_{23} *and* C_{24} *such that*

$$\int_0^{+\infty} (10C_9R^2\|\nabla \boldsymbol{v}(t)\|_3^2 + 250C_{10}Re^4\|\boldsymbol{v}(t)\|_6^4 + 10C_{11}Re^2\|\boldsymbol{v}(t)\|_6^2)\,dt \leqslant M_6\,,$$

$$C_{23} = 10C_8Re^2 \qquad \textit{and} \qquad C_{24} = 250C_7Re^4$$

and h *is any positive number. Moreover,*

$$\|\nabla \boldsymbol{u}(t)\|^2 \leqslant h\,, \qquad \forall t \geqslant T_0\,,$$

$$(3.1)\qquad \|\nabla \boldsymbol{u}(t)\|^2 \leqslant \Big[C_{23}(\|\nabla \boldsymbol{u}(T_0)\|^2(T_0+1)+1)\cdot \cdot\exp\left(M_6 + (C_{23}+hC_{24})\|\boldsymbol{u}_0\|^2\frac{R}{R-Re}\right)^{-1}\Big]\cdot[C_{23}(t+1)]^{-1}\,, \qquad \forall t \geqslant T_0\,.$$

PROOF. Starting from

$$2\int_0^{+\infty} \|\nabla \boldsymbol{u}(t)\|^2\,dt \leqslant \|\boldsymbol{u}_0\|^2\frac{R}{R-Re}$$

by employing an argument previously used several times, we determine

$$T_0\,\varepsilon\left[0, \frac{R\varepsilon^{-1}}{R-Re}\|\boldsymbol{u}_0\|^2\right] \cap \theta$$

such that

$$\|\nabla \boldsymbol{u}(T_0)\|^2 \leqslant \varepsilon .$$

On the other hand, according to Lemma 2.3, the weak solution becomes a strong solution in $[T_0, T_0 + T)$ for some $T > 0$. However, in this time interval the solution is uniformly bounded. In fact, from (2.6) we deduce

$$(3.2) \qquad \frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}\|^2 + (1 - 5\eta)\|P\Delta \boldsymbol{u}\|^2 \leqslant \frac{C_7 Re^4}{\eta^3}\|\nabla \boldsymbol{u}\|^6 + \frac{C_8 Re^2}{\eta}\|\nabla \boldsymbol{u}\|^4 + \\ + \left[\frac{C_9 Re^2}{\eta}\|\nabla \boldsymbol{v}(t)\|_3^2 + \frac{C_{10} Re^4}{\eta^3}\|\boldsymbol{v}(t)\|_6^4 + \frac{C_{11} Re^2}{\eta}\|\boldsymbol{v}(t)\|_6^2\right]\|\nabla \boldsymbol{u}\|^2 .$$

Choosing $\eta = 1/5$ in (3.2), by an obvious meaning of the symbols from (3.2) we deduce

$$(3.3) \qquad \varphi'(t) \leqslant b(t)\varphi(t) = C_{23}\varphi^2(t) + C_{24}\varphi^3(t)$$

where C_{23} and C_{24} denote the coefficients of $\|\nabla \boldsymbol{u}\|^4$ and $\|\nabla \boldsymbol{u}\|^6$ respectively. Since $\varphi_0 < \varepsilon < h/2$ there exists a right neighbourhood N_0 of T_0 such that

$$\varphi'(t) \leqslant b(t)\varphi(t) + (C_{23} + hC_{24})\varphi^2(t) ,$$

Integrating this relation we obtain

$$(3.4) \qquad \varphi(t) \leqslant \varphi_0 \exp\left(2M_6 + (C_{23} + hC_{24})\|\boldsymbol{u}_0\|^2 \frac{R}{R - Re}\right) < h , \\ \forall t \in [T_0, T_0 + T] .$$

Therefore, $\varphi(t)$ is defined and bounded $\forall t \geqslant T_0$. Furthermore, applying the second part of Lemma 1.7, we deduce (3.1).

REMARK. We notice that to obtain a global existence theorem analogous to Theorem 2.2, we may reduce the assumptions on initial data to a condition connecting the Dirichlet norm of $\boldsymbol{u}$ with its L^2-norm. In fact, it is enough to assume

$$\|\nabla \boldsymbol{u}_0\|^2 \leqslant \frac{1}{2} h\left(\exp\left(2M_6 + (C_{23} + hC_{24})\|\boldsymbol{u}_0\|^2 \frac{R}{R - Re}\right)\right)^{-1} .$$

In this case the proof is quite analogous to that of previous lemma and therefore, it will be omitted.

3.2. *Stability of unsteady motions: proof of Theorem* 1.2.

First of all we notice that the Galerkin approximations assuming $\boldsymbol{u}(x, T)$ as initial data $\forall T \geqslant T_0$ verify the estimates proved in Lemma 3.1

$$\|\boldsymbol{U}(s)\|^2 \leqslant h\,, \qquad \forall s \geqslant 0\,. \tag{3.5}$$

Furthermore, the following inequality holds true $\forall T \geqslant T_0$

$$\|\nabla \boldsymbol{U}(s)\|^2 \leqslant \left[C_{23}(\|\nabla \boldsymbol{u}(T)\|^2 + 1)\cdot \right. \\ \left. \cdot \exp\left(2M_6 + (C_{23} + hC_{24})\|\boldsymbol{u}_0\|^2 \frac{R}{R - Re}\right)^{-1}\right] \cdot [C_{23}(s+1)]^{-1}\,, \qquad \forall s \geqslant 0\,. \tag{3.6}$$

Then (3.1) (or equivalently (3.6)) proves (1.23) of Theorem 1.2. In order to show (1.24) we propose the following lemma.

LEMMA 3.2. *Let $\boldsymbol{u}$ be a weak solution and T_0 the instant after which $\boldsymbol{u}$ becomes strong. Then, there exists $T \in [T_0, T_0 + 1]$ such that*

$$\|\boldsymbol{u}(T)\|^2 + \|P\Delta \boldsymbol{u}(T)\|^2 \leqslant 4h + C_{25} \tag{3.7}$$

with

$$C_{25} = \|\boldsymbol{u}_0\|^2 \frac{R}{R - Re}[648C_7 Re^4 h^2 + 18C_8 Re^2 h + 18C_9 Re^2 M_2^2 + \\ + 648C_{10} Re^4 M_3^4 + 18C_{11} Re^2 M_3^2]\,.$$

PROOF. Choosing in (3.2) $\eta = 1/6$ we deduce

$$\int_{T_0}^{+\infty} \|P\Delta \boldsymbol{u}(t)\|^2\, dt \leqslant C_{25} + 3h\,. \tag{3.8}$$

Theorefore, there exists $T \in [T_0, T_0 + 1]$ such that

$$\|P\Delta \boldsymbol{u}(T)\|^2 \leqslant C_{25} + 3h\,. \tag{3.9}$$

In fact, assuming *per absurdum*

$$\|P\Delta u(t)\|^2 > C_{25} + 3h\,, \qquad \forall t\in[T_0, T_0+1]\,,$$

it would follow

$$\int_{T_0}^{T_0+1} \|P\Delta u(t)\|^2\,dt > C_{25} + 3h$$

which contradicts (3.8). Thus, relations (3.5) and (3.9) imply (3.7).

For the proof of the next lemma cf. also [11].

LEMMA 3.3. *Let u, T_0 and T be as in Lemma 3.2. Then there exists a constant A_2' such that*

$$\|u_t(t)\|^2 \leqslant \frac{A_2'}{(t-T+1)}\,, \qquad \forall t \geqslant T\,. \tag{3.10}$$

PROOF. Let us consider equation (2.10) for the m-th Galerkin approximation U^m and relation (2.21) which in the case of unsteady unperturbed motions becomes (the superscript m is dropped for the sake of simplicity)

$$\frac{1d}{2ds}\|U_s\|^2 + \|\nabla U_s\|^2 = \tag{3.11}$$

$$= -Re[(U_s\cdot\nabla U, U_s) + (v_s\cdot\nabla U, U_s) + (U\cdot\nabla v_s, U_s) + (U_s\cdot\nabla v, U_s)]\ (^7)\,.$$

Taking into account (1.8) and the following obvious inequality

$$\frac{1d}{2ds}\|\nabla U\|^2 \leqslant \frac{1}{2}\|\nabla U\|^2 + \frac{1}{2}\|\nabla U_5\|^2\,,$$

from (2.10) we obtain

$$\|U_s\|^2 \leqslant \frac{3}{2}\eta_1\|U_s\|^2 + \left(2\eta_1 + \frac{1}{2}\right)\|\nabla U_s\|^2 + $$
$$+\left(\frac{1}{2} + \frac{C_{13}Re^2}{\eta_1}\|U\|_4^2 + \frac{C_{14}Re^2}{\eta_1}\|\nabla v(t)\|_3^2 + \frac{C_{15}Re^2}{\eta_1}\|v(t)\|_6^2\right)\|\nabla U\|^2\,.$$

(^{7}) The coefficients are evaluated for $t \geqslant T$.

Choosing $\eta_1 = 1/3$ and noticing that

$$\|U\|_4^2 \leqslant 2\|u_0\|^{1/2} h^{3/4}$$

we have

$$\|U_s\|^2 \leqslant \frac{7}{3}\|\nabla U_s\|^2 + (1 + 12C_{13}Re^2\|u_0\|^{1/2}h^{3/4} + \\ + 6C_{14}Re^2 M_3^2 + 6C_{15}Re^2 M_2^2)\|\nabla U\|^2 .$$

Therefore, we also deduce •

$$(3.12) \qquad -\|\nabla U_s\|^2 \leqslant -\frac{3}{7}\|U_s\|^2 + \\ + \frac{3}{7}(1 + 12C_{13}Re^2\|u_0\|^{1/2}h^{3/4} + 6C_{14}Re^2 M_3^2 + 6C_{15}Re^2 M_2^2)\|\nabla U\|^2 .$$

Let us now consider equation (3.11) and increase its right hand side through (1.6) to obtain

$$\frac{1d}{2ds}\|U_s\|^2 + \frac{R - Re}{R}\|\nabla U_s\|^2 \leqslant \\ \leqslant \frac{4Re^4}{\varepsilon_1^3}\|U_s\|^2\|\nabla U\|^4 + \frac{11}{4}\varepsilon_1\|\nabla U_s\|^2 + \frac{M_5^2}{\varepsilon_1}(C_{12} + C_{12})Re^2\|\nabla U\|^2 .$$

Choosing $\varepsilon_1 = (2/11)((R - Re)/R)$ we thus obtain with a suitable choice of constants C_{26}, C_{27}

$$(3.13) \qquad \frac{1d}{2ds}\|U_s\|^2 + \frac{1}{2}\frac{R - Re}{R}\|\nabla U_s\|^2 \leqslant (C_{26}\|U_s\|^2 + C_{27})\|\nabla U\|^2$$

which implies

$$(3.14) \qquad \|U_s(s)\|^2 \leqslant \left[\|U_s(0)\|^2 + 2C_{27}\int_0^s \|\nabla U(\tau)\|^2 \cdot \right. \\ \left. \cdot \exp\left(-2C_{26}\int_0^\tau \|\nabla U(\tau')\|^2 d\tau'\right) d\tau\right] \exp\left(2C_{26}\int_0^s \|\nabla U(\tau)\|^2 d\tau\right) \leqslant \\ \leqslant \left[\|U_s(0)\|^2 + C_{27}\|u_0\|^2 \frac{R}{R - Re}\right] \exp\left(C_{26}\|u_0\|^2 \frac{R}{R - Re}\right), \qquad \forall s \geqslant 0 .$$

With an obvious meaning of symbols, inequalities (3.12)-(3.14) yield

$$\varphi'(s) + C_{28}\varphi(s) \leqslant C_{29}\|\nabla \boldsymbol{U}\|^2 . \tag{3.15}$$

Integrating this last differential inequality, we deduce that for the m-th Galerkin approximation, the following inequality holds

$$\|\boldsymbol{U}_s(s)\|^2 \leqslant \|\boldsymbol{U}_s(0)\|^2 \exp(-C_{28}s) + C_{29}\exp(-C_{28}s)\int_0^s \exp(C_{28}\tau)\|\nabla \boldsymbol{U}(\tau)\|^2 d\tau , \qquad \forall s \geqslant 0 . \tag{3.16}$$

We now prove that (3.10) is a consequence of (3.16). In fact, from (3.6) we have

$$\|\nabla \boldsymbol{U}(s)\|^2 \leqslant \frac{K_0}{(s+1)} \ (^8)$$

where K_0 denotes the coefficient on the right hand side of (3.6). Therefore, from (3.16) we have

$$\|\boldsymbol{U}_s(s)\|^2 \leqslant \|\boldsymbol{U}_s(0)\|^2 \exp(-C_{28}s) + K_0 C_{29}\exp(-C_{28}s)\int_0^s \exp(C_{28}\tau)(\tau+1)^{-1} d\tau , \qquad \forall s \geqslant 0 . \tag{3.17}$$

Since

$$\int_0^s \exp(C_{28}\tau)(\tau+1)^{-1} d\tau = \frac{1}{C_{28}}\exp(C_{28}\tau)(\tau+1)^{-1}\Big|_0^s + \frac{1}{C_{28}}\int_0^{\tau'} \exp(C_{28}\tau)(\tau+1)^{-2} d\tau + \frac{1}{C_{28}}\int_{\tau'}^s \exp(C_{28}\tau)(\tau+1)^{-2} d\tau ,$$

setting

$$\tau' = \begin{cases} 0 & \text{if } C_{28} \geqslant 2 , \\ \dfrac{2-C_{28}}{C_{28}} & \text{if } C_{28} < 2 , \end{cases}$$

(8) The estimate $\|\nabla \boldsymbol{U}(s)\|^2 \leqslant K_0/(s+1)$ is evaluated for $t \geqslant T$, i.e.

$$\boldsymbol{u}(x, T) \equiv \boldsymbol{U}(x, 0) .$$

it follows

$$\int_0^s \exp(C_{28}\tau)(\tau+1)^{-1}\,d\tau < \frac{2}{C_{28}}\left[\exp(C_{28}s)(s+1)^{-1} + \right.$$

$$\left. + \int_0^{\tau'} \exp(C_{28}\tau)(\tau+1)^{-1}\,d\tau\right], \qquad \forall s \geqslant 0 .$$

Hence

$$\|U_s(s)\|^2 \leqslant \|U_s(0)\|^2 \exp(-C_{28}s) + \frac{2}{C_{28}}\frac{K_0}{(s+1)} +$$

$$+ \frac{2K_0}{C_{28}} C_{29} \exp(-C_{26}s) \int_0^{\tau'} \exp(C_{28}\tau)(\tau+1)^{-2}\,d\tau , \qquad \forall s \geqslant 0 .$$

This last relation along with

$$\|U_s(0)\|^2 = \|u_t(T)\|^2 \leqslant (4h + C) \quad \text{and} \quad U_s(x,s) \doteq u_t(x,t),$$

$$\forall s \geqslant 0 \text{ and } t \geqslant T \text{ such that } t = T + s ,$$

imply (3.10).

The previous lemma proves (1.24). To complete the proof of Theorem 1.2 it suffices to proceed exactly as we did for the proof of the stability in the steady case.

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