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Dirichlet Problem for a Linear Elliptic Equation in Unbounded Domains with L^2-Boundary Data.

J. CHABROWSKI

1. Introduction.

The main purposes of this paper are to investigate the Dirichlet problem for the elliptic equation

$$(1) \qquad Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) + \sum_{i=1}^{n} b_i(x)D_i u + c(x)u = f(x)$$

in a half-space and a complement of a bounded open set. We shall refer to the second problem as the exterior Dirichlet problem.

Given an open set $\Omega \subset R_n$ we denote by $W^{1,2}(\Omega)$ the Banach space of functions u in $L^2(\Omega)$ having weak (distributional) derivatives $D_i u$ $(i = 1, ..., n)$ in $L^2(\Omega)$. A norm is introduced by defining

$$\|u\|^2_{W^{1,2}(\Omega)} = \int_\Omega u(x)^2\,dx + \int_\Omega |Du(x)|^2\,dx\,.$$

The closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$ is denoted by $\mathring{W}^{1,2}(\Omega)$. A local space $W^{1,2}_{\text{loc}}(\Omega)$ consists of functions belonging to $W^{1,2}(\Omega')$ for every bounded open set Ω' such that $\bar{\Omega}' \subset \Omega$.

To motivate our approach to the Dirichlet problem assume for simplicity that L is uniformly elliptic and the coefficients a_{ij}, b_i, c

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and f are measurable and bounded on Ω. A function u is said to be a weak solution of the equation (1) if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and u satisfies

$$(2) \qquad \int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x) D_i u D_j v + \sum_{i=1}^n b_i(x) D_i u \cdot v + c(x) uv\right] dx = \int_\Omega f(x) v\, dx$$

for every $v \in W^{1,2}(\Omega)$ with compact support in Ω. Let $\varphi \in L^2(\partial\Omega)$ and assume that there is a function $\varphi_1 \in W^{1,2}(\Omega)$ such that $\varphi = \varphi_1$ on $\partial\Omega$ in the sense of trace. A weak solution in $W^{1,2}(\Omega)$ of the equation (1) is a solution of the Dirichlet problem with the boundary condition $u = \varphi$ on $\partial\Omega$ if $u - \varphi_1 \in W^{1,2}(\Omega)$. The basic results concerning the Dirichlet problem in $W^{1,2}$-framework can be found in Ladyzhenskaja and Ural'ceva [16], Gilbarg and Trudinger [10] and Stampacchia [24], [25]. In the above definition it is assumed that the boundary data φ is a trace of some function belonging to $W^{1,2}(\Omega)$. This condition is rather restrictive, because not every function in $L^2(\partial\Omega)$ is the trace of some function belonging to $W^{1,2}(\Omega)$ (see Lions and Magenes [17] Theorems 7.5 and 9.4 Chapter 1). It is clear that the Dirichlet problem with L^2-boundary data requires a new definition. The first attempt to define the Dirichlet problem with L^2-boundary data has been made by Mikhailov who in a series of articles [11], [18], [19] and [20] examined this problem in a bounded domain under the assumption $a_{ij} \in C^1(\bar{\Omega})$ and b_i, $c \in C(\bar{\Omega})$ (see also Nečas [22] and [23]). Similar results were also obtained by Kapanadze [15]. The author and Thompson [5] extended Mikhailov's results to the equation with coefficients satisfying some general integrability conditions. In the articles mentioned above the boundary $\partial\Omega$ belongs to C^2. For the Laplace equation the Dirichlet problem with L^2-boundary data was solved in bounded Lipschitz domains (see Dahlberg [8], Jerrison and Kenig [12], [13]). We mention here that Jerrison and Kenig also extended their results to bounded non-smooth domains for an equation with C^∞-coefficients (see [14]).

The plan of this paper is as follows. In sections 1-6 we examine the Dirichlet problem in a half space. The main result is an energy estimate (section 3, the inequality (19)) for the equation $Lu + \lambda u = f$, where λ is a sufficiently large parameter. Next applying the result of Bottaro and Marina [3] we establish the existence of a unique solution to the Dirichlet problem with L^2-boundary data for the equation (1) with the condition $c(x) \geqslant \mathrm{Const} > 0$. In sections 7, 8 and 9 similar results are established for the exterior Dirichlet problem. Methods used in both cases are similar and presented in some details. We point out that the assumptions on the coefficients a_{ij} for the Dirichlet problem in the half space are considerably weaker than the corresponding assumptions for the exterior Dirichlet problem. This follows from the fact that the boundary of the half space is flat. Among the papers devoted to study the boundary value problems in unbounded domains we mention the work of Benci and Fortunato [2], in which the Dirichlet problem with zero boundary data in a weighted Sobolev space has been studied.

In this paper we make frequent use of the Sobolev inequality

$$\|u\|_{L^{2^*}(\Omega)} \leqslant S\|Du\|_{L^2(\Omega)} \quad \text{for all } u \in \overset{\circ}{W}{}^{1,2}(\Omega)\,,$$

where

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n} \quad \text{and} \quad S = \frac{2(n-1)}{n(n-2)\sqrt{\pi}}\left(\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right)\right)^{\frac{1}{n}}.$$

The most significant feature of this inequality lies in the independence of the constant S of the domain Ω which makes possible to use it in unbounded domains (see Federer [9]).

2. Traces.

Let $R_n^+ = \{x;\ x \in R_n,\, x_n > 0\}$. We denote a point $x \in R_n^+$ by $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in R_{n-1}$.

Throughout sections 2-6 we make the following assumptions about the operator L:

(A) L is uniformly elliptic in R_n^+, i.e., there exists a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j$$

for all $x \in R_n^+$ and $\xi \in R_n$, moreover $a_{ij} \in L^\infty(R_n^+)$ $(i,j = 1, \ldots, n)$.

(B) (i) There exist positive constants $\varkappa$ and $0 < \alpha < 1$ such that

$$|a_{nn}(x', x_n) - a_{nn}(x', \bar{x}_n)| \leqslant \varkappa |x_n - \bar{x}_n|^\alpha$$

for all $x' \in R_{n-1}$ and all x_n, $\bar{x}_n \in [0, \infty)$.

(ii) $a_{in} \in C^1(R_n^+)$ and $|D_k a_{in}(x)| \leqslant \varkappa_1 x_n^{-\beta}$ for all $x' \in R_{n-1}$ and $x_n \in (0, b]$, where $\varkappa_1$, b and β are positive constants, $0 \leqslant \beta < 1$, and moreover $D_k a_{in} \in L^\infty(R_{n-1} \times [b, \infty))$ $(i, k = 1, \ldots, n,\ i \neq n)$.

(iii) $b_i \in L^\infty(R_n^+)$ $(i = 1, \ldots, n)$ and $c \in L^\infty(R_n^+) + L^n(R_n^+)$.

(C) $\int_{R_n^+} f(x)^2 \min(1, x_n)\, dx < \infty$.

All constants in this paper will be denoted by C_i. The statement « C_i depends on the structure of the operator L » means that C_i depends on n, γ, β, α, b, k, $\varkappa_1$ and the norms of the coefficients Da_{ij}, a_{ij}, b_i and c in appropriate spaces.

Let $\tilde{W}^{1,2}_{\text{loc}}(R_n^+) = \{u;\ u \in W^{1,2}_{\text{loc}}(R_n^+)$ and $\int_T^\infty dx_n \int_{R_{n-1}} u(x)^2\, dx' < \infty$ for every $T > 0\}$.

In the sequel we shall need the following elementary lemmas.

Lemma 1. *If* $u \in L^2(R_n^+)$ *and* $\sup_{0 < x_n \leqslant T} \int_{R_{n-1}} u(x', x_n)^2\, dx' < \infty$ *for certain* $T > 0$, *then* $\int_{R_n^+} [\min(x_n, 1)]^{-\beta} u(x)^2\, dx < \infty$ *for every* $0 \leqslant \beta < 1$.

Lemma 2. *If* $u \in \tilde{W}^{1,2}_{\text{loc}}(R_n^+)$ *and* $\int_{R_n^+} \min(x_n, 1)\, |Du|^2\, dx < \infty$ *then for all* $T > 0$ *and* $0 \leqslant \gamma < 1$

$$\int_\delta^T dx_n \int_{R_{n-1}} \frac{u(x)^2}{(x_n - \delta)^\gamma}\, dx'$$

is bounded independently of $\delta \in (0, T/2]$.

Proof. Integrating by parts

$$\int_\delta^T (x_n - \delta)^{-\gamma} dx_n \int_{R_{n-1}} u(x)^2\, dx' = \frac{(T - \delta)^{1-\gamma}}{1 - \gamma} \int_{R_{n-1}} u(x', T)^2\, dx' -$$

$$- 2 \int_\delta^T dx_n \int_{R_{n-1}} \frac{(x_n - \delta)^{1-\gamma}}{1 - \gamma} D_n u(x) \cdot u(x)\, dx'.$$

Denoting the last integral by J and applying Young's inequality we obtain

$$|J|\leq\frac{2\varepsilon}{1-\gamma}\int_{\delta}^{T}\int_{R_{n-1}}(x_n-\delta)^{-\gamma}u(x)^2\,dx+\frac{2}{\varepsilon(1-\gamma)}\int_{\delta}^{T}\int_{R_{n-1}}(x_n-\delta)^{2-\gamma}|D_nu|^2\,dx\leq$$

$$\leq\frac{2\varepsilon}{1-\gamma}\int_{\delta}^{T}\int_{R_{n-1}}(x_n-\delta)^{-\gamma}u(x)^2\,dx+\frac{2T^{1-\gamma}}{\varepsilon(1-\gamma)}\int_{\delta}^{T}\int_{R_{n-1}}x_n|D_nu|^2\,dx\,.$$

Taking $\varepsilon=(1-\gamma)/4$ the result follows.

We also need the following simple observation.

LEMMA 3. *Let $u\in \tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$ be a solution of* (1). *Then for every $r>0$*

$$\int_{2r}^{\infty}\int_{R_{n-1}}|Du(x)^2\,dx\leq C\left[\int_{r}^{\infty}\int_{R_{n-1}}u(x)^2\,dx+\int_{r}^{\infty}\int_{R_{n-1}}f(x)^2\,dx\right],\tag{3}$$

where a positive constant C depends on the structure of the operator L and r.

PROOF. Let $v=u\Phi^2$, where $\Phi\in C_0^\infty(R_n^+)$. Using v as a test function in (2) we obtain

$$\int_{R_n^+}\sum_{i,j=1}^{n}a_{ij}D_iuD_ju\Phi^2\,dx+2\int_{R_n^+}\sum_{i,j=1}^{n}a_{ij}D_iu\cdot uD_j\Phi\cdot\Phi\,dx+$$

$$+\int_{R_n^+}\sum_{i=1}^{n}b_iD_iu\cdot u\Phi^2\,dx+\int_{R_n^+}cu^2\Phi^2\,dx=\int_{R_n^+}fu\Phi^2\,dx\,.$$

It follows from ellipticity of L and the inequalities of Young and Sobolev that

$$\int_{R_n^+}|Du|^2\Phi^2\,dx\leq C\Big[\int_{R_n^+}u^2(\Phi^2+|D\Phi|^2)\,dx+\int_{R_n^+}f^2\Phi^2\,dx\Big]\,,$$

where a positive constant C depends on the structure of the operator L. Here we have used the fact that $c=c_1+c_2$ with $c_1\in L^n$ and $c_2\in L^\infty$. To complete the proof put $\Phi=\Phi_\nu$, where Φ_ν is an increasing sequence of non-negative functions in $C_0^\infty(R_n^+)$ with the gradient bounded independently of ν and converging as $\nu \to \infty$ to a non-negative function Ψ on R_n^+ equal to 1 for $x_n \geqslant 2r$ and vanishing for $x_n \leqslant r$.

THEOREM 1. *Let* $u \in \tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$ *be a solution of* (1) *in* R_n^+. *Then the following conditions are equivalent*:

$$\text{(I) } \textit{there exists } T \textit{ such that } \sup_{0<x_n<T} \int_{R_{n-1}} u(x', x_n)^2\, dx' < \infty,$$

$$\text{(II) } \int_{R_n^+} \min(1, x_n)\, |Du(x)|^2\, dx < \infty.$$

PROOF. Let $0 < 3\delta_0 < 1$. Define a non-negative function $\eta \in C^2([0, \infty))$ such that $\eta(x_n) = x_n$ for $x_n \leqslant 2\delta_0$, and $\eta(x_n) = 1$ for $x_n \geqslant 3\delta_0$. We may assume that $\eta(x_n) \geqslant \delta$ for all $x_n \geqslant \delta$ and $0 < \delta \leqslant \delta_0$.

Let

$$v(x) = \begin{cases} u(x)(\eta(x_n) - \delta)\, \Phi(x')^2 & \text{for } x_n > \delta, \\ 0 & \text{elsewhere}, \end{cases}$$

where Φ is a non-negative function in $C_0^1(R_{n-1})$. Since for every $\delta < x_n$ $v(\cdot, x_n)$ has a compact support in R_{n-1}, it follows from Lemma 5 that v is an admissible test function in (2), hence

$$\text{(4)} \quad \int_\delta^\infty \int_{R_{n-1}} \sum_{i,j=1}^n a_{ij} D_i u D_j u (\eta - \delta)\, \Phi^2\, dx + \int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in} D_i u \cdot u D_n \eta \Phi^2\, dx +$$
$$+ \int_\delta^\infty \int_{R_{n-1}} a_{nn} D_n u \cdot u \cdot D_n \eta \Phi^2\, dx + 2 \int_\delta^\infty \int_{R_{n-1}} \sum_{i,j=1}^n a_{ij} D_i u \cdot u (\eta - \delta)\, \Phi D_j \Phi\, dx +$$
$$+ \int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^n b_i D_i u \cdot u (\eta - \delta)\, \Phi^2\, dx + \int_\delta^\infty \int_{R_{n-1}} c u^2 (\eta - \delta)\, \Phi^2\, dx =$$
$$= \int_\delta^\infty \int_{R_{n-1}} f \cdot u (\eta - \delta)\, \Phi^2\, dx .$$

Denote the integrals on the left hand side of (4) by $J_1, \ldots, J_6$. It follows from the ellipticity of L that

$$(5)\qquad J_1 \geqslant \gamma^{-1}\int_{\delta}^{\infty}\int_{R_{n-1}} |Du|^2(\eta-\delta)\Phi^2\,dx\,.$$

By Young's inequality

$$(6)\qquad |J_4| \leqslant \frac{\gamma}{5}\int_{\delta}^{\infty}\int_{R_{n-1}} |Du|^2(\eta-\delta)\Phi^2\,dx + C_1\int_{\delta}^{\infty}\int_{R_{n-1}} u^2(\eta-\delta)|D\Phi|^2\,dx\,,$$

where a positive constant C_1 depends on γ and $\|a_{ij}\|_{L^\infty}$. Similarly

$$(7)\qquad |J_5| \leqslant \frac{\gamma}{5}\int_{\delta}^{\infty}\int_{R_{n-1}} |Du|^2(\eta-\delta)\Phi^2\,dx + C_2\int_{\delta}^{\infty}\int_{R_{n-1}} u^2(\eta-\delta)\Phi^2\,dx\,,$$

where a positive constant C_2 depends on $\|b_i\|_{L^\infty}$. Now according to the assumption $(B\,iii)$

$$J_6 = \int_{\delta}^{\infty}\int_{R_{n-1}} c_1 u^2(\eta-\delta)\Phi^2\,dx + \int_{\delta}^{\infty}\int_{R_{n-1}} c_2 u^2(\eta-\delta)\Phi^2\,dx\,,$$

where $c_1 \in L^\infty(R_n^+)$ and $c_2 \in L^n(R_n^+)$. By Hölder's inequality

$$\left|\int_{\delta}^{\infty}\int_{R_{n-1}} c_2 u^2(\eta-\delta)\Phi^2\,dx\right| \leqslant \|c_2\|_{L^n}\left[\int_{\delta}^{\infty}\int_{R_{n-1}} u^2\Phi^2\,dx\right]^{\frac{1}{2}}\left[\int_{\delta}^{\infty}\int_{R_{n-1}} [u(\eta-\delta)\Phi]^{2^*}\,dx\right]^{\frac{1}{2^*}},$$

where $1/2^* = 1/2 - 1/n$. Now by Sobolev's inequality

$$\left\{\int_{\delta}^{\infty}\int_{R_{n-1}} [u(\eta-\delta)\Phi]^{2^*}dx\right\}^{\frac{1}{2^*}} \leqslant S\left[\int_{\delta}^{\infty}\int_{R_{n-1}} |Du|^2(\eta-\delta)^2\Phi^2\,dx + \right.$$

$$\left. + \int_{\delta}^{\infty}\int_{R_{n-1}} u^2|D\eta|^2\Phi^2\,dx + \int_{\delta}^{\infty}\int_{R_{n-1}} u^2(\eta-\delta)^2|D\Phi|^2\,dx\right]^{\frac{1}{2}},$$

where S depends only on n. Since we may assume that $\eta-\delta\leqslant 1$ we obtain by Young's inequality

$$(8)\qquad |J_6|\leqslant\frac{\gamma^{-1}}{5}\int_{\delta}^{\infty}\int_{R_{n-1}}|Du|^2(\eta-\delta)\Phi^2\,dx+C_3\Bigg[\int_{\delta}^{\infty}\int_{R_{n-1}}u^2(\eta-\delta)\Phi^2\,dx+$$
$$+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2\Phi^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2(\eta-\delta)|D\Phi|^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2|D\eta|^2\Phi^2\,dx\Bigg],$$

where a positive constant C_3 depends on n, $\|c_1\|_{L^\infty}$, $\|c_2\|_{L^n}$ and γ. By Green's formula we have

$$J_2=\frac{1}{2}\int_{\delta}^{\infty}\int_{R_{n-1}}\sum_{i=1}^{n-1}a_{in}D_i(u^2)D_n\eta\Phi^2\,dx=-\frac{1}{2}\int_{\delta}^{\infty}\int_{R_{n-1}}\sum_{i=1}^{n-1}D_i(a_{in}\Phi^2D_n\eta)u^2\,dx$$

and by the assumption $(B\,ii)$

$$(9)\qquad |J_2|\leqslant C_4\Bigg[\int_{\delta}^{b}\int_{R_{n-1}}x_n^{-\beta}u^2|D_n\eta|\Phi^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2|D_n\eta|\,\Phi^2\,dx+$$
$$+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2|D_n\eta|\,|D_x\Phi|^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2\Phi^2|D_n^2\eta|\,dx\Bigg],$$

where a positive constant C_4 depends on $\|a_{ij}\|_{L^\infty}$ and $\varkappa_1$. Integrating by parts

$$J_3=\frac{1}{2}\int_{\delta}^{\infty}\int_{R_{n-1}}a_{nn}(x',\delta)D_n\big(u(x)^2\big)\Phi(x')^2D_n\eta(x_n)\,dx+$$
$$+\int_{\delta}^{\infty}\int_{R_{n-1}}[a_{nn}(x)-a_{nn}(x',\delta)]D_nu(x)\cdot u(x)D_n\eta(x_n)\Phi(x')^2\,dx=$$

$$-\frac{1}{2}\int_{R_{n-1}} a_{nn}(x', \delta)\, u(x', \delta)^2 \Phi(x')^2\, dx' - \frac{1}{2}\int_\delta^\infty \int_{R_{n-1}} a_{nn}(x', \delta)\, u(x)^2 \Phi(x')^2 D_n^2\eta(x_n)\, dx +$$

$$+\int_\delta^\infty \int_{R_{n-1}} [a_{nn}(x) - a_{nn}(x', \delta)]\, D_n u(x)\cdot u(x) D_n\eta(x_n)\, \Phi(x')^2\, dx\,.$$

By the assumption (Bi) we obtain

$$\left|\int_\delta^\infty \int_{R_{n-1}} [a_{nn}(x) - a_{nn}(x', \delta)]\, D_n u\cdot u D_n\eta \Phi^2\, dx\right| \leqslant$$

$$\leqslant \int_\delta^\infty \int_{R_{n-1}} \varkappa (x_n - \delta)^\alpha\, |D_n u|\, |u| \Phi^2 |D_n \eta|\, dx \leqslant \frac{\gamma^{-1}}{5}\int_\delta^{2\delta_0} \int_{R_{n-1}} (x_n - \delta)|Du|^2 \Phi^2\, dx +$$

$$+ C_5\left[\int_{2\delta_0}^\infty \int_{R_{n-1}} |Du|^2 \Phi^2\, dx + \int_\delta^\infty \int_{R_{n-1}} \frac{u^2}{(x_n - \delta)^{1-2\alpha}}\, \Phi^2\, dx\right],$$

where a positive constant C_5 depends on $\varkappa$, γ and $\|D\eta\|_{L^\infty}$. From the last inequality we deduce the following estimate for J_3

$$|J_3| \leqslant \frac{\gamma^{-1}}{5}\int_\delta^{2\delta_0} \int_{R_{n-1}} (x_n - \delta)|Du|^2 \Phi^2\, dx + C_6\left[\int_{R_{n-1}} u(x', \delta)^2 \Phi(x')^2\, dx' + \right. \tag{10}$$

$$\left. + \int_\delta^\infty \int_{R_{n-1}} u^2 \Phi^2\, dx + \int_{2\delta_0}^\infty \int_{R_{n-1}} |Du|^2 \Phi^2\, dx + \int_\delta^\infty \int_{R_{n-1}} \frac{u^2}{(x_n - \delta)^{1-2\alpha}}\, \Phi^2\, dx\right],$$

where a positive constant C_6 depends on $\|a_{in}\|_{L^\infty}$, γ, $\varkappa$, $\|D_n^2\eta\|_{L^\infty}$ and $\|D_n\eta\|_{L^\infty}$. Inserting the estimates (5)-(10) into (4) we obtain

$$\int_\delta^{2\delta_0} \int_{R_{n-1}} |Du|^2 (x_n - \delta)\, \Phi^2\, dx \leqslant C_7\left[\int_{R_{n-1}} u(x', \delta)^2 \Phi(x')^2\, dx' + \right. \tag{11}$$

$$+ \int_{2\delta_0}^\infty \int_{R_{n-1}} |Du|^2 \Phi^2\, dx + \int_\delta^\infty \int_{R_{n-1}} u^2(\eta - \delta)\, \Phi^2\, dx + \int_\delta^\infty \int_{R_{n-1}} u^2 \Phi^2\, dx +$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}} u^2(x_n-\delta)^{2\alpha-1}\Phi^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}} x_n^{-\beta}u^2\Phi^2\,dx+$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}}|u^2|D\Phi|^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2(\eta-\delta)|D\Phi|^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}}f^2(\eta-\delta)\Phi^2\,dx\Bigg].$$

If the condition (I) holds then by Lemma 1 the integrals

$$\int_{\delta}^{\infty}\int_{R_{n-1}} x_n^{-\beta}u^2\,dx \quad \text{and} \quad \int_{\delta}^{\infty}\int_{R_{n-1}} u^2(x_n-\delta)^{2\alpha-1}\,dx$$

are bounded independently of δ. Now put $\Phi=\Phi_\nu$, where Φ_ν is an increasing sequence of non-negative functions in $C_0^\infty(R_{n-1})$ tending to 1 as $\nu\to\infty$ with the gradient bounded independently of ν. Letting $\nu\to\infty$ in (11) it follows from Lemma 3 that

$$\text{(12)}\qquad \frac{\gamma^{-1}}{5}\int_{\delta}^{2\delta_0}\int_{R_{n-1}}|Du|^2(x_n-\delta)\,dx\leqslant C_7\Bigg[\int_{R_{n-1}}u(x',\delta)^2\,dx+\int_{2\delta_0}^{\infty}\int_{R_{n-1}}|Du|^2\,dx+$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2(\eta-\delta)\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2dx+\int_{\delta}^{\infty}\int_{R_{n-1}}u^2(x_n-\delta)^{2\alpha-1}\,dx+$$

$$+\int_{\delta}^{\infty}\int_{R_{n-1}}x_n^{-\beta}u^2\,dx+\int_{\delta}^{\infty}\int_{R_{n-1}}f^2(\eta-\delta)\,\Phi^2\,dx\Bigg].$$

The implication «I $\Rightarrow$ II» follows from Lemmas 1 and 3 and the Lebesgue convergence theorem.

To show that «II $\Rightarrow$ I» observe that

$$\frac{1}{2}\int_{R_{n-1}}a_{nn}(x',\delta)\,u(x',\delta)^2\Phi(x')^2\,dx'\leqslant\sum_{\substack{j=1\\ j\neq 3}}^{6}|J_i|+\int_{\delta}^{\infty}\int_{R_{n-1}}|f|\,|u|(\eta-\delta)\,\Phi^2\,dx+$$

$$+\frac{1}{2}\int_{\delta}^{\infty}\int_{R_{n-1}}a_{nn}(x',\delta)\,u(x)^2\Phi(x')^2\,|D_n^2\eta|dx+\varkappa\int_{\delta}^{\infty}\int_{R_{n-1}}(x_n-\delta)^{\alpha}|D_nu|\,|u|\,|D\eta|\Phi^2\,dx.$$

Now by Lemma 2 the condition (II) implies that the integrals

$$\int_{\delta}^{\infty}\int_{R_{n-1}}(x_n-\delta)^{2\alpha-1}u^2\,dx\,,\qquad \int_{\delta}^{\infty}\int_{R_{n-1}}x_n^{-\beta}u^2\,dx\qquad\text{and}\qquad \int_{\delta}^{\infty}\int_{R_{n-1}}u^2\,dx$$

are bounded independently of δ. Repeating the argument from the step « I $\Rightarrow$ II » the result follows.

REMARK 1. It follows from the proof Theorem 1 that the condition (II) implies:

$$\text{for each } T>0\qquad \sup_{0<x_n<T}\int_{R_{n-1}}u(x',x_n)^2\,dx'<\infty\,.$$

As an immediate consequence we obtain

COROLLARY 1. *Let* $u\in\tilde{W}^{1,2}_{\text{loc}}(R_n^+)$ *be a solution of* (1) *in* R_n^+. *Suppose that one of the conditions* (I) *or* (II) *holds. Then there exists a function* $\varphi\in L^2(R_{n-1})$ *and a sequence* $\delta_\nu\to 0$ *as* $\nu\to\infty$ *such that*

$$\lim_{\nu\to\infty}\int_{R_{n-1}}u(x',\delta_\nu)\,\Psi(x')\,dx'=\int_{R_{n-1}}\varphi(x')\,\Psi(x')\,dx'$$

for every $\Psi\in L^2(R_{n-1})$.

THEOREM 2. *Let* $u\in\tilde{W}^{\text{loc}}_{1,2}(R_n^+)$ *be a solution of* (1) *in* R_n^+. *Suppose that one of the conditions* (I) *or* (II) *holds. Then there exists a function* $\varphi\in L^2(R_{n-1})$ *such that*

$$\lim_{\delta\to 0}\int_{R_{n-1}}u(x',\delta)\,\Psi(x')\,dx'=\int_{R_{n-1}}\varphi(x')\,\Psi(x')\,dx'$$

for every $\Psi\in L^2(R_{n-1})$.

PROOF. Since $\int_{R_{n-1}}u(x',\delta)^2\,dx'$ is bounded, say for $\delta\leqslant\delta_0$, and $a_{nn}(x',\delta)$ continuous and bounded it follows from Corollary 1, that

$$\lim_{\nu\to\infty}\int_{R_{n-1}}a_{nn}(x',\delta_\nu)\,u(x',\delta_\nu)\,\Psi(x')\,dx'=\int_{R_{n-1}}a_{nn}(x',0)\,\varphi(x')\,\Psi(x')\,dx'\,.$$

It suffices to show the existence of the limit $\int\limits_{R_{n-1}} u(x', \delta)\Psi(x')\,dx'$ at $\delta = 0$ for $\Psi \in C_0^1(R_{n-1})$. Let $v(x) = \Psi(x')\,(\eta(x_n) - \delta)$ for $x_n > \delta$, $x' \in R_{n-1}$ and $v(x) = 0$ elsewhere, where $\eta \in C^2([0, \infty))$ is the function introduced in the proof of Theorem 1. Taking v as a test function in (2) we obtain

$$\int\limits_{R_{n-1}} a_{nn}(x', \delta)\,u(x', \delta)\,\Psi(x')\,dx' = \int\limits_{\delta}\int\limits_{R_{n-1}} \sum_{i,j=1}^{n} a_{ij}\,D_i u D_j \Psi(\eta - \delta)\,dx -$$

$$-\int\limits_{\delta}^{\infty}\int\limits_{R_{n-1}} a_{nn}(x', \delta)\,u(x)\,\Psi(x')\,D_n^2\eta(x_n)\,dx + \int\limits_{\delta}^{\infty}\int\limits_{R_{n-1}} \sum_{i=1}^{n-1} a_{in} D_i u D_n \eta \Psi\,dx +$$

$$+\int\limits_{\delta}^{\infty}\int\limits_{R_{n-1}} \sum_{i=1}^{n} b_i D_i u \Psi(\eta - \delta)\,dx + \int\limits_{\delta}^{\infty}\int\limits_{R_{n-1}} cu\Psi(\eta - \delta)\,dx +$$

$$+\int\limits_{\delta}^{\infty}\int\limits_{R_{n-1}} [a_{nn}(x', x_n) - a_{nn}(x', \delta)]\,D_n u(x)\,\Psi(x')\,D_n\eta\,dx - \int\limits_{\delta}^{\infty}\int\limits_{R_{n-1}} f\Psi(\eta - \delta)\,dx\,.$$

Since $\int\limits_{R_n^+} \min(x_n, 1)\,|Du|^2 dx < \infty$ and $\int\limits_{R_n^+} x_n^{-\gamma} u^2 dx < \infty$ for every $0 \leqslant \gamma < 1$ and Ψ has a compact support, the Lebesgue dominated convergence theorem implies the continuity of

$$\int\limits_{R_{n-1}} a_{nn}(x', \delta)\,u(x', \delta)\,\Psi(x')\,dx' \quad \text{at } \delta = 0\,.$$

Now

$$\Big|\int\limits_{R_{n-1}} u(x', \delta)\,\Psi(x')\,dx' - \int\limits_{R_{n-1}} \varphi(x')\,\Psi(x')\,dx'\Big| \leqslant$$

$$\leqslant \Big[\int\limits_{R_{n-1}} u(x', \delta)^2\,dx'\Big]^{\frac{1}{2}} \Big[\int\limits_{R_{n-1}} \Big(1 - \frac{a_{nn}(x', \delta)}{a_{nn}(x', 0)}\Big)^2 \Psi(x')^2\,dx'\Big]^{\frac{1}{2}} +$$

$$+\Big|\int\limits_{R_{n-1}} a_{nn}(x', \delta)\,u(x', \delta)\,\frac{\Psi(x')}{a_{nn}(x', 0)}\,dx' - \int\limits_{R_n} \varphi(x')\,\Psi(x')\,dx'\Big|\,.$$

By the Lebesgue dominated convergence theorem and the previous part of the proof, the right hand side of the last inequality tends to 0 as $\delta \to 0$ and this completes the proof.

Our next objective is to establish the L^2-convergence of $u(\cdot, \delta)$ to φ as $\delta \to 0$. To do this we first show that the norm of $u(\cdot, \delta)$ converges to the norm of φ. The result then follows by the uniform convexity of the space L^2.]

THEOREM 3. *Let* $u \in \tilde{W}^{1,2}_{\text{loc}}(R_n^+)$ *be a solution of* (1). *Suppose that one of the conditions* (I) *or* (II) *holds. Then there exists a function* $\varphi \in L^2(R_{n-1})$ *such that*

$$\lim_{\delta \to 0} \int_{R_{n-1}} [u(x', \delta) - \varphi(x')]^2 \, dx' = 0 .$$

PROOF. If $\Psi \in W^{1,2}(R_n^+)$, then by the argument used in the proof of Theorem 1 we obtain

$$\int_{R_{n-1}} a_{nn}(x', 0)\varphi(x')\Psi(x', 0)\, dx' =$$

$$= \int_{R_{n-1}} [a_{nn}(x) - a_{nn}(x', 0)] D_n u(x) \Psi(x) D_n \eta(x_n)\, dx -$$

$$- \int_{R_n^+} D_n\big(a_{nn}(x', 0)\Psi(x') D_n \eta(x_n)\big)\, u(x)\, dx + \int_{R_n^+} \sum_{i=1}^{n-1} a_{in} D_i u D_n \eta \Psi\, dx +$$

$$+ \int_{R_n^+} \sum_{i,j=1}^{n} a_{ij} D_i u D_j \Psi \cdot \eta\, dx + \int_{R_n^+} \sum_{i=1}^{n} b_i D_i u \Psi \eta\, dx + \int_{R_n^+} c u \Psi \eta\, dx -$$

$$- \int_{R_n^+} f \Psi \eta\, dx \equiv \int_{R_n^+} F(\Psi)\, dx ,$$

where η is the function introduced in the proof of Theorem 1. Define

$$v_\delta(x) = \begin{cases} u(x', x_n) & \text{for } x_n \geq \delta , \\ u\left(x', \dfrac{x_n + \delta}{2}\right) & \text{for } 0 < x_n < \delta , \end{cases}$$

it is clear that $v^\delta \in W^{1,2}(R_n^+)$. Thus

$$\int_{R_{n-1}} a_{nn}(x',0)\varphi(x')u\left(x',\frac{\delta}{2}\right)dx' = \int_{R_n^+} F(v_\delta)\,dx = \tag{13}$$

$$= \int_0^\delta \int_{R_{n-1}} F\left(u\left(x',\frac{x_n+\delta}{2}\right)\right)dx + \int_\delta^\infty \int_{R_{n-1}} F(u(x',x_n))\,dx = R_1 + R_2 .$$

We shall show that

$$\lim_{\delta\to 0} R_2 = \lim_{\delta\to 0} \int_{R_{n-1}} a_{nn}(x',0)\,u(x',\delta)^2\,dx \tag{14}$$

and

$$\lim_{\delta\to 0} R_1 = 0 . \tag{15}$$

Indeed

$$\lim_{\delta\to 0} R_2 = \lim_{\delta\to 0}\left\{\int_\delta^\infty \int_{R_{n-1}} \sum_{i,j=1}^n a_{ij}D_iuD_ju(\eta-\delta)\,dx + \int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in}D_iu\cdot u\cdot D_n\eta\,dx - \right.$$

$$-\int_\delta^\infty \int_{R_{n-1}} D_n\big(a_{nn}(x',0)\,u(x)D_n\eta(x_n)\big)\,u(x)\,dx +$$

$$+\int_\delta^\infty \int_{R_{n-1}} \big(a_{nn}(x) - a_{nn}(x',0)\big)D_nu(x)\cdot u(x)D_n\eta(x_n)\,dx +$$

$$\left.+\int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^n b_iD_iu\cdot u(\eta-\delta)\,dx + \int_\delta^\infty \int_{R_{n-1}} cu^2(\eta-\delta)\,dx - \int_\delta^\infty \int_{R_{n-1}} f\cdot u(\eta-\delta)\,dx\right\} =$$

$$= \lim_{\delta\to 0}\left\{-\int_\delta^\infty \int_{R_{n-1}} D_n\big(a_{nn}(x',0)\,u(x)D_n\eta(x_n)\big)\,u(x)\,dx - \right.$$

$$\left.-\int_\delta^\infty \int_{R_{n-1}} a_{nn}(x',0)\,D_nu(x)\cdot u(x)D_n\eta(x_n)\,dx\right\} =$$

$$-\lim_{\delta\to 0}\int_\delta^\infty \int_{R_{n-1}} D_n\big(a_{nn}(x',0)\,D_n\eta(x_n)\,u(x)^2\big)\,dx = \lim_{\delta\to 0}\int_{R_n^+} a_{nn}(x',0)\,u(x',\delta)^2\,dx .$$

Here we have used the fact that u is a solution of (1) and the identity (2) with the test function

$$v(x) = \begin{cases} u(x)(\eta(x_n) - \delta) & \text{for } x_n > \delta\,, \\ 0 & \text{elsewhere}\,, \end{cases}$$

where η is the function introduced in the proof of Theorem 1.

To prove (15) observe that

$$\begin{aligned} R_1 = & \int_0^\delta \int_{R_{n-1}} [a_{nn}(x) - a_{nn}(x', 0)]\, D_n u(x) \cdot u\left(x', \frac{x_n + \delta}{2}\right) D_n \eta(x_n)\, dx - \\ & - \int_0^\delta \int_{R_{n-1}} D_n\left(a_{nn}(x', 0)\, u\left(x', \frac{x_n + \delta}{2}\right) D_n \eta(x_n)\right) u(x)\, dx + \\ & + \int_0^\delta \int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in}(x)\, D_i u(x)\, D_n \eta(x_n)\, u\left(x', \frac{x_n + \delta}{2}\right) dx + \\ & + \int_0^\delta \int_{R_{n-1}} \sum_{i,j=1}^{n} a_{ij}(x)\, D_i u(x)\, D_j u\left(x', \frac{x_n + \delta}{2}\right) \eta(x_n)\, dx + \\ & + \int_0^\delta \int_{R_{n-1}} \sum_{i=1}^{n} b_i(x)\, D_i u(x)\, u\left(x', \frac{x_n + \delta}{2}\right) \eta(x_n)\, dx + \\ & + \int_0^\delta \int_{R_{n-1}} c(x)\, u(x)\, u\left(x', \frac{x_n + \delta}{2}\right) \eta(x_n)\, dx - \\ & \qquad - \int_0^\delta \int_{R_{n-1}} f(x)\, u\left(x', \frac{x_n + \delta}{2}\right) \eta(x_n)\, dx = \sum_{j=1}^{7} J_j\,. \end{aligned}$$

Using the conditions (I), (II) and Lemma 1 one can show that $\lim_{\delta \to 0} J_j = 0$, $j = 1, \ldots, 7$. We only restrict ourselves to the term J_3.

Integrating by parts the term J_3 can be written in the following form

$$J_3 = -\int_0^\delta \int_{R_{n-1}} \sum_{i=1}^{n-1} D_i a_{in}(x) D_n \eta(x_n) u(x) u\left(x', \frac{x_n+\delta}{2}\right) dx -$$

$$-\int_0^\delta \int_{R_{n-1}} \sum_{i=1}^{n-1} a_{in}(x) D_n \eta(x_n) u(x) D_i u\left(x', \frac{x_n+\delta}{2}\right) dx.$$

Hence by the assumption $(B\,ii)$

$$|J_3| \leqslant C\left[\int_0^\delta \int_{R_{n-1}} x_n^{-\beta} |u(x)| \left| u\left(x', \frac{x_n+\delta}{2}\right)\right| dx + \right.$$
$$\left. + \int_0^\delta \int_{R_{n-1}} |u(x)| \left| Du\left(x', \frac{x_n+\delta}{2}\right)\right| dx\right], \tag{16}$$

where a positive constant C depends on $\varkappa_1$, n and $\|a_{in}\|_{L^\infty}$ $(i = 1, \ldots, n-1)$. By Lemma 1 the first integral on the right hand side tends to 0 as $\delta \to 0$. Now by Hölder's inequality

$$\int_0^\delta \int_{R_{n-1}} |u(x)| \left| Du\left(x', \frac{x_n+\delta}{2}\right)\right| dx \leqslant \left[\int_0^\delta \int_{R_{n-1}} \frac{2u(x)^2}{x_n+\delta} dx\right]^{\frac{1}{2}} \cdot$$
$$\cdot \left[\int_0^\delta \int_{R_{n-1}} \left| Du\left(x', \frac{x_n+\delta}{2}\right)\right|^2 \frac{(x_n+\delta)}{2} dx\right]^{\frac{1}{2}}.$$

It follows from the condition (I) that

$$\int_0^\delta \int_{R_{n-1}} \frac{u(x)^2}{x_n+\delta} dx \leqslant \sup_{0<\delta\leqslant\delta_0} \int_{R_{n-1}} u(x', \delta)^2 dx' \int_0^\delta \frac{dx_n}{x_n+\delta} = \ln 2 \sup_{0<\delta\leqslant\delta_0} \int_{R_{n-1}} u(x', \delta)^2 dx'$$

and consequently by the condition (II) the second integral on the right hand side of (16) also tends to 0 as $\delta \to 0$. It follows from (13),

(14) and (15) that

$$\lim_{\delta\to 0}\int_{R_{n-1}} a_{nn}(x',0)\,u(x',\delta)^2\,dx' = \int_{R_{n-1}} a_{nn}(x',0)\,\varphi(x')^2\,dx' .$$

To complete the proof observe that the norms

$$\Big[\int_{R_{n-1}} a_{nn}(x',0)\,\varphi(x')^2\,dx'\Big]^{\frac12} \quad \text{and} \quad \Big[\int_{R_{n-1}} \varphi(x')^2\,dx'\Big]^{\frac12}$$

are equivalent in $L^2(R_{n-1})$.

3. Energy estimate.

Consider the elliptic equation of the form

$$Lu + \lambda u = f(x) \tag{17}$$

in R_n^+, where λ is a real parameter.

The results of section 2 suggest the following definition of the Dirichlet problem.

Let $\varphi \in L^2(R_{n-1})$. A weak solution $u \in \tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$ of (1) (or (17)) is a solution of the Dirichlet problem with the boundary condition

$$u(x',0) = \varphi(x') \text{ on } R_{n-1} \tag{18}$$

if

$$\lim_{\delta\to 0}\int_{R_{n-1}} [u(x',\delta) - \varphi(x')]^2\,dx' = 0 .$$

The main result of this section is the following energy estimate.

THEOREM 4. *Let $u \in \tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$ be a solution of the Dirichlet problem (17), (18). Then there exist positive constants d, λ_0 and C independent of u, such that*

$$\int_{R_n^+} \min(1,x_n)|Du(x)|^2\,dx + \sup_{0<\delta\leqslant d}\int_{R_{n-1}} u(x',\delta)^2\,dx' + \int_{R_n^+} u(x)^2\,dx \leqslant \\ \leqslant C\Big[\int_{R_n^+} \min(1,x_n)\,f(x)^2\,dx + \int_{R_{n-1}} \varphi(x')^2\,dx'\Big], \tag{19}$$

for all $\lambda \geqslant \lambda_0$, where a positive constant C depends on the structure of L.

PROOF. Since $u \in \tilde{W}^{1,2}_{\text{loc}}(R_n^+)$ is a solution of the Dirichlet problem (17), (18), $\sup\limits_{0<\delta\leqslant\delta_0} \int\limits_{R_{n-1}} u(x', \delta)^2 dx' < \infty$ for certain $\delta_0 > 0$ and consequently by Theorem 1 the condition (II) holds. All constants appearing in the proof are indepenent of δ and depend on the structure of L and may vary from line to line. By the same considerations as in the proof of the step « I $\Rightarrow$ II » of Theorem 1 we show that

$$\int\limits_{\delta}^{2\delta_0} \int\limits_{R_{n-1}} |Du|^2 (x_n - \delta)\, dx + \lambda \int\limits_{\delta}^{\infty} \int\limits_{R_{n-1}} u^2(\eta - \delta)\, dx \leqslant C_1 \Big[\int\limits_{R_{n-1}} u(x', \delta)^2 dx' + \int\limits_{\delta_0}^{\infty} \int\limits_{R_{n-1}} |Du|^2 dx + \int\limits_{\delta}^{\infty} \int\limits_{R_{n-1}} u^2(\eta - \delta)\, dx + \int\limits_{\delta}^{\infty} \int\limits_{R_{n-1}} u^2 dx + \int\limits_{\delta}^{\infty} \int\limits_{R_{n-1}} u^2 (x_n - \delta)^{2\alpha - 1} dx + \int\limits_{\delta}^{\infty} \int\limits_{R_{n-1}} x_n^{-\beta} u^2 dx + \int\limits_{\delta}^{\infty} \int\limits_{R_{n-1}} f^2(\eta - \delta)\, dx \Big],$$

for $\delta \leqslant \delta_0$, where η is the function introduced in the proof of Theorem 1. Letting $\delta \to 0$ we deduce from the last inequality

$$\text{(20)} \quad \int\limits_{0}^{2\delta_0} \int\limits_{R_{n-1}} |Du|^2 \min(1, x_n)\, dx + \lambda \int\limits_{R_n^+} u^2 \min(1, x_n)\, dx \leqslant$$

$$\leqslant C_2 \Big[\int\limits_{R_{n-1}} \varphi^2 dx' + \int\limits_{\delta_0}^{\infty} \int\limits_{R_{n-1}} |Du|^2 dx + \int\limits_{R_n^+} u^2 dx + \int\limits_{R_n^+} u^2 [\min(1, x_n)]^{2\alpha - 1} dx + \int\limits_{R_n^+} u^2 [\min(1, x_n)]^{-\beta} dx + \int\limits_{R_n^+} f^2 \min(1, x_n)\, dx \Big].$$

It follows from (20) and (3) (see Lemma 3) that

$$\text{(21)} \quad \int\limits_{0}^{2\delta_0} \int\limits_{R_{n-1}} |Du|^2 x_n\, dx + \lambda \int\limits_{R_n^+} u^2 \min(1, x_n)\, dx \leqslant$$

$$\leqslant C_3\Big[\int\limits_{R_{n-1}}\varphi^2\,dx' + \int\limits_{R_n^+}u^2\,dx + \int\limits_{R_n^+}u^2(\min(1,x_n))^{2\alpha-1}\,dx + \int\limits_{R_n^+}u^2(\min(1,x_n))^{-\beta}\,dx + \int\limits_{R_n^+}f^2\min(1,x_n)\,dx\Big].$$

Now adding the inequalities (3) and (21) we get

$$(22)\qquad \int\limits_{R_n^+}|Du|^2\min(1,x_n)\,dx + \lambda\int\limits_{R_n^+}u^2\min(1,x_n)\,dx \leqslant$$

$$\leqslant C_4\Big[\int\limits_{R_{n-1}}\varphi^2\,dx' + \int\limits_{R_n^+}u^2\,dx + \int\limits_{R_n^+}u^2(\min(1,x_n))^{2\alpha-1}\,dx + \int\limits_{R_n^+}u^2(\min(1,x_n))^{-\beta}\,dx + \int\limits_{R_n^+}f^2\min(1,x_n)\,dx\Big].$$

By a similar manner (see the proof of the step «I $\Rightarrow$ II» of Theorem 1) we obtain

$$(23)\qquad \sup_{0<\delta<d}\int\limits_{R_{n-1}}u(x',\delta)^2\,dx' \leqslant C_5\Big[\int\limits_{R_n^+}|Du|^2\min(1,x_n)\,dx + \lambda\int\limits_{R_n^+}u^2\min(1,x_n)\,dx + \int\limits_{R_n^+}u^2\,dx + \int\limits_{R_n^+}u^2(\min(1,x_n))^{2\alpha-1}\,dx + \int\limits_{R_n^+}u^2(\min(1,x_n))^{-\beta}\,dx + \int\limits_{R_n^+}f^2\min(1,x_n)\,dx\Big].$$

Now inserting (22) into (23) we obtain

$$(24)\qquad \sup_{0<\delta\leqslant d}\int\limits_{R_{n-1}}u(x',\delta)^2\,dx' \leqslant C_6\Big[\int\limits_{R_{n-1}}\varphi^2\,dx' + \int\limits_{R_n^+}u^2\,dx + \int\limits_{R_n^+}u^2(\min(1,x_n))^{2\alpha-1}\,dx + \int\limits_{R_n^+}u^2(\min(1,x_n))^{-\beta}\,dx + \int\limits_{R_n^+}f^2\,dx\Big].$$

Now we make use of the following fact: for every $0 \leqslant \varrho < 1$ and $0 < d < 1$

$$\int_{R_n^+} u^2(\min(1, x_n))^{-\varrho}\, dx \leqslant \frac{d^{1-\varrho}}{1-\varrho} \sup_{0<\delta\leqslant d} \int_{R_{n-1}} u(x', \delta)^2\, dx + d^{-1-\varrho}\int_{R_n^+} u^2 \min(1, x_n)\, dx\,.$$

Thus (19) follows from (22) and (23) provided λ is sufficiently large and d sufficiently small.

4. Dirichlet problem in R_n^+.

We are now in a position to establish the existence of a solution to the Dirichlet problem.

THEOREM 5. *Let $\lambda \geqslant \lambda_0$. Assume that $b_i \in L^n(R_n^+) \cap L^\infty(R_n^+)$ $(i = 1, \ldots, n)$ and that $c \in L^n(R_n^+) + L^\infty(R_n^+)$. Then for every $\varphi \in L^2$ (R_{n-1}) there exists a unique solution of the Dirichlet problem* (17), (18) *in* $\tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$.

PROOF. Let $\{\varphi_m\}$ be a sequence of functions in $C_0^1(R_{n-1})$ converging in $L^2(R_{n-1})$ to the function φ. Put

$$f_m(x) = \begin{cases} f(x) & \text{for } x \in R_{n-1} \times \left(\frac{1}{m}, \infty\right) \\ 0 & \text{for } x \in R_{n-1} \times \left(0, \frac{1}{m}\right) \end{cases}$$

$m = 1, 2, \ldots$. It follows from [3] that there exists a unique solution in $W^{1,2}(R_n^+)$ of the Dirichlet problem

$$Lu_m + \lambda u_m = f_m \quad \text{in } R_n^+\,,$$

$$u_m(x', 0) = \varphi_m(x') \quad \text{on } R_{n-1}\,.$$

By Theorem 4

$$\int_{R_n^+} |Du_q - Du_p|^2 \min(1, x_n)\, dx + \sup_{0<\delta\leq d} \int_{R_{n-1}} (u_q - u_p)^2\, dx +$$
$$+ \int_{R_n^+} (u_p - u_q)^2\, dx \leq C \left[\int_{R_n^+} \min(1, x_n)(f_p - f_q)^2\, dx + \int_{R_{n-1}} (\varphi_q - \varphi_p)^2\, dx' \right],$$

for $\lambda \geq \lambda_0$, where C is a positive constant independent of p and q. Consequently $\{u_m\}$ is the Cauchy sequence in the norm

$$\left[\int_{R_n^+} |Du|^2 \min(1, x_n)\, dx + \sup_{0<\delta\leq d} \int_{R_{n-1}} u(x', \delta)^2\, dx' + \int_{R_n^+} u^2\, dx \right]^{\frac{1}{2}}$$

and the result follows.

THEOREM 6. *Suppose that the essumptions of Theorem 5 hold and moreover $c(x) \geq \text{Const} > 0$ on R_n^+. Then for every $\varphi \in L^2(R_{n-1})$ there exists a unique solution to the Dirichlet problem* (1), (18) *in $\tilde{W}^{1,2}_{\text{loc}}(R_n^+)$ and moreover*

$$\int_{R_n^+} |Du|^2 \min(1, x_n)\, dx + \sup_{0<\delta\leq d} \int_{R_{n-1}} u(x', \delta)^2\, dx' + \int_{R_n^+} u^2\, dx \leq$$
$$\leq C \left[\int_{R_{n-1}} \varphi^2\, dx' + \int_{R_n^+} \min(1, x_n) f^2\, dx \right], \tag{25}$$

where C is a positive constant depending on the structure of the operator L.

PROOF. Let λ_0 be a sufficiently large positive constant. By Theorem 5 the Dirichlet problem

$$Lu_0 + \lambda_0 u_0 = f \quad \text{in } R_n^+$$
$$u_0 = \varphi \quad \text{on } R_{n-1}$$

has a unique solution in $\tilde{W}^{1,2}_{\text{loc}}(R_n^+)$ satisfying the energy estimate (25). On the other hand in virtue of Theorem 1.1 in [3] the Dirichlet problem

$$Lv = \lambda_0 u_0 \quad \text{in } R_n^+ ,$$
$$v = 0 \quad \text{on } R_{n-1}$$

has a unique solution in $W^{1,2}(R_n^+)$ and moreover

$$\|v\|_{\overset{\circ}{W}{}^{1,2}(R_n^+)} \leqslant \lambda_0 C \|u_0\|_{L^2(R_n^+)}, \tag{26}$$

where C is a positive constant. Thus the function $u = u_0 + v$ is a solution of the problem (1), (18) and it is obvious that

$$\int_{R_n^+} |Du|^2 \min(1, x_n)\, dx + \int_{R_n^+} u^2\, dx \leqslant C \left[\int_{R_{n-1}} \varphi^2\, dx' + \int_{R_n^+} \min(1, x_n) f^2\, dx \right], \tag{27}$$

where C is a positive constant. To obtain the estimate (25) it suffices to derive the estimate of the form (23) for $\sup\limits_{0<\delta\leqslant d} \int_{R_{n-1}} u(x', \delta)^2\, dx'$ and then use the final part of the argument of the proof of Theorem 4 and the inequality (27).

The next result establishes the relation between the solution of the Dirichlet problem in $\tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$ and $W^{1,2}(R_n^+)$. We point out here that by a solution of the Dirichlet problem in $W^{1,2}(R_n^+)$ we mean a solution of (1) with the boundary condition in the sense of trace (see Introduction).

THEOREM 7. *Suppose that the assumptions of Theorem* 6 *hold, that* $f \in L^2(R_n^+)$ *and that there exists a function* $\varphi_1 \in W^{1,2}(R_n^+)$ *such that* $\varphi_1(x', 0) = \varphi(x')$ *on* R_{n-1} *in the sense of trace. Then the solution of the Dirichlet problem* (1), (18) *in* $\tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$ *is a solution of the same problem in* $W^{1,2}(R_n^+)$.

PROOF. It follows from [3] that the Dirichlet problem $Lu = f$ on R_n^+ and $u = \varphi$ on R_{n-1} has a solution in $W^{1,2}(R_n^+)$ which is also a solution in $\tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$. The result follows from the uniqueness of the solutions in $\tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$ of the problem (1), (18).

5. Weighted estimate of the gradient.

It is well known that a solution of the Dirichlet problem for the Laplace equation

$$\Delta u = 0 \quad \text{on } R_n^+$$

$$u(x', 0) = \varphi(x') \quad \text{on } R_{n-1},$$

where $\varphi \in L^2(R_{n-1})$ satisfies the following relation $\int\limits_{R_n^+} x_n |Du(x)|^2\, dx = 2^{-2}\|\varphi\|^2_{L^2(R_{n-1})}$. The question arises whether the inequality $\int\limits_{R_n^+} x_n |Du(x)|^2\, dx \leqslant \mathrm{Const}\, \|\varphi\|^2_{L^2(R_{n-1})}$ remains true for the solution of the problem (1), (18) in $\tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$ (see [26] p. 82-83). The following theorem contains a partial answer to this question.

THEOREM 8. *Suppose that* $c(x) \geqslant \mathrm{Const} > 0$ *on* R_n^+, $c \in L^n(R_n^+) + L^\infty(R_n^+)$, $b_i \equiv 0$ $(i = 1, \dots, n)$ *and* $f \equiv 0$ *on* R_n^+. *Let* u *be a solution of the Dirichlet problem* (1), (18) *in* $\tilde{W}^{1,2}_{\mathrm{loc}}(R_n^+)$. *Then*

$$\int\limits_{R_n^+} x_n |Du(x)|^2\, dx + \int\limits_{R_n^+} x_n u(x)^2\, dx \leqslant C \int\limits_{R_{n-1}} \varphi(x')^2\, dx', \tag{28}$$

where a positive constant C *depends on the structure of the operator* L.

PROOF. Let $\{\eta_\nu\}$ be an increasing sequence of functions in $C^2([0, \infty))$ converging to x_n on $[0, \infty)$ with properties: $\eta_\nu(x_n) = x_n$ for $x_n \leqslant 2\delta_0$, $\eta(x_n) \geqslant \delta$ for $x_n > 2\delta_0$ and $\delta \leqslant \delta_0$, $|D\eta_\nu|$ and $|D^2\eta_\nu|$ are bounded independently of ν. η_ν may be chosen to be constant for large x_n. Taking

$$v(x) = \begin{cases} u(x)\big(\eta(x_n) - \delta\big) & \text{for } x_n \geqslant \delta, \\ 0 & \text{elsewhere} \end{cases}$$

as a test function in (2) we obtain

$$\int\limits_\delta^\infty \int\limits_{R_{n-1}} \sum_{i,j=1}^n a_{ij} D_i u D_j u (\eta_\nu - \delta)\, dx + \int\limits_\delta^\infty \int\limits_{R_{n-1}} \sum_{i=1}^{n-1} a_{in} D_i u \cdot u D_n \eta_\nu\, dx +$$

$$+ \int\limits_\delta^\infty \int\limits_{R_{n-1}} a_{nn}(x', \delta) D_n u \cdot u \cdot D_n \eta_\nu\, dx + \int\limits_\delta^\infty \int\limits_{R_{n-1}} [a_{nn}(x) - a_{nn}(x', \delta)] D_n u \cdot u D_n \eta_\nu\, dx +$$

$$+ \int\limits_\delta^\infty \int\limits_{R_{n-1}} c u^2 (\eta_\nu - \delta)\, dx = 0\,.$$

We denote the sum of the second and third integral by J_1 and integrating by parts we obtain

$$|J_1| \leqslant C_1 \Big[\int_{R_{n-1}} u(x', \delta)^2\, dx' + \int_\delta^\infty \int_{R_{n-1}} u(x)^2 x_n^{-\beta}\, dx + \int_\delta^\infty \int_{R_{n-1}} u(x)^2\, dx \Big],$$

where a positive constant C_1 depends on $\varkappa_1$, $\|a_{nn}\|_{L^\infty}$, $\sup_\nu |D\eta_\nu|$ and $\sup_\nu |D^2\eta_\nu|$. Similarly the fourth integral J_3 can be estimated in the following way

$$|J_3| \leqslant C_2 \Big[\int_\delta^{2\delta_0} \int_{R_{n-1}} |Du|^2 (x_n - \delta)\, dx + \int_{2\delta_0}^\infty \int_{R_{n-1}} |Du|^2\, dx + \int_\delta^{2\delta_0} \int_{R_{n-1}} u^2 (x_n - \delta)^{2\alpha - 1}\, dx + \int_{2\delta_0}^\infty \int_{R_{n-1}} u^2\, dx \Big],$$

where a positive constant C_2 depends on $\varkappa_1$, $\|a_{nn}\|_{L^\infty}$ and $\sup_\nu |D\eta_\nu|$. Finally using the ellipticity and the fact that c is bounded below by a positive constant we obtain

$$\int_\delta^\infty \int_{R_{n-1}} |Du|^2 (\eta_\nu - \delta)\, dx + \int_\delta^\infty \int_{R_{n-1}} u^2 (\eta_\nu - \delta)\, dx \leqslant C_3 \Big[\int_{R_{n-1}} u(x', \delta)^2\, dx' + \int_\delta^\infty \int_{R_{n-1}} u^2\, dx + \int_\delta^\infty \int_{R_{n-1}} u^2 x_n^{-\beta}\, dx + \int_\delta^\infty \int_{R_{n-1}} u^2 (x_n - \delta)^{2\alpha - 1}\, dx + \int_\delta^{2\delta_0} \int_{R_{n-1}} |Du|^2 (x_n - \delta)\, dx + \int_{2\delta_0}^\infty \int_{R_{n-1}} |Du|^2\, dx \Big].$$

Letting $\nu \to \infty$ and $\delta \to 0$ we derive from the last inequality

$$\int_{R_n^+} |Du|^2 x_n\, dx + \int_{R_n^+} u^2 x_n\, dx \leqslant \tilde{C}_3 \Big[\int_{R_{n-1}} \varphi^2\, dx' + \int_{R_n^+} u^2\, dx + \int_{R_n^+} u^2 x_n^{-\beta}\, dx + \int_{R_n^+} u^2 x_n^{\alpha - 1}\, dx + \int_{R_n^+} |Du|^2 \min(1, x_n)\, dx,$$

where positive constants C_3 and $\tilde{C}_3$ depend on the structure of the operator L. Now applying the energy estimate (25) we obtain (28).

REMARK 2. The estimate (28) can be extended to the nonhomogeneous equation provided $\int_{R_n^+} f(x)^2 x_n\,dx < \infty$.

6. Estimate of derivatives of the second order.

In this section we replace the assumptions ($B\,i$) and ($B\,ii$) by the following condition

(D) $a_{ij} \in C^1(R_n^+)$, $|Da_{ij}(x| \leqslant \varkappa_1 x_n^{-\beta}$ on $R_{n-1} \times (0, b]$ and $D_k a_{ij} \in L^\infty \cdot$ $\cdot \big(R_n \times [b, \infty)\big)$ $(k, i, j, = 1, \dots, n)$, where $0 \leqslant \beta < 1$, b and $\varkappa_1$ are positive constants.

THEOREM 9. *Let u be a solution in $\tilde{W}^{1,2}_{\text{loc}}(R_n^+)$ of the problem* (1), (18). *Then*

$$\int_{R_n^+} |D^2 u(x)|^2 [\min(1, x_n)]^3\,dx \leqslant \leqslant C\Big[\int_{R_n^+} |Du|^2 \min(1, x_n)\,dx + \int_{R_n^+} u(x)^2\,dx + \int_{R_n^+} f(x)^2 \min(1, x_n)\,dx\Big], \tag{29}$$

where C is a positive constant depending on the structure of L.

PROOF. It follows from Theorem 8.8 in [10] (p. 173) that $u \in W^{2,2}_{\text{loc}}(R_n^+)$. We shall first show that for every $r > 0$

$$\int_{2r}^{\infty} \int_{R_{n-1}} |D^2 u|^2\,dx \leqslant C\Big[\int_r^\infty \int_{R_{n-1}} |Du|^2\,dx + \int_{R_n^+} u^2\,dx + \int_r^\infty \int_{R_{n-1}} f^2\,dx\Big]. \tag{30}$$

Indeed, taking $v = D_k w$, where $w \in W^{2,2}(R_n^+)$ with compact support, as a test function in (2) and integrating by parts we obtain

$$\int_{R_n^+} \sum_{i,j=1}^n D_k a_{ij} D_i u D_j w\,dx + \int_{R_n^+} \sum_{i,j=1}^n a_{ij} D_{ik} u D_j w\,dx - -\int_{R_n^+} \sum_{i=1}^n b_i D_i u D_k w\,dx - \int_{R_n^+} c u D_k w\,dx = -\int_{R_n^+} f D_k w\,dx. \tag{31}$$

Now let $w = D_k u\Phi^2$, where Φ is a non-negative function in $C_0^1(R_n^+)$ such that $\Phi = 0$ for $x_n \leqslant r$ and $x \in R_{n-1}$.

Then

$$\int_{R_n^+} \sum_{i,j=1}^{n} D_k a_{ij} D_i u D_{jk} u \Phi^2 \, dx + 2\int_{R_n^+} \sum_{i,j=1}^{n} D_k a_{ij} D_i u D_k u D_j \Phi \cdot \Phi \, dx +$$
$$+\int_{R_n^+} \sum_{i,j=1}^{n} a_{ij} D_{ik} u D_{jk} u \Phi^2 \, dx + 2\int_{R_n^+} \sum_{i,j=1}^{n} a_{ij} D_{ik} u D_k u D_j \Phi \cdot \Phi \, dx -$$
$$-\int_{R_n^+} \sum_{j=1}^{n} b_i D_i u D_{kk} u \Phi^2 \, dx - 2\int_{R_n^+} \sum_{i=1}^{n} b_i D_i u D_k u D_k \Phi \cdot \Phi \, dx -$$
$$-\int_{R_n^+} c u D_{kk} u \Phi^2 \, dx - 2\int_{R_n^+} c u D_k u \cdot \Phi \cdot D_k \Phi \, dx = -$$
$$-\int_{R_n^+} f D_{kk} u \Phi^2 \, dx - 2\int_{R_n^+} f D_k u \Phi D_k \Phi \, dx .$$

Applying the inequalities of Hölder, Young and Sobolev we deduce from the last equation that

$$\int_{R_n^+} |D^2 u|^2 \Phi^2 dx \leqslant C\Big[\int_{R_n^+} |Du|^2 (\Phi^2 + |D\Phi|^2) \, dx +$$
$$+\int_{R_n^+} u^2 (\Phi^2 + |D\Phi|^2) \, dx + \int_{R_n^+} f^2 \Phi^2 \, dx\Big],$$

where a positive constant C depends on the structure of L. Now put $\Phi = \Phi_\nu$, where Φ_ν is an increasing sequence of non-negative functions in $C_0^1(R_n^+)$ with supports in $R_{n-1} \times [r, \infty)$, $\Phi_\nu \to 1$ on $R_{n-1} \times [2r, \infty)$ and with $|D\Phi_\nu|$ bounded independently of ν. Letting $\nu \to \infty$ the inequality (30) follows. To establish (29) let

$$w(x) = \begin{cases} (\eta(x_n) - \delta)^3 D_k u(x) & x_n \geqslant \delta , \\ 0 & \text{elsewhere} , \end{cases}$$

where η is a function introduced in the proof of Theorem 1. By (30)

w is an admissible test function in (31) and

$$\int_\delta^\infty \int_{R_{n-1}} \sum_{i,j=1}^n D_k a_{ij} D_i u D_{jk} u (\eta - \delta)^3 \, dx +$$

$$+ 3\int_\delta^\infty \int_{R_{n-1}} \sum_{i,j=1}^n D_k a_{in} D_i u D_k u D_n \eta (\eta - \delta)^2 \, dx +$$

$$+\int_\delta^\infty \int_{R_{n-1}} \sum_{i,j=1}^n a_{ij} D_{ik} u D_{jk} u (\eta - \delta)^3 \, dx +$$

$$+ 3\int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^n a_{in} D_{ik} u D_k \, u (\eta - \delta)^2 D_n \eta \, dx -$$

$$-\int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^n b_i D_i u D_k^2 u (\eta - \delta)^3 \, dx - 3\int_\delta^\infty \int_{R_{n-1}} \sum_{i=1}^n b_i \, D_i u D_k u (\eta - \delta)^2 D_n \eta \delta_{kn} \, dx -$$

$$-\int_\delta^\infty \int_{R_{n-1}} c u D_k^2 u (\eta - \delta)^3 \, dx - 3\int_\delta^\infty \int_{R_{n-1}} c u D_k u (\eta - \delta)^2 D_n \eta \delta_{kn} \, dx -$$

$$-\int_\delta^\infty \int_{R_{n-1}} f D_k^2 u (\eta - \delta)^3 \, dx - 3\int_\delta^\infty \int_{R_{n-1}} f D_k u (\eta - \delta)^2 D_n \eta \delta_{nk} \, dx \, .$$

We may assume that $\eta - \delta \leqslant 1$. Applying the inequalities of Young, Hölder and Sobolev we easily derive from the last inequality the following estimate

$$\int_\delta^{2\delta_0} \int_{R_{n-1}} |D^2 u|^2 (x_n - \delta)^3 \, dx \leqslant C \Big[\int_{2\delta_0}^{\infty} \int_{R_{n-1}} |Du|^2 \, dx + \int_{2\delta_0}^{\infty} \int_{R_{n-1}} |D^2 u|^2 \, dx +$$

$$+\int_\delta^{2\delta_0} \int_{n-1} |Du|^2 (x_n - \delta)^2 x_n^{-\beta} \, dx + \int_{2\delta_0}^{\infty} \int_{R_{n-1}} |Du|^2 (\eta - \delta)^3 \, dx +$$

$$+\int_{R_n^+} u^2 \, dx + \int_{R_n^+} f^2 (\eta - \delta) \, dx \Big] \, ,$$

where C is a positive constant depending on the structure of the operator L, and the result follows.

7. Exterior Dirichlet problem. Preliminaries.

Let $\Omega \subset R_n$ be a complement of a bounded closed set with boundary $\partial\Omega$ of class C^2. In Ω we consider the equation (1). Let $x \in \Omega$, we denote by $r(x)$ the distance from x to $\partial\Omega$.

We make the following assumptions

(A_1) There exists a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$$

for all $x \in \Omega$ and $\xi \in R_n$; moreover $a_{ij} \in L^\infty(\Omega) \cap C^1(\Omega)$ $(i, j = 1, \ldots, n)$. We also assume that $|Da_{ij}(x)| < \varkappa r(x)^{-\alpha}$ $(i, j = 1, \ldots, n)$ in some neighbourhood N of the boundary $\partial\Omega$, where $\varkappa > 0$ and $0 < \alpha < 1$ are constants and that $D_k a_{ij} \in L^\infty(\Omega - N)$ $(k, i, j = 1, \ldots, n)$.

(A_2) $b_i \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $c \in L^\infty(\Omega) + L^n(\Omega)$ $(i = 1, \ldots, n)$.

It follows from the regularity of the boundary $\partial\Omega$ that there is a number $\delta_0 > 0$ such that for $\delta \in (0, \delta_0]$ the domain $\Omega_\delta = \Omega \cap \{x; \min_{y\in\partial\Omega} |x - y| > \delta\}$ with boundary $\partial\Omega_\delta$, possesses the following property: to each $x_0 \in \partial\Omega$ there is a unique point $x_\delta(x_0) = x_0 - \delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to $\partial\Omega$ at x_0. According to Lemma 1 in [10] p. 382, the distance $r(x)$ belongs to $C^2(\Omega - \bar{\Omega}_{\delta_0})$ if δ_0 is sufficiently small. We extend $r(x)$ to Ω as positive function of class $C^2(\Omega)$ and denote this extension again by $r(x)$. We may assume that $N \subset \Omega - \Omega_\delta$..

Let x_δ denote an arbitrary point of $\partial\Omega_\delta$. For fixed $\delta \in (0, \delta_0]$ let

$$A_\varepsilon = \partial\Omega_\delta \cap \{x; |x - x_\delta| < \varepsilon\},$$

$$B_\varepsilon = \{x; x = \tilde{x}_\delta + \delta\nu_\delta(\tilde{x}_\delta), \tilde{x}_\delta \in A_\varepsilon\}$$

and

$$\frac{dS_\delta}{dS_0} = \lim_{\varepsilon\to 0} \frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denote the $(n-1)$-dimensional Hausdorff measure of A, and $\nu_\delta(x_\delta)$ is the outward normal to $\partial\Omega_\delta$ at x_δ. Mikhailov [19] proved that there is a positive number γ_0 such that

$$\gamma_0^{-2} \leqslant \frac{dS_\delta}{dS_0} \leqslant \gamma_0^2 . \tag{32}$$

We will use the surface integral

$$M(\delta) = \int_{\partial\Omega_\delta} u(x)^2 \, dS_x ,$$

where $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and the values of $u(x)$ on $\partial\Omega_\delta$ are understood in the sense of trace.

Let R_0 be a positive number such that $R_n - \Omega_{\delta_0} \subset \{|x| < R\}$ for every $R > R_0$. Set

$$\Omega_R = \Omega \cap \{|x| < R\} \qquad \text{and} \qquad \Omega_{\delta,R} = \Omega_\delta \times \{|x| < R\}$$

for all $R > R_0$.

In the sequel we shall need the following estimate: if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and μ is a constant in $[0,1)$, then

$$\int_{\Omega_{\delta,R}} \frac{u(x)^2}{(r(x)-\delta)^\mu} \, dx \leqslant M \Bigg[\delta_0^{-\mu} \int_{\Omega_{\delta,R}} u(x)^2 \, dx + \delta_0^{1-\mu} \int_{\partial\Omega_{\delta_0/4}} u(x)^2 \, dS_x + \\ + \delta_0^{1-\mu} \int_{\Omega_\delta - \Omega_{\delta_0}} |Du(x)|^2 (r(x)-\delta) \, dx \Bigg] , \tag{33}$$

for $\delta \in (0, \delta_0/4]$, where M is a positive constant independent of δ and u. This inequality can be established by the same argument as in the proof of Lemma 2; in the proof we use the inequality (32).

Moreover it is easy to see that if $M(\delta)$ is bounded on $(0, \delta_0]$ then for every $\mu \in [0,1)$

$$\int_{\Omega_{\delta,R}} \frac{u(x)^2}{(r(x)-\delta)^\mu} \, dx \leqslant M_1 \tag{34}$$

for all $\delta \in (0, \delta_0/2]$, where M_1 is a positive constant independent of δ and u.

THEOREM 10. *Let u be a solution of* (1) *belonging to* $W^{1,2}_{\mathrm{loc}}(\Omega)$. *Then the following conditions are equivalent*

(I_1) $M(\delta)$ *is a bounded function on* $(0, \delta_0]$,

(II_1) $\int_{\Omega_R} |Du(x)|^2 r(x)\, dx < \infty$ for every $R > R_0$.

PROOF. We only sketch the proof because it is identical to that of Theorem 1 in [5]. Fix an $R > R_0$ and let Φ be a non-negative function in $C_0^\infty(R_n)$ such that $\Phi(x) = 1$ for $|x| \leqslant R$ and $\Phi(x) = 0$ for $|x| > 2R$.

Suppose that (I_1) holds. Thus $u \in L^2(\Omega_R)$ for every $R > R_0$ and (34) holds. It is clear that

$$v(x) = \begin{cases} u(x)\big(r(x) - \delta\big)\, \Phi(x)^2 & \text{on } \Omega_\delta, \\ 0 & \text{elsewhere} \end{cases}$$

is an admissible test function in (2) and

$$\int_{\Omega_\delta} \sum_{i,j=1}^n a_{ij} D_i u D_j u (r - \delta)\, \Phi^2\, dx + \int_{\Omega_\delta} \sum_{i,j=1}^n a_{ij} D_i u \cdot u \cdot D_j r \Phi^2\, dx + \\ + 2\int_{\Omega_\delta} \sum_{i,j=1}^n a_{ij} D_i u \cdot u (r - \delta)\, \Phi D_j \Phi\, dx + \int_{\Omega_\delta} \sum_{i=1}^n b_i D_i u \cdot u (r - \delta)\, \Phi^2\, dx + \\ + \int_{\Omega_\delta} c u^2 (r - \delta)\, \Phi^2\, dx = \int_{\Omega_\delta} f u (r - \delta)\, \Phi^2\, dx . \tag{35}$$

By Green's formula (see [21], p. 139) we have

$$\frac{1}{2}\int_{\Omega_\delta} \sum_{i,j=1}^n a_{ij} D_i(u^2) D_j r \Phi^2\, dx = \\ = -\frac{1}{2}\int_{\partial\Omega_\delta} \sum_{i,j=1}^n a_{ij} D_i r D_j r u^2\, \Phi^2\, dx - \frac{1}{2}\int_{\Omega_\delta} \sum_{i,j=1}^n D_i(a_{ij} D_j r \Phi^2)\, u^2\, dx \tag{36}$$

and

$$\left|\int_{\Omega_\delta}\sum_{i,j=1}^{n} D_i(a_{ij}D_j r\Phi^2)u^2\,dx\right| \leqslant C_1\left[\int_{\Omega_{\delta,2R}} u^2\,dx + \int_{\Omega_{\delta,2R}} u^2 r^{-\alpha}\,dx\right]. \tag{37}$$

By the ellipticity condition and the inequalities of Young and Sobolev we derive from (35), (36) and (37) that

$$\int_{\Omega_\delta} |Du|^2(r-\delta)\Phi^2\,dx \leqslant C\left[\sup_{0<\delta\leqslant\delta_0} M(\delta) + \int_{\Omega_{\delta,2R}} u^2(r-\delta)\,dx + \right.$$
$$\left. + \int_{\Omega_{\delta,2R}} u^2\,dx + \int_{\Omega_{\delta,2R}} u^2 r^{-\alpha}\,dx + \int_{\Omega_{\delta,2R}} f^2\,dx\right].$$

Now choose $\varphi = \Phi_\nu$, where $\{\Phi_\nu\}$ is an increasing sequence of non-negative functions in $C_0^\infty(R_n)$ converging to 1 for $|x| \leqslant R$ with $|D\Phi_\nu|$ bounded independently of ν. Letting $\delta \to 0$ and $\nu \to \infty$ the result easily follows.

Now suppose that the condition (II_1) holds. By (35) and (36) we have

$$\frac{1}{2}\int_{\partial\Omega_\delta}\sum_{i,j=1}^{n} a_{ij}D_i r D_j r u^2\Phi^2\,dx = -\frac{1}{2}\int_{\Omega_\delta}\sum_{i,j=1}^{n} D_i(a_{ij}D_j r\Phi^2)u^2\,dx +$$
$$+\int_{\Omega_\delta}\sum_{i,j=1}^{n} a_{ij}D_i u D_j u(r-\delta)\Phi^2\,dx + 2\int_{\Omega_\delta}\sum_{i,j=1}^{n} a_{ij}D_i u\cdot u(r-\delta)\Phi D_j\Phi\,dx +$$
$$+\int_{\Omega_\delta}\sum_{i=1}^{n} b_i D_i u\cdot u(r-\delta)\Phi^2\,dx + \int_{\Omega_\delta} cu^2(r-\delta)\Phi^2\,dx - \int_{\Omega_\delta} fu(\varrho-\delta)\Phi^2\,dx.$$

Using (33) and the inequalities of Young, Hölder and Sobolev, it is easy to see that $M(\delta)$ is bounded on $(0, \delta_0]$.

Our next objective is to prove that u has a trace on $\partial\Omega$ in $L^2(\partial\Omega)$. We first state the following preliminary result, which is easy to prove (see Theorems 2 and 3 of Section 2).

THEOREM 11. *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a solution of* (1). *Assume that one of the conditions* (I_1) *or* (II_1) *holds. Then there exists a function*

$\varphi \in L^2(\partial\Omega)$ such that

$$\lim_{\delta\to 0}\int_{\partial\Omega} u(x_\delta(x))\,g(x)\,dS_x = \int_{\partial\Omega}\varphi(x)\,g(x)\,dS_x$$

for every $g \in L^2(\partial\Omega)$.

To prove that $u(x_\delta) \to \varphi$ in $L^2(\partial\Omega)$ we show that $\int_{\partial\Omega} u(x_\delta)^2\,dS_x \to \int_{\partial\Omega}\varphi(x)^2 dS_x$ and the result follows from the uniform convexity of $L^2(\partial\Omega)$.

Fix $R > R_0$, for $\delta \in (0, \delta_0]$ we define the mapping $x^\delta : \bar{\Omega}_{2R} \to \bar{\Omega}_{\delta,2R}$ by

$$x^\delta(x) = \begin{cases} x & \text{for } x \in \Omega_{\delta,2R} \\ x_\delta(x) + \frac{1}{2}\big(x - x_\delta(x)\big) & \text{for } x \in \Omega_R - \Omega_{\delta,2R}, \end{cases}$$

where $x_\delta(x)$ denotes the nearest point to x on $\partial\Omega_\delta$ and $x^\delta(x) = x_{\delta/2}(x)$ for each $x \in \partial\Omega$. Moreover $r\big(x^\delta(x)\big) \geqslant \delta/2$ and x^δ is uniformly Lipschitz continuous. Note that if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ then $u(x^\delta) \in W^{1,2}(\Omega_{2R})$ for each $R > R_0$.

To prove L^2-convergence of $u(x_\delta)$ we shall need the following technical lemmas

LEMMA 4. *If $r^{\frac{1}{2}}f$ and $r^{\frac{1}{2}}g \in L^2(\Omega_R)$ then*

$$\lim_{\delta\to 0}\int_{\Omega-\Omega_\delta} f\big(x^\delta(x)\big)\,g(x)\,r(x)\,dx = 0\,.$$

LEMMA 5. *If $g \in L^2(\Omega_R)$ and $r^{\frac{1}{2}}f \in L^2(\Omega_R)$ and suppose that $\int_{\partial\Omega_\delta} g(x)^2\,dS$ is bounded on $(0, \delta_0]$, then*

$$\lim_{\delta\to 0}\int_{\Omega-\Omega_\delta} f\big(x^\delta(x)\big)\,g(x)\,dx = 0\,.$$

Let $L_1^2 = L^2(\partial\Omega, dS_x)$ with inner product (norm) denoted by $\langle\cdot,\cdot\rangle_1$ ($\|\cdot\|_1$) and $L_2^2 = L^2\big(\partial\Omega, g(x)dS_x\big)$ with inner product (norm) denoted by $\langle\cdot,\cdot\rangle_2$ ($\|\cdot\|_2$), where

$$g(x) = \sum_{i,j=1}^{n} a_{ij}(x)\,D_i r(x)\,D_j r(x)\,.$$

THEOREM 12. *Let* $u \in W^{1,2}_{loc}(\Omega)$ *be a solution of* (1) *such that one of the conditions* (I_1) *or* (II_1) *holds. Then there is a function* $\varphi \in L^2_1(\partial\Omega)$ *such that* $u(x_\delta)$ *converges to* $\varphi \in L^2_1(\partial\Omega)$.

PROOF. The proof is similar to that of Theorem 4 and is the repetition of the argument given in the case of bounded domain (Theorem 4 in [5]).

Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent it sufficient to show that there is $\varphi \in L^2_2$ such that $\lim_{\delta\to 0} u(x_\delta) = \varphi$ in L^2_2. By uniform convexity of L^2_2 it suffices to show that $\lim_{\delta\to 0} \|u(x_\delta)\|_2 = \|\varphi\|_2$.

Let $\Psi \in W^{1,2}(\Omega)$ and $\Psi = 0$ on $\Omega - \Omega_{2R}$. Set

$$F(\Psi(x)) = \sum_{i,j=1}^{n} a_{ij}(x) D_i u D_j \Psi \cdot r - \sum_{i,j=1}^{n} D_i(a_{ij} D_j r \cdot \Psi) u + \\ + \sum_{i=1}^{n} b_i(x) D_i u \cdot \Psi r + c(x) u \Psi r - f \cdot u \cdot \Psi \cdot r .$$

By Green's theorem we find that

$$\langle \varphi, \Psi \rangle_2 = \int_\Omega F(\Psi)\, dx .$$

Let Φ be defined as in the proof of Theorem 10, then

$$u(x^\delta)\, \varphi(x)^2 \in W^{1,2}(\Omega) \text{ and } u(x^\delta)\, \Phi(x)^2 = 0 \text{ on } \Omega - \Omega_{2R}.$$

Consequently we obtain

$$\langle \varphi, u(x^\delta) \rangle_2 = \int_\Omega F(u(x^\delta)\Phi^2)\, dx = \int_{\Omega-\Omega_\delta} F(u(x^\delta))\, dx + \int_{\Omega_\delta} F(u(x)\, \Phi(x)^2)\, dx$$

for $\delta \in (0, \delta_0]$, since $x^\delta(x) = x$ on $\Omega_{\delta,2R}$ and $\Phi(x) = 1$ on Ω_R. We show that

$$\lim_{\delta\to 0} \int_{\Omega-\Omega_\delta} F(u(x^\delta))\, dx = 0 \tag{38}$$

and

$$(39)\quad \lim_{\delta\to 0}\int_{\Omega_\delta} F\big(u(x)\Phi(x)^2\big)\,dx = \lim_{\delta\to 0}\|u(x_\delta)\|_2^2 = \lim_{\delta\to 0}\langle\varphi, u(x^\delta)\rangle_2 = \|\varphi\|_2^2\,,$$

since $x^\delta(x) = x_{\delta/2}(x)$ on $\partial\Omega$.

The relation (38) follows from Lemmas 4 and 5. To establish (39) we use (2) with a test function v given by the following formula

$$v(x) = \begin{cases} u(x)\big(r(x)-\delta\big)\,\Phi(x)^2 & \text{for } x\in\Omega_\delta\,,\\ 0 & \text{for } x\in\Omega-\Omega_\delta\,.\end{cases}$$

Theorem 12 justifies the following definition of the Dirichlet problem.

Let $\varphi\in L^2(\partial\Omega)$. A weak solution $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ of (1) is a solution of the Dirichlet problem with the boundary condition

$$(40)\qquad u(x) = \varphi(x)\quad \text{on } \partial\Omega\,,$$

if

$$(14)\qquad \lim_{\delta\to 0}\int_{\partial\Omega}[u(x_\delta(x)) - \varphi(x)]^2\,dS_x = 0\,.$$

8. Energy estimate for the exterior Dirichlet problem.

Assume that the distance function $r(x)$ is extended into Ω in such a way that

(E) $r\in C^2(\Omega)$, $r(x)>0$ on Ω, $|Dr(x)|\leqslant R_1\,r(x)$ and $|D^2\,r(x)|\leqslant R_1\,r(x)$ on $\Omega\cap(|x|>R_0)$, where R_1 is a positive constant.

Let $\varphi\in L^2(\partial\Omega)$ and consider the following Dirichlet problem in $W^{1,2}_{\mathrm{loc}}(\Omega_R)$.

$$(42)\qquad Lu+\lambda u = f\quad \text{on } \Omega_R\,,$$

$$(43)\qquad u = 0\quad \text{on } |x| = R$$

$$(44)\qquad u = \varphi\quad \text{on } \partial\Omega\,,$$

where λ is a real parameter. If $\lambda \geqslant \lambda_0$ (λ_0 sufficiently large) then by Theorem 6 in [5] there exists a unique solution. The boundary conditions (44) is understood in the sense of L^2-convergence and by (43) we mean that $u\Psi \in \overset{\circ}{W}{}^{1,2}(\Omega_R)$ for every function $\Psi \in C^1(\bar{\Omega}_R)$ such that $\Psi = 1$ in some neighbourhood of $|x| = R$ and $\Psi = 0$ on $\Omega - \Omega_{\delta_0}$.

LEMMA 6. *Suppose that r satisfies the condition* (E) *and let u be a solution in $W^{1,2}_{\mathrm{loc}}(\Omega_R)$ of the Dirichlet problem* (42), (43) *and* (44). *Then there exist $\lambda_0 > 0$* (*sufficiently large*) *and $d > 0$* (*sufficiently small*) *such that*

$$(45)\qquad \int_{\Omega_R} |Du(x)|^2 r(x)\,dx + \int_{\Omega_R} u(x)^2 r(x)\,dx + \sup_{0<\delta\leqslant d} M(\delta) \leqslant$$

$$\leqslant C\left[\int_{\partial\Omega} \varphi(x)^2\,dS_x + \int_{\Omega_R} f(x)^2 r(x)\,dx\right],$$

for all $R \geqslant R_0$, where positive constant λ_0, d and C depend on the structure of the operator and are independent of R.

PROOF. The energy estimate (45) was essentially proved in [5]. We repeat the proof to show that the constants C, d and λ_0 are independent of R. Let

$$v(x) = \begin{cases} u(x)(r(x) - \delta) & \text{on } \Omega_{\delta,R}, \\ 0 & \text{elsewhere}. \end{cases}$$

By Green's theorem

$$(46)\qquad \int_{\Omega_{\delta,R}} \sum_{i,j=1}^n a_{ij} D_i u D_j u (r-\delta)\,dx + \lambda \int_{\Omega_{\delta,R}} u^2(r-\delta)\,dx =$$

$$= \frac{1}{2}\int_{\partial\Omega_\delta} \sum_{i,j=1}^n a_{ij} D_i r D_j r u^2\,dS_x + \int_{\Omega_{\delta,R}} \sum_{i,j=1}^n D_i(a_{ij} D_j r)\, u^2\,dx -$$

$$-\int_{\Omega_{\delta,R}} \sum_{i=1}^n b_i D_i u \cdot u(r-\delta)\,dx - \int_{\Omega_{\delta,R}} c u^2 (r-\delta)\,dx + \int_{\Omega_{\delta,R}} f u (r-\delta)\,dx.$$

Now applying Hölder, Young and Sobolev inequalities we obtain

$$(47)\qquad \int_{\Omega_{\delta,R}} |Du|^2 (r-\delta)\,dx + \lambda \int_{\Omega_{\delta,R}} u^2(r-\delta)\,dx \leqslant$$
$$\leqslant C_1 \Big[M(\delta) + \int_{\Omega_{\delta,R}} u^2(r-\delta)\,dx + \int_{\Omega-\Omega_{\delta_0}} u^2 r^{-\alpha}\,dx +$$
$$+ \int_{\Omega_{\delta,R}} u^2(|Dr| + |D^2 r|)\,dx + \int_{\Omega_{\delta,R}} f^2(r-\delta)\,dx \Big],$$

where a positive constant C_1 is independent of R. On the other hand we have

$$(48)\qquad M(\delta) \leqslant C_2 \Big[\int_{\Omega_{\delta,R}} |Du|^2 (r-\delta)\,dx + \lambda \int_{\Omega_{\delta,R}} u^2(r-\delta)\,dx +$$
$$+ \int_{\Omega_{\delta,R}} u^2(r-\delta)\,dx + \int_{\Omega-\Omega_{\delta_0}} u^2 r^{-\alpha}\,dx + \int_{\Omega_{\delta,R}} u^2(|Dr| + |D^2 r|)\,dx + \int_{\Omega} f^2(r-\delta)\,dx \Big].$$

Letting $\delta \to 0$ in (47) we obtain

$$(49)\qquad \int_{\Omega_R} |Du|^2 r\,dx + \lambda \int_{\Omega_R} u^2 r\,dx \leqslant C_1 \Big[\int_{\partial\Omega} \varphi(x)^2\,dS_x + \int_{\Omega} u^2 r\,dx +$$
$$+ \int_{\Omega-\Omega_{\delta_0}} u^2 r^{-\alpha}\,dx + \int_{\Omega_R} u^2(|Dr| + |D^2 r|)\,dx + \int_{\Omega_R} f^2(r-\delta)\,dx \Big].$$

It follows from (48) and (49) that

$$(50)\qquad \sup_{0<\delta\leqslant d} M(\delta) \leqslant C_3 \Big[\int_{\Omega_R} u^2 r\,dx + \int_{\Omega_R} u^2(|Dr| + |D^2 r|)\,dx +$$
$$+ \int_{\Omega-\Omega_{\delta_0}} u^2 r^{-\alpha}\,dx + \int_{\Omega_R} f^2 r\,dx + \int_{\partial\Omega} \varphi\,dS_x \Big],$$

where C_2 and C_3 are positive constants independent of R. Now observe that by the assumption (E) we have

$$\int_{\Omega_R} u^2(|Dr| + |D^2 r|)\,dx \leqslant \varkappa_2 \int_{\Omega_{R_0}} u^2\,dx + 2R_1 \int_{\Omega_R} u^2 r\,dx,$$

where $\varkappa_2 = \sup_{\Omega_{R_0}} (|Dr| + |D^2 r|)$ and

$$\int_{\Omega_{R_0}} u^2\,dx \leqslant d \sup_{0<\delta\leqslant d} M(\delta) + \int_{\Omega_{R_0}-\Omega_d} u^2\,dx \leqslant d \sup_{0<\delta\leqslant d} M(\delta) + \frac{1}{\alpha_d}\int_{\Omega_R} u^2 r\,dx\,,$$

where $\alpha_d = \inf_{\Omega_{R_0}-\Omega_d} r(x)$ and consequently

$$\int_{\Omega_R} u^2(|Dr| + |D^2 r|)\,dx \leqslant \varkappa_2 d \sup_{0<\delta\leqslant d} M(\delta) + \left(\frac{\varkappa_2}{\alpha_d} + 2R_1\right)\int_{\Omega_R} u^2 r\,dx\,. \tag{51}$$

Similarly

$$\int_{\Omega-\Omega_{\delta_0}} u^2 r^{-\alpha}\,dx \leqslant \frac{d^{1-\alpha}}{1-\alpha}\sup_{0<\delta\leqslant d} M(\delta) + \alpha_d^{-\alpha}\int_{\Omega_{R_0}-\Omega_{\delta_0}} u^2\,dx \leqslant$$
$$\leqslant \frac{d^{1-\alpha}}{1-\alpha}\sup_{0<\delta\leqslant d} M(\delta) + \alpha_d^{-\alpha-1}\int_{\Omega_R} u^2 r\,dx\,. \tag{52}$$

Choosing d sufficiently small and λ_0 sufficiently large the result follows from (48), (49), (50), (51) and (52).

9. Existence of a solution of the exterior Dirichlet problem.

It is clear that one can deduce from Lemma 6 the existence of a solution of the Dirichlet problem (42), (43) and (44) in $W^{1,2}_{\mathrm{loc}}(\Omega)$.

As an immediate consequence of Lemma 6 we obtain

THEOREM 13. *Suppose that the distance function r satisfies the condition (E). Let $\varphi \in L^2(\partial\Omega)$ and $\int_\Omega f(x)^2 r(x)\,dx < \infty$. Then for every $\lambda \geqslant \lambda_0$ there exists a unique solution of the Dirichlet problem* (42), (40) *and* (41) *in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and moreover*

$$\int_\Omega |Du(x)|^2 r(x) + \int_\Omega u(x)^2 r(x)\,dx + \sup_{0<\delta\leqslant d} M(\delta) \leqslant$$
$$\leqslant C\left[\int_{\partial\Omega}\varphi(x)^2\,dS_x + \int_\Omega f(x)^2 r(x)\,dx\right], \tag{53}$$

where a positive constant C depends on the structure of the operator L.

PROOF. For fixed $R > R_0$ consider a solution v_R of the problem (42), (43) and (44) and set $v_R(x) = 0$ for $|x| > R$. It follows from the energy estimate (45) that there exists a sequence v_R tending weakly to a solution of the problem (42), (40) and (41).

From now on introduce the following assumption on the distance function r

(F) The distance function $r(x)$ on $\Omega - \Omega_\delta$ is extended to a function in $C^2(\Omega)$ such that $r(x) = 1$ on $\Omega \cap (|x| \geqslant R_0)$.

It is obvious that condition (F) implies (E).

THEOREM 14. *In addition to the hypotheses* (A_1) *and* (A_2) *assume that* $b_i \in L^n(\Omega)$ $(i = 1, \ldots, n)$, $c \in L^n(\Omega) + L^\infty(\Omega)$ *and* $c > \text{Const.} > 0$ *on* Ω. *Let* $\varphi \in L^2(\partial\Omega)$ *and* $r^{\frac{3}{2}} f \in L^2(\Omega)$. *Then there exists a unique solution* u *of the Dirichlet problem* (1), (40) *and* (41) *in* $W^{1,2}_{\text{loc}}(\Omega)$ *and moreover*

$$\int_\Omega |Du(x)|^2 r(x)\,dx + \int_\Omega u(x)^2 r(x)\,dx + \sup_{0<\delta\leqslant d} M(\delta) \leqslant C\left[\int_{\partial\Omega} \varphi(x)^2\,dS_x + \int_\Omega f(x)^2 r(x)\,dx\right],$$

where a positive constant C *depends on the structure of* L.

The proof is similar to that of Theorem 6 and therefore is omitted. We only point out that we again use here the results of Bottaro and Marina [3].

Under our assumption a solution in $W^{1,2}_{\text{loc}}(\Omega)$ of the problem (1), (40)-(41) belongs to $W^{2,2}_{\text{loc}}(\Omega)$. By the same argument as in the proof of Theorem 9 one can establish the following estimate of the derivative of the second order of a solution in $W^{1,2}_{\text{loc}}(\Omega)$ of the problem (1), (40) and (41).

THEOREM 15. *Suppose that the assumptions of Theorem* 14 *hold. Let* u *be a solution in* $W^{1,2}_{\text{loc}}(\Omega)$ *of the problem* (1), (40) *and* (41). *Then*

$$\int_\Omega |D^2 u(x)|^2 r(x)^3\,dx \leqslant C\left[\int_\Omega |Du|^2 r(x)\,dx + \int_\Omega u(x)^2\,dx + \int_\Omega f(x)^2\,dx\right],$$

where C *is a positive constant depending on the structure of the operator* L.

We mention here that the estimate of the derivative of the second order of a solution of the exterior Dirichlet problem in $\mathring{W}^{2,2}(\Omega)$ was obtained by Acanfora [1].

10. Case $c \geqslant 0$.

To establish the existence and the uniqueness of a solution of the Dirichlet problem we have assumed that $c \geqslant \text{Const} > 0$. If the coefficient c is only non-negative one can also construct a solution but belonging to a different function space. Namely introduce the space $\tilde{W}^{1,2}(R_n^+)$ equipped with the norm

$$\left[\int_{R_n^+} |Du(x)|^2 \min(1, x_n)\, dx + \int_{R_n^+} u(x)^2\, dx\right]^{\frac{1}{2}}$$

and similarly for the exterior Dirichlet problem the space $\tilde{W}^{1,2}(\Omega)$ with the norm

$$\left[\int_{\Omega} |Du(x)|^2 r(x) + \int_{\Omega} u(x)^2\, dx\right]^{\frac{1}{2}},$$

where r satisfies the condition (F). By Theorem 7 a solution of the Dirichlet problem (1), (18) belongs $\tilde{W}^{1,2}(R_n^+)$ and by Theorem 14 a solution of the exterior Dirichlet problem (1), (40) and (41) belongs to $\tilde{W}^{1,2}(\Omega)$.

Now denote by $D(R_n^+)$ $(D(\Omega))$ the completion of $C_0^1(R_n^+)$ $(C_0^1(\Omega))$ with respect to the norm

$$\left[\int_{R_n^+} |Du(x)|^2\, dx\right]^{\frac{1}{2}} \left(\left[\int_{\Omega} |Du(x)|^2\, dx\right]^{\frac{1}{2}}\right).$$

By Sobolev's inequality $D(R_n^+) \subset L^{2^*}(R_n^+)$ $\left(D(\Omega) \subset L^{2^*}(\Omega)\right)$, whence $D(R_n^+) \subset L^2_{\text{loc}}(R_n^+)$ $\left(D(\Omega) \subset L^2_{\text{loc}}(\Omega)\right)$.

THEOREM 16. *Suppose that the assumptions* (A) *and* (B) *hold. Let* $f \in L^2(R_n^+)$ *and* $\varphi \in L^2(R_{n-1})$ *and moreover assume that* $b_i \in L^n(R_n^+)$ $(i = 1, \ldots, n)$, $c \in L^{n/2}(R_n^+)$ *and* $c(x) \geqslant 0$ *on* R_n^+. *Then there exists a solution* u *to the Dirichlet problem* (1), (18), *belonging to the space* $\tilde{W}^{1,2}(R_n^+) + D(R_n^+)$.

Here the boundary condition (18) is satisfied in the following sense:

for every $R > 0$

$$\lim_{\delta\to 0} \int_{|x'|<R} [u(x', \delta) - \varphi(x')]^2 \, dx' = 0 . \tag{54}$$

PROOF. Consider the Dirichlet problem

$$Lu + \lambda_0 u = f \quad \text{in } R_n^+ ,$$
$$u(x', 0) = \varphi(x') \quad \text{on } R_{n-1} ,$$

where the boundary condition is understood inthe sense of L^2-convergence (see section 3). If λ_0 is sufficiently large then by Theorem 5 there exists a unique solution u_0 belonging to the space $\tilde{W}^{1,2}(R_n^+)$. On the other hand by the result of Chicco [7] the Dirichlet problem

$$Lv = \lambda_0 u_0 \quad \text{in } R_n^+$$
$$v(x', 0) = 0 \quad \text{on } R_{n-1}$$

has a solution in $D(R_n^+)$. It is clear that $v + u_0$ is a solution of (1), (18) in $\tilde{W}^{1,2}(R_n^+) + D(R_n^+)$. The L^2-convergence in the sense (54) follows from the fact that

$$v \in W^{1,2}\big((|x'| < R) \times (0, T)\big) \quad \text{for each } R > 0 \text{ and } T > 0.$$

In a similar manner one can establish

THEOREM 17. *Suppose that the assumptions* (A_1) *and* (A_2) *hold. Let* $\varphi \in L^2(\partial\Omega)$ *and moreover assume that* $b_i \in L^n(\Omega)$ $(i = 1, \ldots, n)$, $c \in L^{n/2}(\Omega)$ *and* $c(x) \geqslant 0$ *on* Ω. *Then there exists a solution of the Dirichlet problem* (1), (40) *and* (41) *in* $\tilde{W}^{1,2}(\Omega) + D(\Omega)$.

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