RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

DEREK J. S. ROBINSON
JAMES WIEGOLD

Groups with boundedly finite automorphism classes

Rendiconti del Seminario Matematico della Università di Padova, tome 71 (1984), p. 273-286
<http://www.numdam.org/item?id=RSMUP_1984__71__273_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 71 (1984)

Groups with Boundedly Finite Automorphism Classes.

DEREK J. S. ROBINSON - JAMES WIEGOLD (*)

1. Introduction.

Let G be a group and let Γ be a subgroup of Aut G, the automorphism group of G. Then Γ acts in a natural way on the group and on its set of subgroups. Thus we may speak of Γ-orbits of elements and of subgroups of G. In particular the Aut G-orbits are called *automorphism classes*. Our object here is to analyze the structure of groups whose automorphism classes of elements or subgroups are finite with bounded order.

Similar problems for Inn G-orbits were studied by B. H. Neumann almost thirty years ago. In [4] Neumann proved that the groups with boundedly finite conjugacy classes (or Inn G-orbits) of elements, the so-called *BFC-groups*, are precisely the groups with finite derived subgroup (see also [12]). In a subsequent paper [5] Neumann was able to show that the groups which have finite conjugacy classes of subgroups are exactly the centre by finite groups, that is, the groups with finite inner automorphism group. It was later shown by Eremin [2] that this remains true if one restricts only the conjugacy classes of *abelian* subgroups.

RESULTS. Our first main result is a criterion for a group to have boundedly finite automorphism classes of elements.

(*) Indirizzo degli AA.: D. J. S. ROBINSON: Department of Mathematics, University of Illinois, Urbana, Illinois 61801, U.S.A.; J. WIEGOLD: Department of Pure Mathematics, University College, Cardiff, Wales, U.K.

THEOREM 1. *Let G be a group and let T denote the torsion-subgroup of the centre of G. Then the automorphism classes of elements of G are boundedly finite if and only if T is finite and Aut G induces a finite group of automorphisms in G/T.*

This result may be reformulated in such a way as to refer only to T, G/T and the cohomology class Δ of the central extension $T \rightarrowtail G \twoheadrightarrow \bar{G} = G/T$. Recall that there is a natural left action of Aut $\bar{G}$ and a natural right action of Aut T on $H^2(\bar{G}, T)$. Using these actions we may restate Theorem 1 in the following form.

COROLLARY 1. *The group G has boundedly finite automorphism classes of elements if and only if T and $St_{\mathrm{Aut}\,\bar{G}}(\Delta^{\mathrm{Aut}\,T})$, the (setwise) stabilizer in Aut $\bar{G}$ of the Aut T-orbit containing Δ, are both finite.*

In general the automorphism group can be infinite when the automorphism classes of elements are boundedly finite. In § 4 we give some examples of this phenomenon: we also give necessary and sufficient conditions for the automorphism group to be finite.

Turning to automorphism classes of subgroups we may state our conclusions in the following form.

THEOREM 2. *The following properties of a group G are equivalent.*

(i) *The automorphism classes of subgroups of G are boundedly finite.*

(ii) *The automorphism classes of abelian subgroups of G are boundedly finite.*

(iii) *Either Aut G is finite or there is a direct decomposition $G = G_1 \times G_2$ where G_1 is a locally cyclic torsion group, G_2 is a finite central extension of a direct product of finitely many groups of type p^∞ for different primes p, and G_1 and G_2 do not contain elements of the same prime order.*

Thus the condition on automorphism classes of subgroups comes quite close to forcing the automorphism group to be finite, the only obstacle, so to speak, being locally cyclic torsion groups.

Ultimately Theorems 1 and 2 depend upon information about a group of automorphisms Γ of an FC-group G such that all Γ-orbits of G are boundedly finite. This information is to be found in a paper of Baer [1], (p. 111). We shall give a short and elementary proof of the necessary facts. These can be summed up as follows.

PROPOSITION 1. *Let Γ be a group of automorphisms of an FC-group G. Then the Γ-orbits of G are boundedly finite if and only if there is a finite Γ-invariant normal subgroup E of G such that Γ induces a finite group of automorphisms in G/E.*

We mention that a topological proof of this result has recently been given by Schlichting [10].

REMARKS:

(i) It will be seen that the proof of Theorem 1 provides bounds —albeit complicated ones—for the order of T and the index of $C_{\mathrm{Aut}\,G}(G/T)$ in terms of the least upper bound of the numbers of elements in an automorphism class.

(ii) We leave open the problem of determining the groups in which the automorphism classes of elements (or subgroups) are finite but possibly unbounded.

2. Automorphism groups with boundedly finite orbits.

We begin with some elementary facts.

LEMMA 1. *Let G be a group in which all automorphism classes of elements have at most n elements.*
Then

(i) *Aut G is residually a subgroup of the symmetric group S_n; thus Aut G is residually finite and has finite exponent $m \leqslant n!$,*

(ii) *the group G' is finite and $G/Z(G)$ is centre by finite.*

PROOF. The first statement is true because Aut G permutes the elements of an Aut G-orbit $g^{\mathrm{Aut}\,G}$ and $|g^{\mathrm{Aut}\,G}| \leqslant n$. The remaining statements follows from B. H. Neumann's theorem on BFC-groups and a result of P. Hall (see [7], 4.25).

Our immediate aim is to prove Proposition 1. For this we shall require a lemma which already occurs in Baer's paper.

LEMMA 2. *Let Γ be a group of automorphisms of a group G and assume that all Γ-orbits of G have at most n elements where $n > 1$. Let g be an element of G such that the Γ-orbit of g has exactly n elements and define N to be the subgroup generated by all $(g^x)^\gamma$ where $x \in G$ and $\gamma \in \Gamma$. If $K = C_\Gamma(N)$, then each K-orbit of G/N has at most $n - 1$ elements.*

PROOF. Suppose that on the contrary some x in G has $|K: C_K(xN)| = n$. Then clearly

$$n = |K: C_K(xN)| \leqslant |\Gamma: C_\Gamma(xN)| \leqslant |\Gamma: C_\Gamma(x)| \leqslant n ,$$

so that $C_\Gamma(x) = C_\Gamma(xN)$. Now consider $C_\Gamma(gx)$. If $\sigma \in C_\Gamma(gx)$, then $gx = g^\sigma x^\sigma$ and $x^\sigma \equiv x \bmod N$. Thus $\sigma \in C_\Gamma(xN) = C_\Gamma(x)$ and $g = g^\sigma$. Therefore

$$C_\Gamma(gx) = C_\Gamma(g) \cap C_\Gamma(x) .$$

Consequently

$$n = |\Gamma: C_\Gamma(g)| \leqslant |\Gamma: C_\Gamma(gx)| \leqslant n ,$$

and thus $C_\Gamma(g) = C_\Gamma(gx)$. It follows that $K = C_\Gamma(N) \leqslant C_\Gamma(gx) \leqslant C_\Gamma(x)$ and $C_K(x) = K$. This gives the contradiction $1 = |K: C_K(xN)| = n$

PROOF OF PROPOSITION 1. The sufficiency of the condition is clear. Assume that all Γ-orbits of G have at most n elements. If $n = 1$, then $\Gamma = 1$. Thus we can assume that $n > 1$ and use induction on n. Let g be an element of G with $|g^\Gamma| = n$.

Denote by N the subgroup of G generated by all $(g^x)^\gamma$ where $x \in G$ and $\gamma \in \Gamma$. Set K equal to $C_\Gamma(N)$. Since Γ permutes a finite set of generators of the FC-group N, we conclude that $|\Gamma: K|$ is finite. By Lemma 2 and induction on n there is a finite K-invariant normal subgroup E_0/N of G/N such that K induces a finite group of automorphisms in G/E_0. Define

$$\Gamma_0 = C_K(E_0) \cap C_K(G/E_0) .$$

Since E_0 is a finitely generated FC-group, $|K: \Gamma_0|$ is finite and so $|\Gamma: \Gamma_0|$ is finite.

Clearly $\Gamma_0^m = 1$ where $0 < m \leqslant n!$; since $[G, \Gamma_0] \leqslant E_0$ and $[E_0, \Gamma_0] = 1$, we deduce that $[G, \Gamma_0]^m = 1$. As a torsion subgroup of the finitely generated FC-group E_0, the group $[G, \Gamma_0]$ is finite. Now define E to be $[G, \Gamma_0]^\Gamma$; then E is a finite Γ-invariant normal subgroup of G, while $|\Gamma: C_\Gamma(G/E)|$ is finite since $\Gamma_0 \leqslant C_\Gamma(G/E)$.

From Proposition 1 we can derive a first criterion for a group to have boundedly finite automorphism classes of elements.

LEMMA 3. *Let G be a group. Then the automorphism classes of elements of G are boundedly finite if and only if there is a finite characteristic subgroup E such that Aut G induces a finite group of automorphisms in G/E.*

This follows at once on taking Γ to be Aut G in Proposition 1. (Notice that if we take Γ to be Inn G and apply Schur's Theorem on central by finite groups we recover Neumann's result that a BFC-group has finite derived subgroup.)

COROLLARY 2. *If G is a group with boundedly finite automorphism classes of elements and T is the torsion-subgroup of the centre, then Aut G induces a finite group of auromorphism in G/T.*

PROOF. By Lemma 3 there is a finite characteristic subgroup E which is minimal subject to Aut G inducing a finite group of automorphisms in G/E. It will suffice to show that $E\leqslant Z(G)$. Fix g in G; then $L = C_G(\langle g^{\mathrm{Aut}\, G}\rangle)$ is characteristic with finite index in G. Hence Aut G induces a finite group of automorphisms in $G/L\cap E$. Thus $E\leqslant L$ by minimality of E and $E\leqslant Z(G)$.

COROLLARY 3. *The automorphism group of a group with boundedly finite autcmorphism clas9es of elements is abelian by finite.*

The point here is that the subgroup Γ_0 which appeared in the proof of Proposition 1 is abelian since it satisfies $[G, \Gamma_0, \Gamma_0] = 1$.

3. The torsion-subgroup of the centre.

In this section we shall prove Theorem 1. The sufficiency of the condition in the theorem is clear. Thus in view of Corollary 2 all that remains to be done is to show that if G is a group with boundedly finite automorphism classes of elements, the torsion-subgroup of the centre, T, is finite. We shall establish this in a series of five steps. Let C denote the centre of G and set Q equal to G/C. Write Δ for the cohomology class of the extension $C \rightarrowtail G \twoheadrightarrow Q$.

(i) *There is a positive integer e such that $eH^2(Q, C) = 0$.* By the Universal Coefficients Theorem

$$H^2(Q, C) \simeq \mathrm{Ext}\,(Q_{ab}, C) \oplus \mathrm{Hom}\,(M(Q), C)\ .$$

Here $Q_{ab} = Q/Q'$ and $M(Q)$ is the multiplicator of Q. Now $Q \simeq \operatorname{Inn} G$, so we may deduce from Lemma 1 that Q has finite exponent. Hence $\operatorname{Ext}(Q_{ab}, C)$ is certainly bounded.

Next $Q/Z(Q)$ is finite by Lemma 1 (ii). It is now clear from the six term homology sequence for the extension $Z(Q) \rightarrowtail Q \twoheadrightarrow Q/Z(Q)$ that $M(Q)$ is bounded ([11], p. 105). Therefore Hom $(M(Q), C)$ is bounded, and we deduce that $H^2(Q, C)$ is bounded, as required.

(ii) *The subgroup T is reduced.* Otherwise we can write $C = C_1 \times C_2$ where C_1 is a p^∞-group for some prime p. Then the mappings $c_1 \mapsto c_1^{1+pe}$, $c_2 \mapsto c_2$, $(c_i \in C_i)$, determine an automorphism γ of C. Now $\Delta\gamma = \Delta$ because $e\Delta = 0$. Hence γ extends to an automorphism of G; however this automorphism has infinite order, in contradiction to Lemma 1.

(iii) *If p is any prime, the p-component T_p of T has finite exponent.* Suppose that $C = C_1 \times C_2$ where C_1 is a cyclic group of order p^l. The assignments $c_1 \mapsto c_1^{1+p^2e}$, $c_2 \mapsto c_2$, $(c_i \in C_i)$, determine an automorphism of C whose order is p^{l-m-2} or 1 if $l < m + 2$; here p^m is the p-share of e. This automorphism extends to an automorphism of G, so its order is $\leqslant n!$ by Lemma 1. Hence $p^{l-2} \leqslant e(n!)$, and there is an upper bound for l. It follows from (ii) and well-known facts about basic subgroups of abelian p-groups ([3], VI) that T_p has finite exponent.

(iv) *If p is any prime, the p-component T_p is finite.* The group Q is centre by finite with finite exponent by Lemma 1. From this it is easy to show that

$$Q = Q_0 \times Q_1$$

where Q_0 is finite and Q_1 is abelian. Write $Q_0 = G_0/C$ and $Q_1 = G_1/C$.

Suppose first that Q_1, and hence Q_{ab}, has an element of order p. Since Hom (Q_{ab}, T_p) is isomorphic with a subgroup of Aut G, the group T_p cannot have rank greater than n; this is because an automorphism class cannot contain $n+1$ elements. By (iii) the group T_p is finite.

Now assume that Q_1 is a p'-group. We can write $G_0 = XC$ with X a finitely generated subgroup. Now $|X : X \cap C|$ is finite, so $X \cap C$ is finitely generated. Since T_p has finite exponent, there are decompositions $C = T_p \times S$ and $T_p = T_0 \times T_1$ where $X \cap C \leqslant T_0 \times S$ and T_0 is finite. Combining these equations we obtain

$$G_0 = XC = (XT_0S) \times T_1 .$$

The next step is to prove that T_1 is also a direct factor of G. To accomplish this we consider the Lyndon-Hochschild-Serre spectral sequence for the extension

$$G_0/T_1 \rightarrowtail G/T_1 \twoheadrightarrow Q_1 .$$

Now our extension $T_1 \rightarrowtail G \twoheadrightarrow G/T_1$ restricts to a split extension $T_1 \rightarrowtail G_0 \twoheadrightarrow G_0/T_1$ and so comes from the kernel of the restriction. Thus we need only consider the terms E_2^{11} and E_2^{20} in the spectral sequence. In fact both of these vanish.

In the first place

$$E_2^{20} = H^2(Q_1, T_1) = 0$$

because T_1 is a p-group and Q_1 is an abelian p'-group—here it is relevant that T_1 is a trivial Q_1-module. Next consider

$$E_2^{11} = H^1\big(Q_1, H^1(G_0/T_1, T_1)\big);$$

of course

$$H \equiv H^1(G_0/T_1, T_1) = \mathrm{Hom}\,(G_0/T_1, T_1)\,,$$

which is a bounded abelian p-group. The action of G_1 on H is the following; if $\theta \in H$ and $g \in G_1$, then θ^g sends xT_1 in G_0/T to $\big(x[x, g^{-1}]T_1\big)\theta$. Now $[G_0, G_1] \leqslant C$, and since G_1/C is a p'-group, so is $[G_0, G_1]$. Therefore $[G_0, G_1]T_1/T_1$ is mapped by θ to the identity. It follows that $\theta^g = \theta$ and G_1 acts trivially on H, so that H is a trivial Q_1-module. Hence $H^1(Q_1, H) = \mathrm{Hom}\,(Q_1, H) = 0$. Consequently $E_2^{11} = 0$ and thus G splits over T_1. This means that T_1 is a direct factor of G. Since T_1 has finite exponent, we conclude that T_1, and hence T_p, must be finite.

(v) *The subgroup T is finite.* Let p be a prime greater than $1 + n!$ and assume that $T_p \neq 1$. Now Q has exponent at most $n!$, so it must be a p'-group. In addition T_p is a direct factor of C by (iii). We may argue as before that T_p is a direct factor of G by appealing to the spectral sequence for the extension $C/T_p \rightarrowtail G/T_p \twoheadrightarrow Q$. Thus G has a cyclic direct factor of some order $p^k > 1$. However this implies that $p - 1 \leqslant n!$. By this contradiction $T_p = 1$ if $p > 1 + n!$. It now follows from (iv) that T is finite.

PROOF OF COROLLARY 1. Recall that T is the torsion-subgroup of the centre of G and $\bar{G} = G/T$. We know that G has boundedly finite automorphism classes of elements if and only if both T and

$$I = \mathrm{Im}\,(\mathrm{Aut}\, G \to \mathrm{Aut}\, \bar{G})$$

are finite. Let $\varkappa \in \mathrm{Aut}\, \bar{G}$; then $\varkappa \in I$ if and only if there is a τ in Aut T such that $\varkappa\Delta = \Delta\tau$ ([11], p. 22). This is precisely the condition for $\varkappa$ to leave the orbit $\Delta^{\mathrm{Aut}\, T}$ fixed as a set. Hence $I = St_{\mathrm{Aut}\, \bar{G}}(\Delta^{\mathrm{Aut}\, T})$.

4. Some groups with infinite automorphism group.

We begin with a result which tells us exactly when a group with boundedly finite automorphism classes of elements has finite automorphism group.

PROPOSITION 2. *Let G be a group with boundedly finite automorphism classes of elements. Let $C = Z(G)$, $Q = G/C$, and let T be the torsion-subgroup of C; write $\bar{C} = C/T$. Then Aut G is finite if and only if Q_{ab}/Q_{ab}^t and $\bar{C}/\bar{C}^t$ are finite where $t = |T|$.*

PROOF. By Theorem 1 the image of the canonical homomorphism

$$\mathrm{Aut}\, G \to \mathrm{Aut}\, T \times \mathrm{Aut}\,(G/T)$$

is finite, while its kernel is clearly isomorphic with $H = \mathrm{Hom}((G/T)_{ab}, T)$. The given conditions imply that H, and hence Aut G, is finite. The converse is easy.

REMARK. It is not difficult to show that if Aut G is infinite, it will be uncountable: in this connection see [6] and [9]. The relation between finiteness of automorphism classes and compactness of the automorphism group is discussed in [6], a work in which our investigations had their origin.

LEMMA 4. *Let p be any prime and let F be a torsion-free abelian group such that Aut F is finite but F/F^p is infinite. Then $G = F \oplus \mathbf{Z}_p$ is a group with boundedly finite automorphism classes of elements but which has infinite automorphism group.*

This follows at once from Proposition 2. Such an abelian group F can be constructed in the following way. Let $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ be two sequences of distinct primes. Let $\{v_1, v_2, \ldots\}$ be a basis for a rational vector space V of countably infinite dimension. Define F to be the subgroup of V generated by all elements of the form

$$p_i^m v_i \quad \text{and} \quad q_i^m(v_i + v_{i+1})$$

where $m \in \mathbf{Z}$ and $i = 1, 2, \ldots$. It is straightforward to prove that $|\mathrm{Aut}\, F| = 2$, while it is clear that F/pF is infinite.

REMARK. The form of this example is typical of an abelian group with boundedly finite automorphism classes. Indeed by Theorem 1 such a group is always a direct product $F \times T$ where $|T|$ is finite, F is torsion-free and $\mathrm{Aut}\, F$ is finite.

A more challenging problem is to find a torsion group with boundedly finite automorphism classes of elements which has infinite automorphism group.

CONSTRUCTION. If p is any odd prime, let G be the group with generators

$$a, b\,, \qquad x_1, x_2, \ldots$$

and the following relations:

$$[a, x_i] = [b, x_i] = 1\,,$$

$$[x_{2i-1}, x_{2i}] = a\,, \qquad [x_{2i}, x_{2i+1}] = b\,,$$

$$[x_i, x_j] = 1 \quad \text{in other cases},$$

$$x_1^p = b\,, \qquad x_i^p = 1 \quad \text{if } i > 1\,.$$

It is clear that G is an infinite nilpotent p-group of class 2 and exponent p^2. Furthermore $C \equiv Z(G) = \langle a \rangle \times \langle b \rangle$ and $Q = G/C$ are elementary abelian: the latter has a basis $\{x_i C | i = 1, 2, \ldots\}$. Concerning this group we shall prove the following.

PROPOSITION 3. *The least upper bound of the orders of the automorphism classes of elements of G equals $(p(p-1))^2$. However Aut G is infinite.*

PROOF. We shall merely sketch the proof, omitting most of the computations. For any g in G we have $|G:C_G(g)| \leqslant p^2$, and of course $|G:C_G(x_1)| = p$. The latter property characterizes x_1 in the following sense.

(i) *If* $|G:C_G(g)| = p$, *then* $g \in \langle x_1, C\rangle$. *Hence* $\langle x_1, C\rangle$ *is characteristic in* C. Write $g \equiv x_1^{u_1} \dots x_k^{u_k} \bmod C$ where $u_i \in \mathbf{Z}_p$ and $u_k \neq 0$; let $k > 1$. If $h \in C_G(g)$, we can write $h = x_1^{v_1} \dots x_{k+1}^{v_{k+1}}$ with $v_i \in \mathbf{Z}_p$ at the expense of deleting elements which are known to commute with g. Then $1 = [g, h]$ yields

$$1 = [x_1, x_2]^{u_1v_2 - u_2v_1} \dots [x_{k-1}, x_k]^{u_{k-1}v_k - u_kv_{k-1}} [x_k, x_{k+1}]^{u_kv_{k+1}} .$$

Now use the defining relations to obtain two linear equations for $v_1, \dots, v_{k+1}$; the final terms are

$$\begin{cases} \cdots + (-u_k)v_{k-1} + u_{k-1}v_k = 0 \\ \cdots\cdots\cdots\cdots\cdots + u_kv_{k+1} = 0 . \end{cases}$$

Since $u_k \neq 0$ and v_{k+1} occurs only in the second equation, this a linearly independent system. But this implies that $|G:C_G(g)| = p^2$, a contradiction.

(ii) *The subgroups* $\langle a\rangle$ *and* $\langle b\rangle$ *are characteristic in* G. Since $\langle x_1, C\rangle$ is characteristic, we have $x_1^\alpha \in \langle x_1, C\rangle$ for all $\alpha \in \operatorname{Aut} G$. Hence $b^\alpha = (x_1^\alpha)^p \in \langle b\rangle$. Also $a^\alpha = [x_1^\alpha, x_2^\alpha] \in \langle a\rangle$ since $[x_1, x_2] = a$ and $x_1^\alpha \in \langle x_1, C\rangle$.

(iii) *The subgroup* $\langle x_1, x_3, \dots, x_{2i-1}, C\rangle$ *is characteristic in* G *for all* $i > 0$. Clearly $C_1 = C_G(\langle x_1, C\rangle) = \langle x_1, x_3, x_4, \dots, C\rangle$ is characteristic; also $Z(C_1) = \langle x_1, C\rangle$. By the argument of (i) we have $\langle x_3, Z(C_1)\rangle$ characteristic in C_1 and hence in G. Thus $\langle x_1, x_3, C\rangle$ is characteristic in G. The same method can be applied to $C_3 = C_G(\langle x_1, x_3, C\rangle)$ etc.

In what follows α is an arbitrary automorphism of G. Then we can write

$$x_i^\alpha \equiv x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \dots \bmod C$$

with $\alpha_{ij} \in \mathbf{Z}$. Considerations of order show that $\alpha_{i1} = 0$ if $i > 1$. Also $x_{2i-1}^\alpha \equiv x_3^{\alpha_{2i-1,3}} x_5^{\alpha_{2i-1,5}} \dots x_{2i-1}^{\alpha_{2i-1,2i-1}} \bmod C$ if $i > 0$ by (iii).

(iv) *For each $i > 0$ the subgroup $\langle x_{2i-1}, C\rangle$ is characteristic and* $x_{2j}^{\alpha} x_{2j}^{-\alpha_{2j,2j}} \equiv x_3^{\alpha_{2j,3}} \dots x_{2i-1}^{\alpha_{2j,2i-1}} x_{2i}^{\alpha_{2j,2i}} \dots \bmod C$ for $j = 1, 2, \dots, i-1$. This is proved by induction on i: the cases $i = 1$ or 2 are clear, so let $i > 2$. By applying α to $[x_{2j}, x_{2i-1}]$ and using the defining relations one obtains $\alpha_{2j,2i} = 0$ for $j = 1, 2, \dots, i-1$. One can then apply α to $[x_{2j}, x_{2i+1}]$ to show that $\alpha_{2i+1,2j+1} = 0$ for $j < i$. It follows that $\langle x_{2i+1}, C\rangle$ is characteristic in G. Finally from $[x_{2i}, x_{2j+1}]$ one obtains that $\alpha_{2i,2j} = 0$ for $j = 1, 2, \dots, i-1$. This completes the induction.

So far we know that

$$x_{2i}^{\alpha} x_{2i}^{-\alpha_{2i,2i}} = x_3^{\alpha_{2i,3}} x_5^{\alpha_{2i,5}} \dots, \qquad (i \geqslant 1).$$

By applying α to the commutators $[x_2, x_4]$, $[x_2, x_6]$, $[x_2, x_8]$, ..., we obtain $\alpha_{23} = \alpha_{25} = \alpha_{27} = \dots = 0 = \alpha_{43} = \alpha_{63} = \alpha_{83} = \dots$. We can treat x_4, x_6 etc. in a similar fashion. Thus we conclude that

(v) *Each $\langle x_i, C\rangle$ is characteristic in G.* From this it is straightforward to identify Aut G.

(vi) *There is a split exact sequence.*

$$\mathrm{Hom}\,(Q, C) \rightarrowtail \mathrm{Aut}\, G \twoheadrightarrow \mathbf{Z}_{p-1} \oplus \mathbf{Z}_{p-1}.$$

It is now easy to deduce Proposition 3.

Remark. Proposition 3 is still valid if $p = 2$, as slight changes in the argument show.

5. Automorphism classes of subgroups.

In this section we shall prove Theorem 2. Of course it is clear that condition (i) implies condition (ii). The main difficulty in the proof is to show that (ii) implies (iii). This will be accomplished in several steps.

(*a*) We assume that no automorphism class of abelian subgroups of G has more than k elements where k is some positive integer. Write $C = Z(G)$ and $Q = G/C$, and let T be the torsion-subgroup of C. By the theorem of Eremin quoted in the introduction Q is finite, with order q let us say. If Δ is the cohomology class of the central extension $C \rightarrowtail G \twoheadrightarrow Q$, then $q\Delta = 0$.

(*b*) *The group T is the direct product of a locally cyclic group L and a group of finite exponent E.* We consider a direct decomposition $C = P_1 \times P_2 \times C_0$ where the p-groups P_1 and P_2 are cyclic or of type p^∞ and $|P_1| \leqslant |P_2|$. Let $\theta \in \text{Hom}(P_1, P_2)$; then the assignments $x \mapsto x^{1+q\theta}$, $(x \in P_1)$, $x \mapsto x$, $(x \in P_2 \times C_0)$, determine an automorphism γ of C. Since $q\Delta = 0$, we have $\Delta\gamma = \Delta$, so that γ extends to an automorphism of G, and in consequence there are at most k subgroups of the form P_1^γ. Now $P_1^{1+q\theta} = P_1^{1+q\theta'}$ implies that $q\theta = q\theta'$. Hence $q\,\text{Hom}(P_1, P_2)$ has order at most k. This tells us at once that there is at most one subgroup of type p^∞ for each prime p.

Now suppose that P_1 is cyclic of order p^t and write p^m for the p-share of q. Then $p^{t-m} \leqslant k$ and so $p^t \leqslant qk$ for all primes p. Hence $t = 0$ for almost all p. It follows from well-known facts about basic subgroups ([3], VI) that the p-component C_p of C is locally cyclic for almost all p. Moreover C_p is at worst the direct product of a group of finite exponent and a locally cyclic group. This establishes (*b*).

(*c*) *If G is a torsion group, then $G = G_1 \times G_2$ where G_1 and G_2 are as described in the statement of the theorem.* Since Q is finite, we can write $G = XC$ where X is finitely generated and therefore finite. There are direct decompositions $E = E_0 \times E_1$ and $L = L_0 \times L_1$ where E_0 is finite, L_0 satisfies the minimal condition and $X \cap C \leqslant E_0 \times L_0$. Then $G = XC = (XE_0L_0) \times E_1 \times L_1$. Obviously E_1 inherits the properties of G, so E_1 is finite, being an abelian group of finite exponent. It is now clear that G has a direct decomposition of the type claimed.

From this point on we shall suppose that G is *not* a torsion group.

(*d*) *The subgroup T is finite.* We observe first that the group $\bar{C} = C/T$ cannot be p-divisible for any prime p. For suppose that this is the case. By (*b*) we can write $C = C_p \times D$ for some subgroup D which is evidently p-divisible. Moreover the mapping $x \mapsto x^p$ is an automorphism of D. There is a positive integer l such that $p^l \equiv 1 \bmod q$. Let γ be the automorphism of C determined by the assignments $x \mapsto x^{p^l}$, $(x \in D)$, $x \mapsto x$, $(x \in C_p)$. Then $\Delta\gamma = \Delta$, and so γ extends to an automorphism of G. Let x be an element of infinite order in D. Then there are only finitely many subgroups of the form $\langle x \rangle^{p^{li}}$, $i = 1, 2, \dots$, which is plainly absurd.

We conclude that $\bar{C}^{p^i} > \bar{C}^{p^{i+1}}$ for all $i \geqslant 0$. Now suppose that C_p contains an element of order p^t. Let $\theta \in \mathrm{Hom}\,(C/C^{p^t}T, C_p)$; then the mapping $x \mapsto x(xC^{p^t}\,T)^{q\theta}$ is an automorphism of C which extends to an automorphism γ of G. Writing $C = C_p \times D$ as before, we conclude that there are at most k subgroups of the form D^γ. Now if $D^\gamma = D^{\gamma'}$, then, in the obvious notation, $q\theta$ and $q\theta'$ take the same values on $DC^{p^t}T/C^{p^t}T = C/C^{p^t}T$, that is, $q\theta = q\theta'$. It follows that $q\,\mathrm{Hom}\,(C/C^{p^t}T, C_p)$ has order at most k. Hence $p^{t-m} \leqslant k$ and $p^t \leqslant kq$ where p^m is the p-share of q. Hence T has finite exponent, and we may write $C = F \times T$ with F torsion-free. Also $G = XC$ where X is a finitely generated subgroup. Hence we have $T = T_0 \times T_1$ where T_0 is finite and $X \cap C \leqslant T_0 \times F$. Then $G = (XT_0F) \times T_1$. It follows that T_1, and hence T, is finite.

(*e*) Hom $(G/T, T)$ *is finite*. Let $\theta \in \mathrm{Hom}\,(G/T, T)$; then the mapping $x \mapsto x(xT)^\theta$ is an automorphism γ of G. Writing $C = F \times T$ as in (*d*), we can be sure that there are only finitely many subgroups F^γ. From this it follows that Hom $(G/T, T)$ is finite.

(*f*) Aut G *is finite*. Let Γ denote the group of automorphisms of C induced by Aut G, and let $x \in C$. There are at most k subgroups $\langle x \rangle^\gamma$ with γ in Γ. If x has infinite order, there are therefore not more than $2k$ possibilities for x^γ; it follows that the Γ-orbits of $\bar{C} = C/T$ are boundedly finite. This allows us to deduce from Proposition 1 that Γ, and hence Aut G, induces a finite group of automorphisms in $\bar{C}$.

Define K to be the group of automorphisms of G which operate trivially on T, $\bar{C}$ and Q. Then $|\mathrm{Aut}\,G : K|$ is finite by (*d*) and the previous paragraph. It is easy to see that K must operate trivially on G/T. Hence $K \simeq \mathrm{Hom}\,(G/T, T)$, which is finite by (*e*). Therefore Aut G is finite, and we have shown that (ii) implies (iii).

Final step. To complete the proof of Theorem 2 we must show that (iii) implies (i). Consider a group $G = G_1 \times G_2$ with G_1 and G_2 as described in the theorem. It is clear that we can find a characteristic locally cyclic subgroup N such that $N \leqslant Z(G)$ and G/N is finite, of order n say. Let H be an arbitrary subgroup of G. Then $H \cap N$ is characteristic in G, so we can assume that $H \cap N = 1$. Hence H is finite of order dividing n. Let $\Gamma = C_{\mathrm{Aut}\,G}(G/N)$; then $|\mathrm{Aut}\,G : \Gamma| \leqslant n!$. If $\gamma \in \Gamma$, the mapping $x \mapsto [x, \gamma]$ is a homomorphism from H onto $[H, \gamma]$ since $[H, \gamma] \leqslant N \leqslant Z(G)$. Hence $[H, \gamma]$ has order dividing n and so it is contained in $N[n] = \{x \in N \mid x^n = 1\}$. But $N[n]$ has finite order r, say, and it is clear that there are at most r^n possibilities for H^γ. Hence the Aut G-orbit containing H has at most $r^n(n!)$ elements.

REFERENCES

[1] R. BAER, *Automorphismengruppen von Gruppen mit endlichen Bahnen gleichmässig beschränkter Mächtigkeit*, J. reine angew. Math., **262** (1973), pp. 93-119.

[2] I. I. EREMIN, *Groups with finite classes of conjugate abelian subgroups*, Mat. Sb., **47** (1959), pp. 45-54.

[3] L. FUCHS, *Infinite Abelian Groups*, Academic Press, New York (1971-73).

[4] B. H. NEUMANN, *Groups covered by permutable subsets*, J. London Math. Soc., **29** (1954), pp. 235-248.

[5] B. H. NEUMANN, *Groups with finite classes of conjugate subgroups*, Math. Z., **63** (1955), pp. 76-96.

[6] P. PLAUMANN, *Automorphismengruppen diskreter Gruppen als topologische Gruppen*, Arch. Math. (Basel), **29** (1977), pp. 32-33.

[7] D. J. S. ROBINSON, *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin (1972).

[8] D. J. S. ROBINSON, *A contribution to the theory of groups with finitely many automorphisms*, Proc. London Math. Soc., (3), **35** (1977), pp. 34-54.

[9] D. J. S. ROBINSON - S. E. STONEHEWER - J. WIEGOLD, *Automorphism groups of FC-groups*, Arch. Math. (Basel) **40** (1983), 401-404.

[10] G. SCHLICHTING, *Operationen mit periodischen Stabilisatoren*, Arch. Math. (Basel), **34** (1980), pp. 97-99.

[11] U. STAMMBACH, *Homology in group theory*, Lecture Notes in Mathematics, Vol. 359, Springer, Berlin (1973).

[12] J. WIEGOLD, *Groups with boundedly finite classes of conjugate elements*, Proc. Roy. Soc. London Ser. A, **238** (1957), pp. 389-401.

Manoscritto pervenuto in redazione il 1° febbraio 1983.