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Remarks on Existentially Closed Fields and Diophantine Equations.

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Introduction.

We begin with a simple proposition: an infinite field is always existentially closed in any purely transcendental extension. This leads to the consideration of solutions of diophantine equations in fields $K(t)$. In this respect, we extend a result of Natanson about Catalan's equation, to a much wider class of diophantine equations.

In the last section, we show that a field K, with a non-henselian valuation and algebraically closed residue field $\bar{K}$ cannot be existentially closed in any henselian valued field extension $\tilde{K}$. This leads to the conclusion that (whatever be $\bar{K}$), $\tilde{K}/K$ is not a purely transcendental extension. As corollaries, we obtain anew: $K((X))$ is not a purely transcendental extension of $K(X)$ and the p-adic field $\mathbf{Q}_p$ is not a purely transcendental extension of $\mathbf{Q}$.

1. Let S be a commutative ring with identity, let R be a subring of S. We say (see [1]) that R is *existentially closed* in S when every system of polynomial equations and inequations

$$f_1(X_1, \dots, X_n) = 0, \dots, f_k(X_1, \dots, X_n) = 0 ,$$

$$g_1(X_1, \dots, X_n) \neq 0, \dots, g_l(X_1, \dots, X_n) \neq 0$$

$$(\text{where } n \geqslant 1,\ f_i, g_j \in R[X_1, \dots, X_n])$$

which has a solution in S^n has also a solution in R^n.

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It is immediate that is $R \subset S \subset T$ are rings and subrings, if R is existentially closed in S and S is existentially closed in T, then R is existentially closed in T.

It is also easy to see ([1]): Let S be an integral domain, let R be a subring of S, let K be the field of quotients of R and L the field of quotients of S. If R is existentially closed in S then K is existentially closed in L.

The following proposition is practically trivial:

PROPOSITION 1. If K is an infinite field then K is existentially closed in every purely transcendental extension of K.

PROOF. By transfinite induction and transitivity of the property of being existentially closed, it suffices to show that K is existentially closed in the purely transcendental extension $K(t)$. By the above remark, it suffices to show that K is existentially closed in $K[t]$.

Let $n \geq 1$, $f_1, \ldots, f_k, g_1, \ldots, g_l \in K[X_1, \ldots, X_n]$, and assume that there exist $u_1(t), \ldots, u_n(t) \in K[t]$ such that

$$P_i(t) = f_i(u_1(t), \ldots, u_n(t)) = 0 \qquad (i = 1, \ldots, k)$$

and

$$Q_j(t) = g_j(u_1(t), \ldots, u_n(t)) \neq 0 \qquad (j = 1, \ldots, l)\,.$$

Since K is infinite, there exists an element $a \in K$ such that $Q_j(a) \neq 0$ (for $j = 1, \ldots, l$). Moreover, since $P_i(t) = 0$ then $P_i(a) = 0$ (for $i = 1, \ldots, k$). Thus, $(u_1(a), \ldots, u_n(a)) \in K^n$ is a solution of the given system of equations and inequations, proving that K is existentially closed in $K[t]$. □

We deduce:

COROLLARY 1. Let K be any infinite field. If $f(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$ has a non-trivial solution $(u_1, \ldots, u_n)$ (with each $u_i \neq 0$) in a purely transcendental extension of K, then it has already a non-trivial solution in K.

PROOF. Let $g(X_1, \ldots, X_n) = X_1 X_2, \ldots, X_n$. We need only to apply the proposition to the system

$$\begin{cases} f(X_1, \ldots, X_n) = 0\,, \\ g(X_1, \ldots, X_n) \neq 0\,. \end{cases} \qquad \square$$

For example, we may take $f(X, Y, Z) = X^n + Y^n - Z^n$. Noting that this polynomial is homogeneous, we deduce that if Fermat's equation with the exponent n has only the trivial solution in $\mathbb{Z}$, then it has only the trivial solution in any purely transcendental extension of $\mathbb{Q}$.

2. In this respect, much more is known concerning Fermat's equation.

In 1879, Liouville ([3]) proved that if $K = \mathbb{C}$ (the field of complex numbers), if $\mathbb{C}(t)|\mathbb{C}$ is a purely transcendental extension, if $n > 2$ and if $f(t), g(t), h(t) \in \mathbb{C}[t]$ satisfy $f(t)^n + g(t)^n = h(t)^n$ and $gcd(f(t), g(t), h(t)) = 1$ then $f(t), g(t), h(t) \in \mathbb{C}$.

This result was generalized by Greenleaf [2], who showed that it holds when K is any field whose characteristic does not divide the exponent n of Fermat's equation.

Concerning Catalan's equation, Natanson proved in [4] the following result:

PROPOSITION 2. Let m, n be integers greater than 2 and not divisible by the characteristic of the field K. If $f, g \in K(t)$ (purely transcendental extension of K) and $f^m - g^n = 1$ then $f, g \in K$.

We use the very same method to extend Natanson's result to a wider class of equations.

PROPOSITION 3. Let m, n be integers greater than 2 and n not divisible by the characteristic of the field K. Let $P(X) \in K[X]$ have degree m and distinct roots. If $f, g \in K(t)$ (purely transcendental extension of K) and $g^n = P(f)$ then $f, g \in K$.

PROOF. We may assume without loss of generality that K is algebraically closed. Indeed, assuming the proposition true for such fields, if $\bar{K}$ is the algebraic closure of K, then $f, g \in \bar{K} \cap K(t) = K$.

If $f \in K$ then $g^n = c \in K$; since K is algebraically closed, there exists $d \in K$ such that $c = d^n$, hence $g \in K$, because K contains the n-th roots of 1. Similarly, if $g = c \in K$ and $Q(X) = P(X) - c^n$ then f is a root of $Q(X)$; since K is algebraically closed then $f \in K$.

Let

$$P(X) = a_0 X^m + a_1 X^{m-1} + \ldots + a_m = a_0 \prod_{i=1}^{m} (X - r_i) ,$$

where $a_i, r_i \in K$ $(i = 1, \dots, m)$, all the r_i are distinct and $a_0 \in K$, $a_0 \neq 0$.

Let $f = f_1/f_0$, $g = g_1/g_0$ with $f_0, f_1, g_0, g_1 \in K[t]$ and

$$gcd(f_0, f_1) = 1 , \qquad gcd(g_0, g_1) = 1 .$$

Then

$$g_1^m f_0^m = (a_0 f_1^m + a_1 f_1^{m-1} f_0 + \dots + a_{m-1} f_1 f_0^{m-1} + a_m f_0^m) g_0^n .$$

From $gcd(f_0, a_0 f_1^m + a_1 f_1^{m-1} f_0 + \dots + a_m f_0^m) = 1$ it follows that

$$g_0^n = h f_0^m , \qquad \text{with } h \in K[t] .$$

From $gcd(g_0, g_1) = 1$ it follows that

$$a_0 f_1^m + a_1 f_1^{m-1} f_0 + \dots + a_m f_0^m = h' g_1^n , \qquad \text{with } h' \in K[t] .$$

Hence $hh' = 1$, in particular $h, h' \in K$.

Let $d \in K$ be such that $d^n = h'$. Then

$$(dg_1)^n = a_0 \prod_{i=1}^{m} (f_1 - r_i f_0) .$$

Since the roots r_i are all distinct then the polynomials $f_1 - r_i f_0$ are pairwise relatively prime, hence each is a n-th power:

$$f_1 - r_i f_0 = h_i^n \ (i = 1, \dots, m) , \qquad \text{with } h_i \in K[t] .$$

Since $m \geqslant 3$ the elements $f_1 - r_1 f_0$, $f_1 - r_2 f_0$, $f_1 - r_2 f_0$, which are in the K-subspace of $K[t]$ generated by f_0, f_1, must be linearly dependent. So there exist $b_i \in K$ $(i = 1, 2, 3)$ not all equal to 0, such that

$$\sum_{i=1}^{3} b_i (f_1 - r_i f_0) = 0 .$$

Actually b_1, b_2, b_3 are all not zero, since $gcd(f_0, f_1) = 1$.

Let $c_i \in K$ be such that $c_i^n = b_i$ $(i = 1, 2, 3)$. Then

$$(c_1 h_1)^n + (c_2 h_2)^n + (c^r h^r)^n = 0 .$$

By Greenleaf's result on Fermat's equation, quoted above,

$$h_1, h_2, h_3 \in K, \qquad \text{that is } f_1 - r_i f_0 \in K \ (i = 1, 2, 3).$$

This implies that $(r_1 - r_2) f_0 \in K$ hence $f_0, f_1 \in K$ and this is against the hypothesis. $\square$

It is quite easy to provide many applications of the above proposition.

If $P(X) = X^m - 1$ and m is not divisible by the characteristic of K, we have Natanson's result.

If $P(X) = 1 - X^n$ we have Greenleaf's result.

If $P(X) = 1 + X + X^2 + \dots + X^{m-1} + X^m$ and n, $m+1$ are not divisible by the characteristic of K, we may apply the proposition. Etc. ....

3. In this section, we shall indicate some results about valued fields; the valuations are not required to be of height 1.

PROPOSITION 4. Let (K, v) be a valued field which is not henselian, having algebraically closed residue field. If $(\tilde{K}, \tilde{v})$ is a henselian valued field, extension of (K, v), then K is not existentially closed in $\tilde{K}$.

PROOF. Let A_v denote the valuation ring of v, let $\bar{K}$ be the residue field of (K, v). For each polynomial $f \in A_v[X]$ let $\bar{f}$ denote its canonical image in $\bar{K}[X]$.

Since (K, v) is not henselian, there exist monic polynomials $f, g, h \in A_v[X]$ such that $\bar{f} = \bar{g}\bar{h}$, $gcd(\bar{g}, \bar{h}) = 1$, $\deg(g) > 0$, $\deg(h) > 0$, and such that there does not exist polynomials $g', h' \in A_v[X]$ such that $f = g'h'$, $\bar{g}' = \bar{g}$, $\bar{h}' = \bar{h}$, $\deg(g') = \deg(g)$. We choose f of minimal degree with the above property.

We show that f has no roots in K. Indeed, if $b \in K$ and $f(b) = 0$ then b is integral over A_v, hence $b \in A_v$. So $f = (X - b)^r f_1$, with $r \geq 1$, $f_1 \in K[X]$ and $f_1(b) \neq 0$; in particular, f_1 is monic. Let v^* be the natural extension of v to $K(X)$, defined by

$$v^*(a_0 X^m + a_1 X^{m-1} + \dots + a_m) = \min_{0 \leq i \leq m} \{v(a_i)\}.$$

Then $v^*(f) = 0$, $v^*(X - b) = 0$ so $v^*(f_1) = 0$, thus $f_1 \in A_v[X]$. We have therefore $\bar{g}\bar{h} = \bar{f} = (X - \bar{b})^r \bar{f}_1$ and, say, $\bar{h}(\bar{b}) = 0$, hence $\bar{g}(\bar{b}) \neq 0$. So $\bar{h} = (X - \bar{b})^r \bar{k}$ where $k \in A_v[X]$ is a monic polynomial. Therefore $\bar{f}_1 = \bar{g}\bar{k}$, with $gcd(\bar{g}, \bar{k}) = 1$.

We have $\deg \bar{k} > 0$. Indeed if $\deg \bar{k} = 0$ then $k = 1$ so

$$f = (X - b)^r f_1 , \qquad \text{with } (X - \bar{b})^r = \bar{h} \, , \; \bar{f}_1 = \bar{g} \, ,$$

which is against the hypothesis. By the minimality of f, there exist polynomials g_1', $k_1' \in A_v[X]$, such that

$$f_1 = g_1' k_1' , \quad \bar{g}_1' = \bar{g} , \quad \bar{k}_1' = \bar{k} , \quad \deg(g_1') = \deg(g) .$$

It follows that $f = g_1 (X - b)^r k_1'$ with g_1',

$$(X - b)^r k_1' \in A_v[X] \, , \; \bar{g}_1' = \bar{g} \, , \; (X - \bar{b})^r \bar{k}_1' = \bar{h} \, , \; \deg(g_1') = \deg(g) \, ,$$

which is a contradiction. So f has no roots in K.

On the other hand, the residue field $\bar{\tilde{K}}$ contains $\bar{K}$, which is algebraically closed. Since $\bar{f}$ has a root in $\bar{K} \subseteq \bar{\tilde{K}}$ and $(\tilde{K}, \tilde{v})$ is henselian, then f has a root in $\tilde{K}$.

This shows that K is not existentially closed in $\tilde{K}$. □

PROPOSITION 5. Let (K, v) be a valued field which is not henselian. If $(\tilde{K}, \tilde{v})$ is a henselian valued field, extension of (K, v), then $\tilde{K}|K$ is not a purely transcendental extension.

PROOF. We assume first that the residue field $\bar{K}$ is algebraically closed. By Proposition 4, K is not existentially closed in $\tilde{K}$. Since K is not finite (otherwise v is trivial and (K, v) would be henselian), by Proposition 1, $\tilde{K}|K$ is not a purely transcendental extension.

Now we assume that $\bar{K}$ is not algebraically closed. Let $\bar{K}^a$ denote the algebraic closure of $\bar{K}$. We claim:

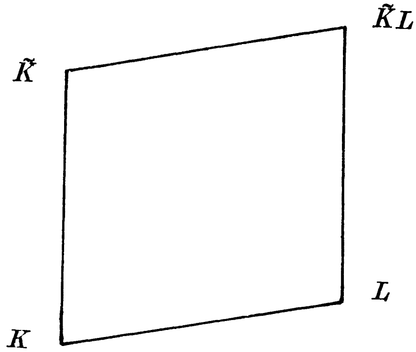
(*) There exists an algebraic extension $L|K$ such that:

1) v has a unique extension w to L,

2) $\bar{L} = \bar{K}^a$.

Assuming (*), we continue the proof. If $\tilde{K}|K$ is a purely transcendental extension, then $\tilde{K}L|L$ is purely transcendental and $\tilde{K}L|\tilde{K}$

is algebraic. Let $\tilde{w}$ be the unique extension of $\tilde{v}$ to $\tilde{K}L$, so $(\tilde{K}L, \tilde{w})$ is again a henselian valued field. Moreover, the restriction of $\tilde{w}$ to L must be equal to w, which is the only extension of v to L.



Now we observe that (L, w) is not henselian. Indeed, since (K, v) is not henselian, there exist at least two distinct extensions v_1^*, v_2^* of v to the algebraic closure K^a of K; we may assume $K^a \supseteq L$. Since the restrictions of v_1^*, v_2^* must be equal to w then (L, w) is not henselian.

Since $\bar{L}$ is algebraically closed, by Proposition 4 L is not existentially closed in $\tilde{K}L$, hence by Proposition 1 $\tilde{K}L/L$ is not purely transcendental, which is a contradiction.

It remains to establish the claim $(*)$, which is in fact well-known. We include the proof for completeness.

Consider the family of all algebraic extensions L of K (contained in a given algebraic closure K^a), such that

1) v has a unique extension w to L,

2) $\bar{L} \subseteq \overline{K^a}$.

It is immediate that this family has a maximal element, which we still denote by L. We show that $\bar{L} = \overline{K^a}$.

If there exists $\gamma \in \overline{K^a}$, $\gamma \notin \bar{L}$, let $f \in A_w[X]$ be a monic polynomial such that $\bar{f} \in \bar{L}[X]$ is the minimal polynomial of γ over $\bar{L}$; let $n = \deg(\bar{f}) > 1$. Therefore f is irreducible in $A_w[X]$, and since A_w is a Bézout domain, f is also irreducible in $L[X]$.

Let $c \in L^a$ (algebraic closure of L) be a root of f, let $L' = L(c)$ and let w' be any extension of w to L'. Then $w'(c) \geqslant 0$, because $f \in A_w[X]$ and f is monic. Thus the residue field $\bar{L}'$ contains $\bar{L}(\bar{c})$. From $\bar{f}(\bar{c}) = 0$, it follows that $[\bar{L}(\bar{c}):\bar{L}] = n$ so $[\bar{L}':\bar{L}] \geqslant n$. This implies that $\bar{L}' = \bar{L}(\bar{c})$ and w' is the only extension of w to L'.

From the decomposition of f into linear factors (in its splitting field L''), $f = \prod_{i=1}^{n} (X - c_i)$ if w'' is a valuation of L'' extending v, then each $c_i \in A_{w''}$. Hence $\bar{f} = \prod_{i=1}^{n} (X - \bar{c}_i)$, so $\bar{c}_1, \ldots, \bar{c}_n$ are all the roots of $\bar{f}$, hence there exists i such that $\bar{c}_i = \gamma$. The above consideration (with $c = c_i$) shows that $\bar{L}' = \bar{L}(\gamma) \subseteq \bar{K}^a$, against the maximality of L. This concludes the proof. $\square$

As corollaries, we have the following results already established in [6]:

Corollary 1. If K is any field, then $K((X))$ is not a purely transcendental extension of $K(X)$.

Proof. The field $K(X)$ is not henselian with respect to the X-adic valuation. On the other hand, $K((X))$ is the completion of $K(X)$, relative to the X-adic valuation. So it is a henselian field, with respect to the natural extension of the X-adic valuation. By the proposition, $K((X))$ is not a purely transcendental extension of $K(X)$. $\square$

Corollary 2. If p is any prime number, the field $\mathbf{Q}_p$ of p-adic numbers is not a purely transcendental extension of $\mathbf{Q}$.

Proof. $\mathbf{Q}_p$ is the completion of $\mathbf{Q}$, with respect to the p-adic valuation. Since $\mathbf{Q}$ is not henselian, while $\mathbf{Q}_p$ is henselian (with respect to the p-adic valuation), then $\mathbf{Q}_p$ is not a purely transcendental extension of $\mathbf{Q}$. $\square$

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