Rendiconti
del
Seminario Matematico
della
Università di Padova

Daniel Simson

Right pure semisimple l-hereditary PI-rings

Rendiconti del Seminario Matematico della Università di Padova, tome 71 (1984), p. 141-175

<http://www.numdam.org/item?id=RSMUP_1984__71__141_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1984, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

Numdam

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Rend. Sem. Mat. Univ. Padova, Vol. 71 (1984)

Right Pure Semisimple l-Hereditary PI-Rings.

Daniel Simson (*) (**)

We recall from [14] that an artinian ring is said to be *l-hereditary* if any local one-sided ideal of R is projective (l for local ideals). A ring R is *right pure semisimple* if every right R-module is a direct sum of finitely generated modules (see [24, 25, 28]). We say that a ring R is a *PI-ring* if R satisfies a polynomial identity. It is well-known that an artinian ring R is a PI-ring if and only if the endomorphism ring of any simple R-module is finite dimensional over its its center.

There is an open problem if any right pure semisimple ring is of finite representation type (see [27, 28]). In [28] the problem was solved for a class of hereditary rings including hereditary PI-rings.

In the present paper we give a positive solution of the problem for l-hereditary PI-rings. In particular we show that the Bautista's diagrammatic characterization of l-hereditary artin algebras of finite representation type [3, 4] remains also true for l-hereditary PI-rings. A complete list of indecomposable modules is given for any non-homogeneous l-hereditary PI-ring of finite representation type.

We recall from [5, 7] that a module M is said to be *l-hereditary* if all local submodules of M are projective. It is easy to see that if R is an l-hereditary right QF-2 artinian ring then an R-module M is l-hereditary if and only if soc (M) is projective.

We recall that an artinian ring R is said to be a *right QF-2 ring* if every indecomposable projective right ideal in R has a simple socle.

(*) The author was partially supported by Consiglio Nazionale delle Ricerche, Comitato Nazionale per le Scienze Matematiche.

(**) Indirizzo dell'A.: Institute of Mathematics, Nicholas Copernicus University ul. Chopina 12/18, 87-100 Toruń, Poland.

The main tool we use in the study of right pure semisimple l-hereditary PI-rings are special Schurian vector space categories $\mathbf{K}_F$ and the functor

$$\Phi\colon \mathfrak{U}(\mathbf{K}_F) \to l\,\mathrm{her}\,(\boldsymbol{R}_{\mathbf{K}})$$

defined in [31] where $\boldsymbol{R}_K$ is an l-hereditary right QF-2 semiperfect ring associated to $\mathbf{K}_F$ and $l\,\mathrm{her}\,(\boldsymbol{R}_{\mathbf{K}})$ is the category of finitely generated right $\boldsymbol{R}_{\mathbf{K}}$-modules (see § 1). The general idea of the proof is similar to that one in [20] and is the following. For any right pure semisimple l-hereditary PI-ring R we construct two l-hereditary PI-rings R' and R'' such that

1) R' is a proper factor ring of R;

2) R'' is a right QF-2 artinian ring of the form $\boldsymbol{R}_{\mathbf{K}}$ where $\mathbf{K}_F$ is a special Schurian vector space category associated to R;

3) $\#\,\mathrm{mod}\,(R) = \#\,\mathrm{mod}\,(R') + \#\,l\,\mathrm{her}\,(R'')$ where $\#\,\mathfrak{B}$ denotes the number of pairwise nonisomorphic indecomposable objects in $\mathfrak{B}$.

Since by an inductive assumption we can suppose that $\#\,\mathrm{mod}\,(R')$ is finite then the problem is reduced to the question if $\#\,l\,\mathrm{her}\,(R'')$ is finite. Fortunately there is only few types of l-hereditary right QF-2 artinian PI-rings which appear as the rings R'' above and for any such a ring we are able to determine its indecomposable l-hereditary modules. We do it again by applying to R'' the above reduction method and by using some of the results of Dlab and Ringel [8, 9].

The paper essentially depends on the results of Bautista [3, 4], Bautista and Martinez [5], Dlab and Ringel [8, 9], Dowbor and Simson [10], Loupias [15, 16] and of the author [24, 25, 28].

The results were announced in [30] and a part of them was presented during the Trento Meeting on Abelian Groups and Modules Theory in May 1980.

Throughout this paper R denotes an indecomposable basic l-hereditary artinian ring of the triangular matrix form

$$R = \begin{bmatrix} F_1 & {}_1M_2 & \dots & {}_1M_{n-1} & {}_1M_n \\ & F_2 & \dots & {}_2M_{n-1} & {}_2M_n \\ & & \ddots & \vdots & \vdots \\ & 0 & & F_{n-1} & {}_{n-1}M_n \\ & & & & F_n \end{bmatrix}$$

where $F_1, \dots, F_n$ are division rings, ${}_iM_j$ are F_i-F_j-bimodules finite dimensional over F_i and over F_j, and the multiplication in R is given by F_i-F_j-bimodule maps

$$c_{ijk}\colon {}_iM_j \otimes {}_jM_k \to {}_iM_k$$

with the property that $c_{ijk}(a \otimes b) = 0$ if and only if either $a = 0$ or $b = 0$. We know from [14, Lemma 1] that every l-hereditary basic artinian ring R has such a triangular matrix form.

We associate with R a *valued poset* $(I_R, \boldsymbol{d})$ where $I_R = \{1, \dots, n\}$, $i < j \Leftrightarrow {}_iM_j \neq$ and $\boldsymbol{d} = (d_{ij})$ is a matrix with

$$d_{ij} = \dim\,({}_iM_j)_{F_j}\,, \qquad d_{ji} = \dim_{F_i}({}_iM_j) \qquad \text{for } i \neq j\,.$$

We will write

$$\overset{\bullet}{i} \xrightarrow{(d_{ij},\, d_{ji})} \overset{\bullet}{j}$$

if $i < j$ and there is no k in I_R such that $i < k < j$; if $d_{ij} = d_{ji} = 1$ we write simply $i \to j$. The valued poset $(I_R, \boldsymbol{d})$ is said to be *homogeneous* if $d_{ij} \neq 0$ implies $d_{ij} = 1$.

It is easy to see that an indecomposable l-hereditary artinian ring R is a right QF-2 ring if and only if the valued poset $(I_R, \boldsymbol{d})$ has a unique maximal element m and $d_{jm} = 1$ for every j.

Throughout this paper $E(X)$ denotes the injective envelope of the module X and $P(X)$ denotes the projective cover of X. We denote by mod (R) the category of finitely generated right R-modules and by l her (R) the category of finitely generated l-hereditary right R-modules. The reader is referred to [5, 7] for basic properties of l-hereditary modules.

1. Preliminaries.

We recall from [31] that an additive category $\mathbf{K}$ together with an additive faithful functor

$$|-|\colon \mathbf{K} \to \operatorname{mod}(F)\,,$$

where F is a division ring, is denoted by $\mathbf{K}_F$ and it is called *special*

Schurian vector space category if $\mathbf{K}$ is a Krull-Schmidt category, $\mathbf{K}$ has only a finite number of pairwise nonisomorphic indecomposable objects and $\dim |X|_F = 1$ as well as $\operatorname{End}(X)$ is a division ring for any indecomposable object X in $\mathbf{K}$.

We recall from [20, 21] that the *subspace category* $\mathcal{U}(\mathbf{K}_F)$ of $\mathbf{K}_F$ is defined as follows. The objects of $\mathcal{U}(\mathbf{K}_F)$ are triples (U, X, φ) where U is a finite dimensional vector space over F, X is an object in $\mathbf{K}$ and $\varphi: U \to |X|_F$ is an F-linear map. The map from (U, X, φ) into (U', X', φ') in $\mathcal{U}(\mathbf{K}_F)$ is a pair (u, h) where $u \in \operatorname{Hom}_F(U, U')$ and $h: X \to X'$ is a map in $\mathbf{K}$ such that $|h|\varphi = \varphi' u$. It is clear that $\mathcal{U}(\mathbf{K}_F)$ is an additive Krull-Schmidt category.

Throughout we suppose that $\mathbf{K}_F$ is a special Schurian vector space category, we fix a complete set $X_1, \ldots, X_n$ of pairwise nonisomorphic indecomposable objects in $\mathbf{K}$ and we put

$$F_{n+1} = F \quad \text{and} \quad F_i = \operatorname{End}(X_i) \quad \text{for } i = 1, \ldots, n\,.$$

For any $i, j \leqslant n+1$ we consider F_i-F_j-bimodules defined by formula

$$\begin{aligned} {}_iN_j &= \operatorname{Hom}(X_j, X_i) && \text{for } i, j \leqslant n\,, \\ &= {}_{F_i}|X_i|_F && \text{for } j = n+1\,. \end{aligned}$$

Since $\dim |X_i|_F = 1$ and F_i are division rings for all i then ${}_iN_j \neq 0$ implies ${}_jN_i = 0$. Therefore without loss of generality we can suppose that $i \leqslant j$ whenever ${}_iN_j \neq 0$.

In [31] we have associated to $\mathbf{K}_F$ the triangular matrix ring

$$R_{\mathbf{K}} = \begin{bmatrix} F_1 & {}_1N_2 & \cdots & {}_1N_n & {}_1N_{n+1} \\ & F_2 & \cdots & {}_2N_n & {}_2N_{n+1} \\ & & \ddots & \vdots & \vdots \\ & 0 & & F_n & {}_nN_{n+1} \\ & & & & F_{n+1} \end{bmatrix}$$

where the multiplication is given by F_i-F_k-bilinear maps

$$c_{ijk}\colon {}_iN_j \otimes {}_jN_k \to {}_iN_k \left(\otimes = \otimes_{F_j}\right)$$

defined by the formula

$$c_{ijk}(f \otimes g) = fg \quad \text{for } k \leqslant n\,,$$
$$= |f|(g) \quad \text{for } k = n+1\,.$$

We denote by $\mathcal{U}_0(\mathbf{K}_F)$ (resp. by $l\,\mathrm{her}_0\,(\boldsymbol{R}_\mathbf{K})$) the full subcategory of $\mathcal{U}(\mathbf{K}_F)$ (resp. of $l\,\mathrm{her}\,(\boldsymbol{R}_\mathbf{K})$) consisting of objects having no direct summands of the form $(F, 0, 0)$ (resp. no simple projective summands).

We have the following result proved in [31].

THEOREM 1.1. *Let $\mathbf{K}_F$ be a special Schurian vector space category such that the ring $\boldsymbol{R}_\mathbf{K}$ is artinian. Then $\boldsymbol{R}_\mathbf{K}$ is an l-hereditary QF-2 ring and there exists a full and dense additive functor*

$$\Phi\colon \mathcal{U}(\mathbf{K}_F) \to l\,\mathrm{her}_0\,(\boldsymbol{R}_\mathbf{K})$$

with the following properties:

(*a*) *If A is an indecomposable object in $\mathcal{U}(\mathbf{K}_F)$ then $\Phi(A) = 0$ if and only if A has one of the following forms $(F, 0, 0)$, $\hat{X}_i = (|X_i|, X_i, id)$, $i = 1, \ldots, n$.*

(*b*) *If A and B are objects in $\mathcal{U}(\mathbf{K}_F)$ having no summands of the form $(F, 0, 0)$, $\hat{X}_1, \ldots, \hat{X}_n$ then every isomorphism from $\Phi(A)$ into $\Phi(B)$ has the form $\Phi(h)$ where $h\colon A \to B$ is an isomorphism.*

Theorem 1.1 will be used in the next Section.

2. Indecomposable modules over l-hereditary artinian PI-rings.

Throughout this section R will denote a basic l-hereditary artinian PI-ring of the triangular matrix form as in the introduction. We keep the terminology and notation in the introduction and Section 1.

A module X over R will be identified with a system $(X_i, {}_j\varphi_i)$ where X_i is a vector space over F_i and ${}_j\varphi_i\colon X_i \otimes {}_iM_j \to X_j$, $j = 1, \ldots, n$, are F_j-linear maps satisfying the usual commutativity and associativity conditions (see [26, Sec. 3]). If no confusion will arrises we will write simply (X_i) instead of $(X_i, {}_j\varphi_i)$.

For every j we consider the simple R-module $\tilde{F}_j = (X_i)$ with

$X_j = F_j$ and $X_i = 0$ for $i \neq j$. It is clear that $\tilde{F}_1, \ldots, \tilde{F}_{n+1}$ is a complete list of simple R-modules.

A full additive subcategory $\mathcal{A}$ of mod (R) is said to be *cofinite* if all but a finite number of indecomposable objects in mod (R) belong to $\mathcal{A}$. We denote by $\# \mathcal{A}$ the number of isomorphisms classes of indecomposable objects in $\mathcal{A}$.

Finally, we recall that an additive functor is a *representation equivalence* if it is full, dense and reflects isomorphisms.

Throughout this section $\boldsymbol{R}(\mathrm{K})$ denotes the ring $\boldsymbol{R}_{\mathrm{K}}$ associated to the vector space category K_F.

The aim of this section is to prove Theorem 2.5. Before we formulate it we prove some preliminary results.

The following simple lemma will be frequently used in this paper.

LEMMA 2.1. *Let R be an l-hereditary right QF-2 artinian ring and let m be the unique maximal element in the valued poset $(I_R, \boldsymbol{d})$.*

(i) *If $(I_R, \boldsymbol{d})$ has a unique minimal element a such that $d_{am} = d_{ma} = 1$ and $R = P(\tilde{F}_a) \oplus P'$ then $R' = \mathrm{End}\,(P')$ is an l-hereditary right QF-2 artinian and $\# l\,\mathrm{her}\,(R) = 1 + \# l\,\mathrm{her}\,(R')$.*

(ii) *Suppose that $(I_R, \boldsymbol{d})$ has an element m' such that there is an arrow $m' \to m$ in $(I_R, \boldsymbol{d})$ and $s \leqslant m'$ for all s in I_R, $s \neq m$. Let $R = \tilde{F}_m \oplus P''$. Then $R'' = \mathrm{End}\,(P'')$ is an l-hereditary right QF-2 artinian ring and $\# l\,\mathrm{her}\,(R) = 1 + \# l\,\mathrm{her}\,(R'')$.*

PROOF. (i) If $X = (X_i)$ is an indecomposable module in $l\,\mathrm{her}\,(R)$ with $X_a \neq 0$ then there is an epimorphism $X \to \tilde{F}_a$. Hence $X = P(\tilde{F}_a)$ because the module $P(\tilde{F}_a)$ is projective-injective. Consequently (i) follows.

(ii) Consider the functor $T\colon l\,\mathrm{her}\,(R'') \to l\,\mathrm{her}\,(R)$ defined by $(X_i, {}_j\varphi_i)_{i,j \leqslant m'} \mapsto (Y_i, {}_j\psi_i)_{i,j \leqslant m}$ with $Y_i = X_i$ for $i \leqslant m'$ and $Y_m = X_m$, ${}_m\psi_{m'} = id$. It is easy to see that every indecomposable module in $l\,\mathrm{her}\,(R)$ except the simple projective module $\tilde{F}_m$ belongs to the image of T and the proof is complete.

Following Ringel [22] we define vector space categories arrising from simple injective and simple projective modules. Let R be an l-hereditary artinian ring and $(I_R, \boldsymbol{d})$ the valued poset of R. Given a minimal point a in I_R (corresponding to the simple injective module $\tilde{F}_a$) we take the projective cover $P(\tilde{F}_a)$ and consider a decomposition

$R = P(\tilde{F}_a) \oplus P'$. We put

$$R_a = \mathrm{End}\,(P')\,, \qquad M^a = \mathrm{rad}\, P(\tilde{F}_a) \quad \text{and} \quad F = F_a\,.$$

Then M^a is an F-R_a-bimodule and there is a ring isomorphism

$$R = \begin{bmatrix} F & {}_F M^a_{R_a} \\ 0 & R_a \end{bmatrix}.$$

Since R is l-hereditary the R_a-module M^a is l-hereditary and we have two vector space categories

$$\mathbf{K}^a_F = \mathrm{Hom}_{R_a}\big(M^a, \mathrm{mod}\,(R_a)\big) \quad \text{and} \quad \tilde{\mathbf{K}}^a_F = \mathrm{Hom}_{R_a}\big(M^a, l\,\mathrm{her}\,(R_a)\big)$$

which are the images of the categories $\mathrm{mod}\,(R_a)$ and $l\,\mathrm{her}\,(R_a)$ under the functor $\mathrm{Hom}_{R_a}(M^a, -)$ (see [22, p. 200]). Let us denote by $\mathrm{mod}\,(R)_a$ the full subcategory of $\mathrm{mod}\,(R)$ consisting of modules having no summands isomorphic to $\tilde{F}_a$. We have the following useful result.

PROPOSITION 2.2. *If the vector space category* $\mathbf{K}^a_F$ *is special Schurian and* $\boldsymbol{R}(\mathbf{K}^a)$ *is the l-hereditary right QF-2 artinian ring associated to* $\mathbf{K}^a_F$ *then*

(i) *There exists a full additive functor* $H\colon \mathrm{mod}\,(R) \to l\,\mathrm{her}\,\big(\boldsymbol{R}(\mathbf{K}^a)\big)$ *which establishes a representation equivalence between a subcategory of* $\mathrm{mod}\,(R)$ *and a cofinite subcategory of* $l\,\mathrm{her}\,\big(\boldsymbol{R}(\mathbf{K}^a)\big)$.

(ii) $\#\,\mathrm{mod}\,(R) = \#\,\mathrm{mod}\,(R_a) + \#\,l\,\mathrm{her}\,\big(\boldsymbol{R}(\mathbf{K}^a)\big)$.

If $\tilde{\mathbf{K}}^a_F$ *is special Schurian then*

$$\#\,l\,\mathrm{her}\,(R) = \#\,l\,\mathrm{her}\,(R_a) + \#\,l\,\mathrm{her}\,\big(\boldsymbol{R}(\tilde{\mathbf{K}}^a)\big) - 1\,.$$

PROOF. It is easy to see that $\mathrm{mod}\,(R)$ (resp. $l\,\mathrm{her}\,(R)$) is equivalent to the category of triples (V_F, X, t) where X is a finitely generated right R_a-module (resp. l-hereditary R_a-module) and $t\colon V \otimes_F M^a \to X$ is an R_a-homomorphism (resp. such that the map $t'\colon V_F \to \mathrm{Hom}_{R_a}({}_F M^a, X)$ adjoint to t is injective). Let H be the composed functor

$$\mathrm{mod}\,(R)_a \xrightarrow{H'} \mathcal{U}(\mathbf{K}^a_F) \xrightarrow{\Phi} l\,\mathrm{her}\,\big(\boldsymbol{R}(\mathbf{K}^a)\big)$$

where H' is given by $(V_F, X, t) \mapsto \big(V_F, \mathrm{Hom}_{R_a}(M^a, X), t'\big)$. It follows from [22, § 2.5, Lemma 2] and from the properties of the functor $\dot{\Phi}$ that H has the required properties. Futhermore, if s is the number of all indecomposable modules X such that $\mathrm{Hom}_{R_a}(M^a, X) \neq 0$ then obviously $\# \mathrm{mod}\,(R_a) = s + \# \mathbb{K}_F^a$. Hence the equality (ii) is a consequence of Theorem 1.1 and [22, § 2.5, Lemma 2]. The second equality follows in a similar way.

Now we assume that c is a maximal point in $(I_R, \boldsymbol{d})$. Then the simple module $\tilde{F}_c$ is projective and we have a right module decomposition $R = \tilde{F}_c \oplus P''$. Hence there is a ring isomorphism

$$R = \begin{bmatrix} {}_cR & N^c \\ 0 & F \end{bmatrix}$$

where ${}_cR = \mathrm{End}\,(P'')$, $N^c = \mathrm{Hom}_R\,(\tilde{F}_c, P'')$ and $F = F_c$. We define a vector space category

$$\mathbb{L}_F^c = \mathrm{mod}\,({}_cR) \otimes_{{}_cR} N_F^c$$

as the image of $\mathrm{mod}\,({}_cR)$ under the functor $- \otimes_{{}_cR} N_F^c$.

Proposition 2.3. *If* $\mathbb{L}_F^c$ *is special Schurian then*

(i) *There exists a full additive functor* $H\colon \mathrm{mod}\,(R) \to l\,\mathrm{her}\,(\boldsymbol{R}(\mathbb{L}^c))$ *which establishes a representation equivalence between an additive subcategory of* $\mathrm{mod}\,(R)$ *and a cofinite additive subcategory of* $l\,\mathrm{her}\,(\boldsymbol{R}(\mathbb{L}^c))$.

(ii) $\# \mathrm{mod}\,(R) = \# \mathrm{mod}\,({}_cR) + \# l\,\mathrm{her}\,(\boldsymbol{R}(\mathbb{L}^c))$.

Proof. It is clear that $\mathrm{mod}\,(R)$ is equivalent to the category of triples (X, V_F, t) where X is in $\mathrm{mod}\,({}_cR)$ and $t\colon X \otimes_{{}_cR} N_F^c \to V_F$ is a linear map. Since the category $\mho(\mathbb{L}_F^c)$ is equivalent to the category of triples (X, V_F, t) where $t\colon |X|_F \to V_F$ is a linear map and $|X|_F = X \otimes_{{}_cR} N_F^c$ then there is a full and dense additive functor $H'\colon \mathrm{mod}\,(R) \to \mho(\mathbb{L}_F^c)$ such that the composition of H' and the functor $\mathfrak{G}\colon \mho(\mathbb{L}_F^c) \to l\,\mathrm{her}\,(\boldsymbol{R}(\mathbb{L}^c))$ has the properties required for H. The equality (ii) can be proved similarly as (ii) in Proposition 2.2 (see [31; 1.5]).

In our discussion of pure semisimple rings we will need the following result.

Proposition 2.4. *Suppose R and S are right artinian rings,* $\mathfrak{B}$ *is a full additive subcategory of* $\mathrm{mod}\,(R)$ *and let* $T\colon \mathfrak{B} \to \mathrm{mod}\,(S)$ *be an* *additive functor which is full and reflects isomorphisms. If* $\operatorname{Im} T$ *is cofinite in* $\operatorname{mod}(S)$ *and the ring* R *is right pure semisimple then* S *is right pure semisimple too.*

PROOF. Let $X_1 \xrightarrow{f_1} X_2 \to \ldots \to X_n \xrightarrow{f_n} X_{n+1} \to \ldots$ be a sequence of nonzero monomorphisms between indecomposable modules X_j in $\operatorname{mod}(S)$. We shall show that there is an integer m such that f_j is an isomorphism for $j > m$. This is obviously the case whenever there are infinitely many indices j such that X_j does not belong to $\operatorname{Im} T$. Then we can suppose that X_j belongs to $\operatorname{Im} T$ for all j. By our assumption there is a sequence in $\operatorname{mod}(R)$

$$Y_1 \xrightarrow{g_1} Y_2 \to \ldots \to Y_n \xrightarrow{g_n} Y_{n+1} \to \ldots$$

where Y_j is indecomposable, $T(Y_j) = X_j$ and $T(g_j) = f_j$ for all j. Since R is right pure semisimple then by [29, Theorem 1.3] and [25, Theorem 6.3] there is an integer m such that g_j are isomorphisms for $j > m$. Hence our claim follows and by [29, Theorem 1.3] S is right pure semisimple as we required.

Following [15] we call a surjective map $f\colon (I_R, \boldsymbol{d}) \to (I_{R'}, \boldsymbol{d}')$ of valued posets a *contraction* if $f^{-1}(j)$ is connected an homogeneous for all j in $I_{R'}$.

Now we are able to prove the main result of this section.

THEOREM 2.5. *Let* R *be a basic l-hereditary artinian PI-ring and suppose that the valued poset* $(I_R, \boldsymbol{d})$ *is connected. Then the following statements are equivalent*:

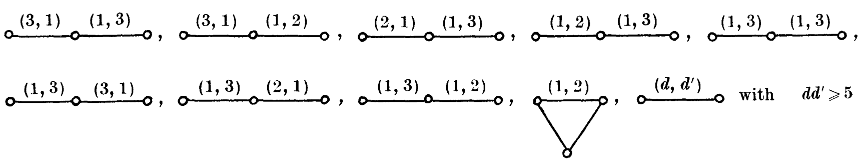
(1) R *is of finite representation type.*

(2) R *is right pure semisimple.*

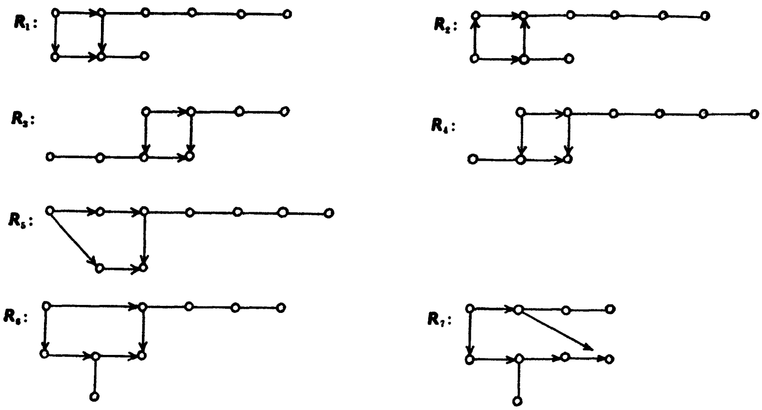
(3) *Neither* $(I_R, \boldsymbol{d})$ *nor its dual has no contractions and no full subposets of one of the following forms*:

a) the extended Dynkin diagrams [9];

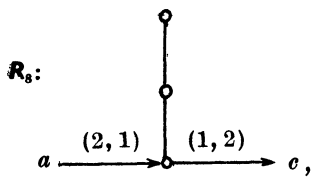
b) the minimal wild valued graphs:

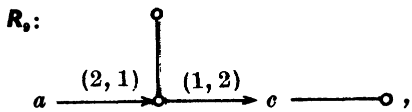


c) *the crucial Loupias' homogeneous posets* [15, 16]:



d) *the critical Bautista's valued posets* [4]:





with $d_{ac} = d_{ca} = 1$.

(4) *The valued poset* $(I_R, \boldsymbol{d})$ *or its dual is of one of the forms*:

a) *the Dynkin diagrams* [9];

b) *the Loupias' homogeneous posets of finite representation type* [16];

c) *the Bautista's valued posets of finite representation type* [4]:

${}_nC_n$: $1 \xrightarrow{(2,1)} 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow n \xrightarrow{(1,2)} n+1, \ d_{1n+1} = d_{n+11} = 1,$

${}_nBB_m$: $-n \longrightarrow -(n-1) \longrightarrow \dots \longrightarrow 0 \xrightarrow{(2,1)} 1 \xrightarrow{(1,2)} 2 \longrightarrow \dots \longrightarrow m, \ d_{02} = d_{20} = 1,$

${}_n^sBB^m$:
$$\begin{array}{ccccccccccc}
-n & \longrightarrow -(n-1) \longrightarrow & \dots & \longrightarrow 0 & \xrightarrow{(2,1)} & 1 & \xrightarrow{(1,2)} & 2 \longrightarrow \dots \longrightarrow & m \\
\uparrow & & & & & & & & \uparrow \\
m+s+1 & \longrightarrow m+s \longrightarrow & \dots & & & & & \dots \longrightarrow m+2 \longrightarrow & m+1
\end{array}, \ d_{02} = d_{20} = 1,$$

${}^{n}B_{s}B^{m}$:

$$\begin{tikzcd}
-(n+1) \arrow[r] \arrow[drrr] & -n \arrow[r] & \cdots \arrow[r] & 0 \arrow[r, "{(2,1)}"] & 1 \arrow[r, "{(1,2)}"] & 2 \arrow[r] & \cdots \arrow[r] & m \\
 & & & m+1 \arrow[urrrr] \arrow[d, dash] & & & & \\
 & & & \vdots \arrow[d, dash] & & & & \\
 & & & m+s & & & &
\end{tikzcd}$$

$m+s\,,\quad d_{uv} = d_{vu} = 1\,,$

C_3C:

$$\begin{tikzcd}
a \arrow[r, "{(2,1)}"] & \circ \arrow[r, "{(1,2)}"] \arrow[d, dash] & c \\
 & \circ &
\end{tikzcd}$$

$,\quad d_{ac} = d_{ca} = 1\,,$

$\widehat{C_3C}$:

$$\begin{tikzcd}
2 \arrow[r, "{(2,1)}"] & 4 \arrow[r, "{(1,2)}"] & 5 \\
1 \arrow[u] \arrow[r, "{(2,1)}"] & 3 \arrow[u] &
\end{tikzcd}$$

$,\quad d_{25} = d_{52} = 1\,,$

${}_3BC_4$:

$$\begin{tikzcd}
1 \arrow[r, "{(2,1)}"] \arrow[d] & 2 \arrow[dr] & \\
 & & 3 \arrow[dl] \\
5 \arrow[r, "{(2,1)}"] & 4 &
\end{tikzcd}$$

$,$

${}_3BF_4$:

$$\begin{tikzcd}
 & 0 \arrow[r, "{(2,1)}"] \arrow[dl] & 4 \arrow[dd] \\
1 \arrow[dr] & & \\
 & 2 \arrow[r, "{(2,1)}"] & 3
\end{tikzcd}$$

$,$

${}_2G_2$:

$$\begin{tikzcd}
a \arrow[r, "{(3,1)}"] & \circ \arrow[r, "{(1,3)}"] & c
\end{tikzcd}$$

$d_{ac} = d_{ca} =$

${}_4F_4$:

$$\begin{tikzcd}
 & a \arrow[r, "{(2,1)}"] \arrow[dl] & \circ \arrow[d] \\
\circ \arrow[r] & c & \circ \arrow[l, "{(2,1)}"']
\end{tikzcd}$$

$,\quad d_{ac} = d_{ca} = 1\,,$

F_4F:

$$\begin{tikzcd}
\circ \arrow[r, dash] & a \arrow[r, "{(2,1)}"] & \circ \arrow[r] & \circ \arrow[r, "{(1,2)}"] & c \arrow[r, dash] & \circ
\end{tikzcd}$$

$,\quad d_{ac} = d_{ca} = 1$

${}_4FF_4$:

$$\begin{tikzcd}
 & \circ \arrow[d, dash] & \\
a \arrow[r, "{(2,1)}"] \arrow[dr] & \circ \arrow[r, "{(1,2)}"] & c \\
 & \circ \arrow[ur] &
\end{tikzcd}$$

$,\quad d_{ac} = d_{ca} = 1\,,$

where $\circ - \circ$ means either $\circ \to \circ$ or $\circ \leftarrow \circ$.

REMARK 1. Let us recall that Bautista [3, 4] has proved the equivalence of the statements (1), (3) and (4) under the assumption that R is a finite dimensional l-hereditary algebra over a field. He succesfuly applies the almost split sequences. We can not follow the Bautista's proof in our more general situation because usually we have no almost split sequences. We will apply Propositions 2.2 and 2.3.

The proof of the implication (4)$\to$(1) will be divided into several lemmas. We start with the following useful remark.

REMARK 2. If F and G are division rings, and ${}_F M_G$ is an F-G-bimodule such that $\dim_F M = 1$, then for any nonzero element m in M there are a ring isomorphism $s\colon G \to F$ defined by $mx = s(x)m$ and an F-G-bimodule isomorphism ${}_F M_G \cong {}_F F_G$. If, in addition, $\dim M_G = 1$ then s is an isomorphism [3, 4].

LEMMA 2.6. *Let R be an l-hereditary artinian PI-ring. If $(I_R, \boldsymbol{d})$ is of one of the forms ${}_n\mathbf{C}_n$, ${}_2\mathbf{G}_2$, ${}_4\mathbf{F}_4$ then $\#\,\mathrm{mod}\,(R)$ is equal $n^2 + \frac{1}{2}n(n+1) + 1$, 12 and 45, respectively. Moreover, if $X = (X_i)$ is an indecomposable non projective-injective module then $X_e = 0$ provided that e is either a maximal or a minimal element in I_R.*

PROOF. Suppose $(I_R, \boldsymbol{d})$ has the form ${}_n\mathbf{C}_n$. It follows from Remark 2 that without loss of generality we can suppose that R has the form

$$R = \begin{bmatrix} F & F & F & \dots & F & F \\ & G & G & \dots & G & F \\ & & G & \dots & G & F \\ & & & \ddots & \vdots & \vdots \\ & 0 & & & G & F \\ & & & & & F \end{bmatrix}.$$

where $G \subset F$ are division rings such that $\dim_G F = \dim F_G = 2$. We prove the lemma by applying Proposition 2.3 to R and $c = n+1$. First we note that $(I_R, \boldsymbol{d})$ is the poset ${}_n\mathbf{C}_n$ with the natural linear ordering. The ring ${}_cR$ is obtained from R by omitting the last column, whereas N^c is the last column of R without the lower term F. The indecomposable objects in the vector space category $\mathbb{L}_F^c$ are determined by indecomposable ${}_cR$-modules X for which $X \otimes N_F^c \neq 0$ or equivalently $0 \neq (X \otimes N_F^c)^* = \mathrm{Hom}_F(X \otimes N_F^c, F) = \mathrm{Hom}_{{}_cR}(X, (N^c)^*)$. It is clear that the ${}_cR$-module $(N^c)^*$ is isomorphic to the unique indecomposable projective-injective module P_1 which is represented by the upper row in ${}_cR$. Since ${}_cR$ is hereditary then $X \otimes N_F^c \neq 0$ if and only if X is one of the indecomposable projectives $(N_F^c)^* = P_1, P_2, \ldots, P_n$ in mod $({}_cR)$ which correspond to the rows of ${}_cR$. It follows that $P_1 \otimes N_F^c \cong {}_FF_F$ and $P_i \otimes N_F^c \cong {}_GF_F$ for $i = 2, \ldots, n$. Now it is easy to see that the vector space category $\mathbf{L}_F^c$ is special Schurian and the ring $\boldsymbol{R}(\mathbf{L}^c)$ associated to $\mathbf{L}_F^c$ is isomorphic to R.

We know from [10] that $\#\,\mathrm{mod}\,({}_cR) = n^2$. By Lemma 2.1

$$\#\, l\,\mathrm{her}\,(R) = 1 + \#\, l\,\mathrm{her}\,(R')$$

where R' is obtained from R by omitting the upper row. From the definition of l-hereditary modules follows that the category $l\,\mathrm{her}\,(R')$ is equivalent to the category of $\mathbf{S}_{n-1}\,(G)$-spaces in the sense of Dlab and Ringel [8] and therefore [8, Proposition 2.5] yields $\#\, l\,\mathrm{her}\,(R') = \frac{1}{2}n(n+1)$. Then by Proposition 2.3 we have $\#\,\mathrm{mod}\,(R) = n^2 + \frac{1}{2}n(n+1) + 1$, as required.

In order to prove the second part of the lemma we note that by [10] there are n^2 of those indecomposable R-modules $X = (X_i)$ for which $X_1 = 0$ and n^2 of those for which $X_{n+1} = 0$. Since we have calculated twice the modules X with $X_1 = 0 = X_{n+1}$ then together with the unique indecomposable projective-injective module $(F, F, \ldots, \ldots, F)$ we have a list of $2n^2 - \frac{1}{2}n(n-1) + 1 = \#\,\mathrm{mod}\,(R)$ indecomposable R-modules. Hence the lemma follows in the case ${}_n\mathbf{C}_n$. In the remaining cases the proof is similar and we leave it to the reader (use [8, Propositions 3.1 and 4.2]).

We note that the second part of the lemma together with [10] gives a complete classification of indecomposable R-modules.

LEMMA 2.7. *Let $G \subset F$ be division rings such that* $\dim_G F = \dim F_G = 2$ *and let*

$$R = \begin{bmatrix} F & 0 & F & F \\ & G & G & F \\ 0 & & G & F \\ & & & F \end{bmatrix}, \quad T = \begin{bmatrix} F & 0 & 0 & F & F \\ & G & G & G & F \\ & & G & G & F \\ 0 & & & G & F \\ & & & & F \end{bmatrix}, \quad T' = \begin{bmatrix} F & F & 0 & F & F \\ & F & 0 & F & F \\ & & G & G & F \\ 0 & & & G & F \\ & & & & F \end{bmatrix},$$

$$E=\begin{bmatrix} F & F & F & F & F & F \\ & G & G & F & F & F \\ & & G & 0 & G & F \\ & & & F & F & F \\ & 0 & & & G & F \\ & & & & & F \end{bmatrix}, \qquad E'=\begin{bmatrix} F & 0 & 0 & F & F & F \\ & G & G & F & F & F \\ & & G & 0 & G & F \\ & & & F & F & F \\ & 0 & & & G & F \\ & & & & & F \end{bmatrix}.$$

(1) *R, T, T', E and E' are l-hereditary right QF-2 artinian rings and*

(I_R, d):
$$\begin{tikzcd} 2 \arrow[r, "{(2,1)}"] & 4 \arrow[r, "{(1,2)}"] & 5 \\ & 3 \arrow[u] & \end{tikzcd}, \quad d_{25}=d_{52}=1\,,$$

(I_T, d):
$$\begin{tikzcd} & & 1 \arrow[d, "{(2,1)}"'] & \\ 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r, "{(1,2)}"] & 5 \end{tikzcd}, \quad d_{15}=d_{51}=1\,,$$

$(I_{T'}, d)$:
$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r, "{(2,1)}"] & 4 \arrow[r, "{(1,2)}"] & 5 \\ & & 3 \arrow[u] & \end{tikzcd}, \quad d_{25}=d_{32}=1\,,$$

(I_E, d):
$$\begin{tikzcd} 1 \arrow[r, "{(2,1)}"] & 2 \arrow[r, "{(1,2)}"] \arrow[dr] & 4 \arrow[r, "{(2,1)}"] & 5 \arrow[r, "{(1,2)}"] & 6 \\ & & 3 \arrow[ur] & & \end{tikzcd}, \quad d_{16}=d_{61}=d_{14}=d_{41}=d_{64}=d_{46}=1$$
$$d_{25}=d_{52}=2\,,$$

$(I_{E'}, d)$:
$$\begin{tikzcd} & 1 \arrow[d] & & \\ 2 \arrow[r, "{(1,2)}"] \arrow[dr] & 4 \arrow[r, "{(2,1)}"] & 5 \arrow[r, "{(1,2)}"] & 6 \\ & 3 \arrow[ur] & & \end{tikzcd}, \quad d_{25}=d_{52}=2\,, \quad d_{46}=d_{64}=1\,,$$

(2) $\#\,\mathrm{her}\,(R)=10$, $\#\,l\,\mathrm{her}\,(T)=24$, $\#\,l\,\mathrm{her}\,(T')=15$,
$\#\,l\,\mathrm{her}\,(E)=14$, $\#\,l\,\mathrm{her}\,(E')=28$ *and* $\#\,\mathrm{mod}\,(R)=23$

(3) Any indecomposable R-module $X=\begin{pmatrix} X_2 & X_4 & X_5 \\ & X_3 & \end{pmatrix}$ with $X_2\neq 0$, $X_3\neq 0$ *and* $X_5\neq 0$ *is of one of the following forms*:

$$D_1 = \begin{pmatrix} F & F & F \\ & G & \end{pmatrix}, \quad D_2 = \begin{pmatrix} F & F & F \\ & F & \end{pmatrix}, \quad D_3 : \begin{tikzcd}[row sep=small] (1, f)F \arrow[r] & F \oplus F \arrow[r, "\mathrm{id}"] & F \oplus F\, , \\ & G \oplus G \arrow[u] & \end{tikzcd}$$

$$D_4 : \begin{tikzcd}[row sep=small] F \arrow[r, "u"] & F \oplus G \arrow[r, "p"] & F \oplus 0\, , \\ & G \oplus 0 \arrow[u] & \end{tikzcd} \qquad D_5 : \begin{tikzcd}[row sep=small] F \arrow[r, "u"] & F \oplus G \arrow[r, "p"] & F \oplus 0\, , \\ & F \oplus 0 \arrow[u] & \end{tikzcd}$$

$$D_6 : \begin{tikzcd}[row sep=small] (1, 1)F \arrow[r] & F \oplus F \arrow[r, "p"] & 0 \oplus F\, , \\ & 0 \oplus F \arrow[u] & \end{tikzcd} \qquad D_7 : \begin{tikzcd}[row sep=small] F \oplus F \arrow[r, "w"] & F \oplus F \arrow[r, "p"] & 0 \oplus F\, , \\ & 0 \oplus F \arrow[u] & \end{tikzcd}$$

where $\{1, f\}$ *is a fixed basis of* ${}_GF$ *as well as of* F_G *over* G, p *denote the natural projections and* $u(x + fy) = (x + fy, x)$, $w(x + fy) = (2x + fy, x + 2fy)$, $x, y \in G$. *Moreover* $\operatorname{End}(D_1) \cong G$ *and* $\operatorname{End}(D_j) \cong F$ *for* $j = 2, \dots, 7$.

(4) *Every indecomposable module* $X = \begin{pmatrix} X_1 & X_2 & X_4 & X_5 & X_6 \\ & & X_3 & & \end{pmatrix}$ *in* $l\,\mathrm{her}\,(E)$ *with* $X_2 \neq 0$ *is of one of the forms* $\begin{pmatrix} F & F & F & F & F \\ & & F & & \end{pmatrix}$, $\begin{pmatrix} 0 & F & F & F & F \\ & & F & & \end{pmatrix}$, $\begin{pmatrix} 0 & G & F & F & F \\ & & F & & \end{pmatrix}$, $\begin{pmatrix} 0 & G & F & F & F \\ & & G & & \end{pmatrix}$.

PROOF. The proof of the statement (1) is obvious.

(2) First we will apply Proposition 2.2 to the ring R and $a = 2$. Note that the ring R_2 is hereditary, the valued graph of R_2 is $3 \to 4 \xrightarrow{(1,2)} 5$ and it follows from [8, Proposition 2.5] that the indecomposable R_2-modules form a diagram

$$(*) \qquad \begin{tikzcd}[column sep=tiny] & & (G\ G\ F) \arrow[dr] & & (0\ G\ 0) \arrow[dr] & & (G\ 0\ 0) \\ & (0\ G\ F) \arrow[ur] \arrow[dr] & & (G\ F\ F) \arrow[ur] \arrow[dr] & & (G\ G\ 0) \arrow[ur] & \\ (0\ 0\ F) \arrow[ur] & & (0\ F\ F) \arrow[ur] & & (F\ F\ F) \arrow[ur] & & \end{tikzcd}$$

Since $M^2 = (0\ F\ F)$ then the vector space category $\tilde{\mathbf{K}}^2_F$ has 3 indecomposable objects corresponding to the R_2-modules M^2, $(G\ F\ F)$ and

$(F\ F\ F)$. It follows that $\tilde{\mathbf{K}}_F^2$ is special Schurian,

$$\boldsymbol{R}(\tilde{\mathrm{K}}^2) \cong \begin{bmatrix} F & F & F & F \\ & G & F & F \\ 0 & & F & F \\ & & & F \end{bmatrix}$$

and the valued poset of $\boldsymbol{R}(\tilde{\mathrm{K}}^2)$ has the form $a \xrightarrow{(2,1)} \circ \xrightarrow{(1,2)} c \to \circ$, $d_{ac} = d_{ac} = 1$. By Lemma 2.1 and [8, Proposition 2.5] $\# l\,\mathrm{her}\,(\boldsymbol{R}(\tilde{\mathrm{K}}^2)) = 1 + 1 + 3 = 5$. Then Proposition 2.2 yields $\# l\,\mathrm{her}\,(R) = 6 + 5 - 1 = 10$, as we required.

Now we note that the category $\mathbf{K}_F^2$ has 5 indecomposable objects corresponding to the modules M^2, $(G\ F\ F)$, $(F\ F\ F)$, $(0\ G\ 0)$ and $(G\ G\ 0)$. It is easy to check that $\mathbf{K}_F^2$ is special Schurian, $\boldsymbol{R}(\mathrm{K}^2)$ has the form

$$R' = \begin{bmatrix} G & G & F & F & F & F \\ & G & 0 & G & F & F \\ & & F & F & F & F \\ & & & G & F & F \\ & 0 & & & F & F \\ & & & & & F \end{bmatrix}$$

and its valued poset has the form

$$\begin{tikzcd} 2 \arrow[r, "{(1,2)}"] \arrow[dr] & 4 \arrow[r, "{(2,1)}"] & 5 \arrow[r, "{(1,2)}"] & 6 \arrow[r] & 7 \\ & 3 \arrow[ur] & & & \end{tikzcd}$$

with $d_{46} = d_{64} = 1$ and $d_{25} = d_{52} = 2$. By Proposition 2.2 $\#\,\mathrm{mod}\,(R) = 9 + \# l\,\mathrm{her}\,(R')$. Since we know from Lemma 2.1 that $\# l\,\mathrm{her}\,(R') = \# l\,\mathrm{her}\,(E)$ it remains to prove that this number equals 14. To prove it we consider the hereditary ring

$$S = \begin{bmatrix} F & 0 & 0 & F \\ & G & G & F \\ 0 & & G & F \\ & & & F \end{bmatrix}.$$

The valued graph of S is $2 \to 5 \xleftarrow{(2,1)} 4 \leftarrow 3$. Now we will apply Proposition 2.2 to $R = S$ and $a = 2$. Note that $S_2 = R_2$, $M^2 = (0\ 0\ F)$ and hence the category $\tilde{\mathbf{K}}^2_F$ has 6 indecomposable objects corresponding to the modules of the form $(\cdot\ \cdot\ F)$ in the diagram $(*)$. It follows that $\tilde{\mathbf{K}}^2_F$ is special Schurian, the ring $\boldsymbol{R}(\tilde{\mathbf{K}}^2)$ has the form

$$S' = \begin{bmatrix} F & F & F & F & F & F & F \\ & G & G & F & F & F & F \\ & & G & 0 & G & F & F \\ & & & F & F & F & F \\ & & & & F & F & F \\ & 0 & & & & F & F \\ & & & & & & F \end{bmatrix}$$

and the valued poset of S' is of the form

$$(**)\qquad \begin{tikzcd} 1 \arrow[r, "{(2,1)}"] & 2 \arrow[r, "{(1,2)}"] \arrow[dr] & 4 \arrow[r, "{(2,1)}"] & 5 \arrow[r, "{(1,2)}"] & 6 \arrow[r] & 7 \\ & & 3 \arrow[ur] & & & \end{tikzcd}$$

with $d_{16} = d_{61} = d_{14} = d_{41} = d_{46} = d_{64} = 1$ and $d_{25} = d_{52} = 2$. Since Lemma 2.1 yields $\# l\,\mathrm{her}\,(S') = 1 + \# l\,\mathrm{her}\,(E)$ then by Proposition 2.2 and [8, Proposition 3.1] we have $20 = \# l\,\mathrm{her}\,(S) = 6 + \# l\,\mathrm{her}\,(E)$ as we required. Consequently, $\#\,\mathrm{mod}\,(R) = 23$ and $\# l\,\mathrm{her}\,(E) = 14$.

In order to prove (2) for T we will apply Proposition 2.2 to $R = T$ and $a = 1$. The valued poset of the ring T_1 has the form $2 \to 3 \to 4 \xrightarrow{(1,2)} 5$ and $M^1 = (0\ 0\ F\ F)$. By [8, Proposition 2.5] those indecomposable modules X in $l\,\mathrm{her}\,(T_1)$ for which $\mathrm{Hom}_{T_1}(M^1, X) \neq 0$ form the following diagram

$$\begin{tikzcd} & & (G\ G\ F\ F) \arrow[dr] & & \\ (0\ 0\ F\ F) \arrow[r] & (0\ G\ F\ F) \arrow[ur] \arrow[dr] & & (G\ F\ F\ F) \arrow[r] & (F\ F\ F\ F). \\ & & (0\ F\ F\ F) \arrow[ur] & & \end{tikzcd}$$

It follows that the vector space category $\tilde{\mathbf{K}}^1_F$ is special Schurian, $\boldsymbol{R}(\tilde{\mathbf{K}}^1) \cong S'$ and the valued poset of $\boldsymbol{R}(\tilde{\mathbf{K}}^1)$ has the form $(**)$. Then by Lemma 2.1, Proposition 2.2 and [8, Proposition 2.5] we get

$$\# l \operatorname{her}(T) = 10 + \# l \operatorname{her}(S') - 1 = 24 .$$

Now we will prove (2) for the ring T' by applying Proposition 2.2 to $R = T'$ and $a = 1$. Since $T'_1 = R$ and $M^1 = \begin{pmatrix} F & F & F \\ & 0 & \end{pmatrix}$ then by the statement (3) proved below the indecomposable modules X in l her (R) for which $\operatorname{Hom}_R(M^a, X) \neq 0$ form a chain $M^a \to D_3 \to D_1 \to D_2$. It follows that the vector space category $\tilde{\mathbf{K}}^1_F$ is special Schurian and the ring $\boldsymbol{R}(\tilde{\mathbf{K}}^1)''$ (see Lemma 2.1) is isomorphic to the ring $\boldsymbol{R}(\tilde{\mathbf{K}}^2)$ below the diagram $(*)$. Hence $\# l \operatorname{her}\big(\boldsymbol{R}(\tilde{\mathbf{K}}^1)''\big) = 6$ and by Prosposition 2.2 we have $\# l \operatorname{her}(T') = 10 + 6 - 1 = 15$.

Finally we prove (2) for the ring E' by applying Proposition 2.2 to $R = E'$ and $a = 1$. First we note that E'_1 is obtained from E by omitting the upper row and E'_1-modules are those E-modules X for which $X_1 = 0$. Since $M^1 = \begin{pmatrix} 0 & 0 & F & F \\ & F & & \end{pmatrix}$ then by (3) and (4) proved below the indecomposable modules X in l her (E'_1) for which $\operatorname{Hom}(M^1, X) \neq 0$ form the following diagram

$$\begin{array}{ccccccc} M^1 \to \begin{pmatrix} 0 & (1,f) & F & F\oplus F & F\oplus F \\ & & G\oplus G & & \end{pmatrix} & \to & \begin{pmatrix} 0 & F & F & F \\ & G & & \end{pmatrix} & \to & \begin{pmatrix} 0 & F & F & F \\ & F & & \end{pmatrix} & & \\ & & \downarrow & & \downarrow & & \\ & & \begin{pmatrix} G & F & F & F \\ & G & & \end{pmatrix} & \to & \begin{pmatrix} G & F & F & F \\ & F & & \end{pmatrix} & \to & \begin{pmatrix} F & F & F & F \\ & F & & \end{pmatrix} . \end{array}$$

It follows that $\tilde{\mathbf{K}}^1_F$ is special Schurian and the ring $\boldsymbol{R}(\tilde{\mathbf{K}}^1)$ satisfies the conditions in Lemma 2.1 (ii) with $\boldsymbol{R}(\tilde{\mathbf{K}}^1)'' = S'$. Consequently, by Lemma 2.1 and Proposition 2.2 we get

$$\# l \operatorname{her}(E') = \# l \operatorname{her}(E) + \# l \operatorname{her}(S') - 1 = 28$$

and the statement (2) is proved.

(3) Let $X = \begin{pmatrix} X_2 & X_4 & X_5 \\ & X_3 & \end{pmatrix}$ be an indecomposable R-module. It

follows from [8, Proposition 2.5] that there are exactly 9 indecomposable modules X for which $X_2 = 0$. They are presented in the diagram (*). Since the modules $\begin{pmatrix} F & 0 & 0 \\ & 0 & \end{pmatrix}$, $\begin{pmatrix} F & G & 0 \\ & 0 & \end{pmatrix}$, $\begin{pmatrix} F & G & 0 \\ & G & \end{pmatrix}$, $\begin{pmatrix} F & F & 0 \\ & G & \end{pmatrix}$, $\begin{pmatrix} F & F & 0 \\ & 0 & \end{pmatrix}$, $\begin{pmatrix} F & F & 0 \\ & F & \end{pmatrix}$, $\begin{pmatrix} F & F & F \\ & 0 & \end{pmatrix}$ are indecomposable and $\#\,\mathrm{mod}\,(R) = 23$ then (3) follows.

(4) We know from (3) that there are exactly 10 indecomposable modules X in $l\,\mathrm{her}\,(E)$ for which $X_2 = 0$. Since the modules presented in (4) are indecomposable then (4) follows and the proof of the lemma is complete.

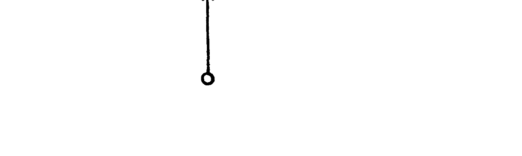
LEMMA 2.8. *Let R be an l-hereditary artinian PI-ring. If the valued poset of R has the form* $\widehat{\mathbf{C}_3\mathbf{C}}$ *then* $\#\,\mathrm{mod}\,(R) = 35$.

PROOF. By Remark 2 we can suppose that there is a pair of division rings $G \subset F$ such that $\dim_G F = \dim F_G = 2$, $F_1 = F_2 = F_5 = F$, $F_3 = F_4 = G$, ${}_2M_4 = {}_FF_G$, ${}_4M_5 = {}_GF_F$, ${}_2M_5 = {}_FF_F$ and ${}_3M_4 = {}_GG_G$. It is clear that (in the notation of Proposition 2.2) R_1 is the ring R in Lemma 2.7 and $M^1 = \begin{pmatrix} F & F & F \\ & F & \end{pmatrix}$.

First we note that the R_1-module $\begin{pmatrix} F & F & 0 \\ & F & \end{pmatrix}$ is the unique indecomposable noninjective epimorphic image of M^1. Thus, if X is indecomposable then $\mathrm{Hom}_{R_1}(M^1, X) \neq 0$ if and only if either X is injective or X contains $\begin{pmatrix} F & F & 0 \\ & F & \end{pmatrix}$ and by Lemma 2.7 (3) all such modules X form a diagram

$$\begin{array}{ccccccc} M^1 \to & \begin{pmatrix} F & F & 0 \\ & F & \end{pmatrix} & \to & \begin{pmatrix} F & G & 0 \\ & G & \end{pmatrix} & \to & \begin{pmatrix} F & 0 & 0 \\ & 0 & \end{pmatrix} \\ & & & \downarrow & & \\ & & & \begin{pmatrix} 0 & 0 & 0 \\ & G & \end{pmatrix}. & & \end{array}$$

It follows that the vector space category $\mathbf{K}^1_F$ is special Schurian, the valued poset of $\boldsymbol{R}(\mathbf{K}^1)$ has the form



with $d_{ac} = d_{ca} = 1$ and if we apply Lemma 2.1 (ii) to $\boldsymbol{R}(\mathrm{K}^1)$ twice we get the ring R in Lemma 1.7. Hence $\# l\,\mathrm{her}\,(\boldsymbol{R}(\mathrm{K}^1)) = 12$ and the required equality follows from Proposition 2.2 and Lemma 2.7.

LEMMA 2.9. *Let R be an l-hereditary artinian PI-ring. If the valued poset of R has the form*

$$1 \to 2 \xrightarrow{(2,1)} 3 \to 4 \xrightarrow{(1,2)} 5 \to 6\,, \qquad d_{25} = d_{52} = 1\,,$$

then $\# \mathrm{mod}\,(R) = 49$. *If (X_i) is an indecomposable R-module then either $X_2 = 0$ or $X_5 = 0$, or else (X_i) is of one of the forms:* $(F\ F\ F\ F\ F\ F)$, $(F\ F\ F\ F\ F\ 0)$, $(0\ F\ F\ F\ F\ F)$, $(0\ F\ F\ F\ F\ 0)$.

PROOF. By Remark 2 we can suppose that there is a pair of division rings $G \subset F$ such that $\dim_G F = \dim F_G = 2$, $F_3 = F_4 = G$ $F_1 = F_2 = F_5 = F_6 = F$, ${}_1M_2 = {}_2M_5 = {}_5M_6 = {}_FF_F$, ${}_2M_3 = {}_FF_G$, ${}_3M_4 = {}_GG_G$ and ${}_4M_5 = {}_4M_6 = {}_GF_F$.

The ring $S = R_1$ is l-hereditary and its valued poset is of the form

$$2 \xrightarrow{(2,1)} 3 \to 4 \xrightarrow{(1,2)} 5 \to 6\,, \qquad d_{25} = d_{52} = 1\,,$$

and $M^1 = (F\ F\ F\ F\ F)$. We will prove that $\# \mathrm{mod}\,(S) = 32$ by applying Proposition 2.2 to $R = S$ and $a = 2$. The ring S_2 is hereditary, its valued graph has the form $3 \to 4 \xrightarrow{(1,2)} 5 \to 6$ and $M^2 = (F\ F\ F\ F) = E(\tilde{F}_6)$. Hence the vector space category K_F^2 is special Schurian and has 4 indecomposable objects corresponding to the indecomposable injective S_2-modules. Then the valued poset of the ring $\boldsymbol{R}(\mathrm{K}^2)$ has the form

$$3 \to 4 \xrightarrow{(1,2)} 5 \to 6 \to 7$$

and by Lemma 2.1 and [8, Proposition 2.5] $\# l\,\mathrm{her}\,(\boldsymbol{R}(\mathrm{K}^2)) = 8$. Then Proposition 2.2 and [10, Theorem 1.1] yield $\# \mathrm{mod}\,(S) = 32$, as we claimed.

Since the following S-modules are indecomposable

$$W_1 = (F\ 0\ 0\ 0\ 0)\,, \quad W_4 = (F\ F\ 0\ 0\ 0)\,, \quad W_7 = (F\ F\ F\ F\ 0)\,,$$
$$W_2 = (F\ G\ 0\ 0\ 0)\,, \quad W_5 = (F\ F\ G\ 0\ 0)\,, \quad W_8 = (F\ F\ F\ F\ F)\,,$$
$$W_3 = (F\ G\ G\ 0\ 0)\,, \quad W_6 = (F\ F\ F\ 0\ 0)\,,$$

and $\# \operatorname{mod}(S_2) = 24$ then any indecomposable S-module X for which $X_2 \neq 0$ is of the form W_i, $i \leqslant 8$. Since $\dim \operatorname{Hom}_S({}_F M^1_S, W_j)_F = 1$ for all j and $\operatorname{Hom}_S(M^1, X) \neq 0$ if and only if $X_1 \neq 0$, then $\mathbf{K}^1_F$ is special Schurian and has 8 indecomposable objects corresponding to the modules W_j, $j \leqslant 8$. It is easy to check that

$$\boldsymbol{R}(\mathrm{K}^1) \cong \begin{bmatrix} F & F & F & F & F & F & F & F & F \\ & G & G & F & F & F & F & F & F \\ & & G & 0 & G & F & F & F & F \\ & & & F & F & F & F & F & F \\ & & & & F & F & F & F & F \\ & & & & & F & F & F & F \\ & & & & & & F & F & F \\ & 0 & & & & & & F & F \\ & & & & & & & & F \end{bmatrix}.$$

Then, applying Lemma 2.1 (ii) and Lemma 2.7, we have

$$\# \, l \operatorname{her}\left(\boldsymbol{R}(\mathrm{K}^1)\right) = 17 .$$

Now, it follows from Proposition 2.2 that $\# \operatorname{mod}(R) = 32 + 17 = 49$.

In order to prove the second part of the lemma one can apply the same type of arguments as in the final part of the proof of Lemma 2.6.

LEMMA 2.10. *Let R_{nm}, $n \geqslant 0$, $m \geqslant 1$, be an l-hereditary artinian PI-ring such that the valued poset of R_{nm} has the form*

$$-n \to -(n-1) \to \ldots \to 0 \xrightarrow{(2,1)} 1 \xrightarrow{(1,2)} 2 \to \ldots \to m , \qquad d_{02} = d_{20} = 1 ,$$

and suppose $F_j = F$ for $j \neq 1$, $F_1 = G \subset F$. Then

(i) $\# \operatorname{mod}(R_{nm}) = m^2 + (n+1)(m+n+2)$.

(ii) *Every indecomposable R_{nm}-module $X = (X_i)_{-n \leqslant i \leqslant m}$ with $X_{-n} \neq 0$ is of one of the forms*: $H = E(\tilde{F}_1) = (F \ldots FG0 \ldots 0)$,

$$H^n_j = (\underbrace{F \ldots F}_{j \text{ terms}} 0 \ldots 0) , \qquad -n \leqslant j \leqslant m , \qquad Q^n_i = (\underbrace{F \ldots F}_{i \text{ terms}} F^2 \ldots F^2 G^2 0 \ldots 0) ,$$

$$i = 1, \ldots, n ,$$

where the maps $F \to F^2 = F \oplus F$, $F^2 \to G^2$ *are defined by* $1 \mapsto (1, f)$ *and* $(x + fy, x' + fy') \mapsto (x, y')$, *respectively, and* $\{1, f\}$ *is a basis of* F_G.

PROOF. In view of Remark 2 we can suppose that ${}_iM_j = F$ for all i and j. We prove the lemma by induction on n. The case $n = -1$ follows from [10]. Suppose $n \geqslant 0$ and apply Proposition 2.2 to $R = R_{nm}$ and $a = -n$. Note that $R_a = R_{n-1m}$ and $M^a = H^{n-1}_{m+n}$. By the inductive assumption the indecomposable R_a-modules Y for which $\mathrm{Hom}(M^a, Y) \neq 0$ form a chain

$$H_1^{n-1} \leftarrow \ldots \leftarrow H_n^{n-1} \leftarrow H \leftarrow Q_1^{n-1} \leftarrow \ldots \leftarrow Q_{n-1}^{n-1} \leftarrow H_{n+1}^{n-1} \leftarrow \ldots \leftarrow H_{n+m}^{n-1}\,.$$

It is obvious that $\mathrm{End}(H_j^{n-1}) \cong F \cong \mathrm{End}(Q_i^{n-1})$ for all i, j and $\mathrm{End}(H) \cong G$. Now it is easy to check that the vector space category $\mathbf{K}^a_Y$ is special Schurian and the valued poset of the ring $\boldsymbol{R}(\mathbf{K}^a)$ has the form

$$1 \to \ldots \to n \xrightarrow{(2,1)} 0 \xrightarrow{(1,2)} 1' \to \ldots \to n' - 1 \to n + 1 \to \ldots \to n + m + 1\,,$$

with $d_{n0} = d_{0n} = 1$. Thus, by Lemma 2.1, $\#\, l\,\mathrm{her}\big(\boldsymbol{R}(\mathbf{K}^a)\big) = m + 2n + 2$ and the equality (i) is a consequence of Proposition 2.2 and the inductive assumption. Since (ii) is an immediate consequence of (i) and the inductive assumption the proof of the lemma is complete.

LEMMA 2.11. *Let R be an l-hereditary artinian PI-ring. Then*

(i) *If the valued poset of R has the form* ${}^n_s\mathbf{B}\mathbf{B}^m$ *then*

$$\#\,\mathrm{mod}(R) = (m + s)^2 + (n + 1)(m + n + s + 2) + \\ + (s + 1)\big(m + 2n + \tfrac{1}{2}(s + 6)\big) + 1\,.$$

If X is an indecomposable R-module nonisomorphic to $P(\tilde{F}_{m+s+1})$ then either $X_{m+s+1} = 0$ or $X_m = 0$.

(ii) *If the valued poset of R has the form* ${}^m\mathbf{B}_s\mathbf{B}^m$ *then R is of finite representation type.*

If X is an indecomposable R-module then either $X_{-n-1} = 0$ or $X_m = 0$, *or else X has the form*

$$\begin{matrix} (FF \dots F \dots FF) & & \\ \left.\begin{matrix} F^2 \\ \vdots \\ F^2 \end{matrix}\right\} & i \text{ terms} & \\ \left.\begin{matrix} F \\ \vdots \\ F \end{matrix}\right\} & j \text{ terms} & i, j = 1, 2, \dots, s,\ i + j = s\,. \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & & \end{matrix}$$

PROOF. (i) In view of Remark 2 we can suppose that $F_j = F$ for $j \neq 1$, $F_1 = G \subset F$ and ${}_iM_j = F$ for $i \neq m + s + 1$. We will prove (i) by applying Proposition 2.2 to the ring R and $a = m + s + 1$. The valued poset of the ring R_a has the form

$$(*) \qquad \begin{matrix} -n \to -(n-1) \to \dots \to 0 \xrightarrow{(2,1)} 1 \xrightarrow{(1,2)} 2 \to \dots \to m \\ \downarrow \\ m + s \to \dots \to m + 2 \to m + 1, \ d_{02} = d_{20} = 1\,, \end{matrix}$$

and $M^a = \begin{pmatrix} F & F & \dots & F & F \\ & F & \dots & F & F \end{pmatrix}$. Let $X = \begin{pmatrix} X_{-n} & X_{-n+1} & \dots & X_m \\ & X_{m+s} & \dots & X_{m+1} \end{pmatrix}$ be an indecomposable R_a-module.

First we will prove that

$$\# \operatorname{mod}(R_a) = (m + s)^2 + (n + 1)(n + m + s + 2)\,.$$

For this purpose we construct a sequence $B_1, \dots, B_t$ of l-hereditary artinian PI-rings and a partial Coxeter functors

$$\operatorname{mod}(R_a) \rightleftarrows \operatorname{mod}(B_1) \rightleftarrows \dots \rightleftarrows \operatorname{mod}(B_t)$$

in the sense of [2, § 4] (see also [28]) such that the following conditions are satisfied:

(a) $\# \operatorname{mod}(R_a) = \# \operatorname{mod}(B_1) = \dots = \# \operatorname{mod}(B_t)$;

(b) the valued poset of the ring B_t is obtained from the one of R_a by inverting the arrows between points $m, m + 1, \dots, m + s$;

(*c*) if X' denotes the image of X in mod (B_t) under the composition of partial Coxeter functors then $X'_j = X_j$ for $j = -n, \dots, m$.

The partial Coxeter functors we are looking for can be defined as functors S_k^-, S_k^+ [28, § 1] where k is either a sink or a source in the graph $m+s - \dots - k - \dots - m+1$ (contained in a corresponding poset $(I_{B_t}, \boldsymbol{d})$) with respect to an appropriate orientation. Then (*c*) immediately follows from the definition of the functors S_k^- and S_k^+, whereas (*a*) follows from the properties $\boldsymbol{c}_1^+ - \boldsymbol{c}_4^+$ and $\boldsymbol{c}_1^- - \boldsymbol{c}_4^-$ in [28, § 1]. Now, the required equality follows from (*a*) and Lemma 2.10.

Suppose that X is not isomorphic to M^a and there is a nonzero map $f\colon M^a \to X$. Since $M^a = E(\tilde{F}_m)$ then Im f contains an epimorphic image of $M^a/\tilde{F}_m = E(\tilde{F}_{m-1}) \oplus E(\tilde{F}_{m+1})$. It is easy to see that indecomposable epimorphic images of the module $E(\tilde{F}_{m+1})$ are of the forms

$$T_j = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ & & \underbrace{F & \dots & F}_{j \text{ terms}} & & 0 & \dots & 0 \end{pmatrix}, \quad j = 1, \dots, s,$$

whereas the indecomposable epimorphic images of $E(\tilde{F}_{m-1})$ are the following

$$L = E(\tilde{F}_1) = \begin{pmatrix} F & \dots & F & \dots & F & G & 0 & \dots & 0 \\ & & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$L_j = \begin{pmatrix} \overbrace{F & \dots & F & \dots & F}^{j \text{ terms}} & 0 & \dots & 0 \\ & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}, \quad j = 1, \dots, n+m.$$

The modules L_j are injective for $j \neq n+2$.

Now suppose $L_{n+2} \subset X$. Then, by (*c*), $L'_{n+2} \subset X'$ and in view of Lemma 2.10 either $X' = Q_j^n$ for some j or $X' = H_{n+2}^n$. Hence, applying again (*c*), we conclude that either $X = L_{n+2}$ or is of one of the forms

$$Q'_j = \begin{pmatrix} \overbrace{F & \dots & F & \dots & F}^{j \text{ terms}} & F^2 & \dots & F^2 & G^2 & 0 & \dots & 0 \\ & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad j = 1, \dots, n.$$

It follows that indecomposable R_a-modules X for which Hom $(M^a,$

$X) \neq 0$ form the following diagram

$$\begin{array}{l} L_1 \leftarrow \dots \leftarrow L_{n+1} \leftarrow L \leftarrow Q_1' \leftarrow \dots \leftarrow Q_n' \leftarrow L_{n+2} \leftarrow \dots \leftarrow L_{n+m} \nwarrow \\ \hspace{25em} M^a . \\ \hspace{17em} T_1 \leftarrow \dots \leftarrow T_s \quad \swarrow \end{array}$$

Note that End $(L) \cong G$ and End $(L_i) \cong$ End $(Q_j') \cong$ End $(T_k) \cong F$ for all i, j, k. Now it is easy to check that the vector space category $\mathbf{K}_F^a$ is special Schurian and the valued poset of $\tilde{R}_a = \boldsymbol{R}(\mathrm{K}^a)$ has the form

$$\begin{array}{l} 1 \to \dots \to n+1 \xrightarrow{(2,1)} 0 \xrightarrow{(1,2)} 1' \to \dots \to n' \to n+2 \to \dots \to n+m \searrow \\ \hspace{27em} a' \to a'' , \\ \hspace{20em} 1'' \to \dots \to \quad s'' \quad \nearrow \end{array}$$

with $d_{n+11'} = d_{1'n+1} = 1$. We know from Lemma 2.1 that $\# l \operatorname{her} (\tilde{R}_a) = 1 + \# l \operatorname{her} (S)$ where $S = \tilde{R}_a'$ and the valued poset of S is obtained from the above one by omitting the point a''. Since by (c) and Lemma 2.10 for every $j = 1, \dots, n+1$ there are exactly $s+1$ of indecomposable modules X in $l \operatorname{her} (S)$ for which $X_{j-1} = 0$ and $X_j = F$, then

$$\# l \operatorname{her} (\tilde{R}_a) = 1 + (n+1)(s+1) + \# l \operatorname{her} (B)$$

where B is a hereditary artinian PI-ring and the valued graph of B has the form

$$\begin{array}{l} 0 \xrightarrow{(1,2)} 1' \to \dots \to n' \to n+2 \to \dots \to n+m \searrow \\ \hspace{20em} a' . \\ \hspace{13em} 1'' \to \dots \to \quad s'' \quad \nearrow \end{array}$$

Applying Proposition 2.2 we can easily prove by induction on s'' that $\# l \operatorname{her} (B) = (s+1)(m+n-1) + \frac{1}{2}(s+1)(s+6)$. This can be also establish either by the method used in [8, § 2] or by the inspection of the positive roots of the Dynkin diagram $\mathbf{B}_{m+s+1}$. Consequently, we have

$$\# l \operatorname{her} (\tilde{R}_a) = 1 + (s+1)\big(m + 2n + \tfrac{1}{2}(s+6)\big)$$

and the required equality in (i) follows from Proposition 2.2.

In order to prove the second part of (i) we note that by Lemma 2.10 there are exactly $(m+s)^2 + (n+1)(m+n+s+2)$ of those indecomposable R-modules X for which $X_{m+s+1} = 0$ and exactly $(m-1)^2 + (n+s+2)(n+m+s+4)$ of those for which $X_m = 0$. Since we have calculated twice $(m-1)^2 + (n+1)(m+n+1) + \frac{1}{2}s(s+1)$ modules then the number of modules X for which either $X_{m+s+1} = 0$ or $X_m = 0$ is equal $\#\operatorname{mod}(R) - 1$, as required.

(ii) Suppose the ordering of points $m+1, \ldots, m+s$ in the poset $(I_R, \boldsymbol{d})$ is the following $m+s \to \ldots \to m+2 \to m+1$. The general case can be reduced to this particular one by applying the partial Coxeter functors arguments used in the proof of part (i). Furthermore, in view of Remark 2 we can suppose that $F_j = F$ for $j \neq 1$, $F_1 = G \subset F$ and ${}_iM_j = F$ for $i \neq -(n+1)$.

We will apply Proposition 2.2 to the ring R and $a = -(n+1)$. The valued poset of R has the form $(*)$ and $M^a = \begin{pmatrix} F \ldots F \ldots F & F \\ & 0 \ldots 0 & F \end{pmatrix}$. Using the same type of arguments as in the proof of the statement (i) we show that the indecomposable R_a-modules X for which $\operatorname{Hom}(M^a, X) \neq 0$ form a diagram

$$\begin{tikzcd}[column sep=small]
L_1 & \ldots \arrow[l] & L_{n+1} \arrow[l] & L \arrow[l] & Q'_1 \arrow[l] & \ldots \arrow[l] & Q'_n \arrow[l] & L_{n+2} \arrow[l] & \ldots \arrow[l] & L_{n+m} \arrow[l] & & & \\
& & & & & & & & & & T''_s & \ldots \arrow[l] & T''_1 \arrow[l] \\
& & & & & & & & & T'_s \arrow[uu] \arrow[ur] & \ldots \arrow[l] \arrow[ur] & T'_1 \arrow[l] \arrow[ur] \arrow[urr] &
\end{tikzcd}$$

where

$$T'_j = \begin{pmatrix} F \ldots F \ldots F & F \ldots F \\ 0 \ldots 0 & \underbrace{F \ldots F}_{j\ \text{terms}} \end{pmatrix}, \quad T''_j = \begin{pmatrix} 0 \ldots 0 \ldots 0 & 0 \ldots 0 \\ 0 \ldots 0 & \underbrace{F \ldots F}_{j\ \text{terms}} \end{pmatrix} \qquad j = 1, \ldots, s,$$

It is easy to check that the vector space category $\mathbf{K}^a_F$ is special Schurian and the valued poset of $\boldsymbol{R}(\mathbf{K}^a)$ has the form

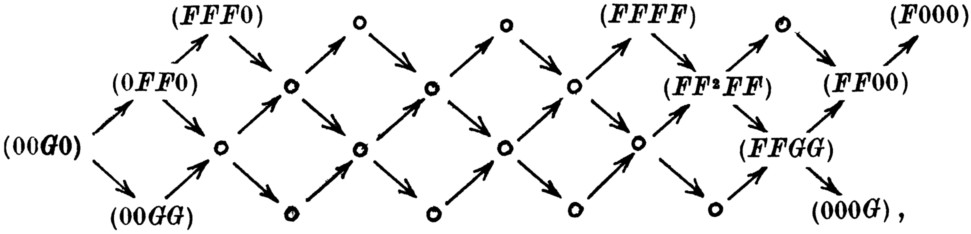
$$\begin{tikzcd}[column sep=small]
1 \arrow[r] & \ldots \arrow[r] & n+1 \arrow[r, "(2,1)"] & 0 \arrow[r, "(1,2)"] & 1' \arrow[r] & \ldots \arrow[r] & n' \arrow[r] & n+2 \arrow[r] & \ldots \arrow[r] & n+m \arrow[dd] & & & \\
& & & & & & & & & & s'' \arrow[r] \arrow[dl] & \ldots \arrow[r] \arrow[dl] & 1'' \arrow[dl] \arrow[dll] \\
& & & & & & & & & \hat{s} \arrow[r] & \ldots \arrow[r] & \hat{1} \arrow[r] & \hat{0}
\end{tikzcd},$$

with $d_{n+1\,1'} = d_{1'\,n+1} = 1$. Now, applying Lemma 2.1, Proposition 2.2 and the arguments used in the final part of the proof of (i) we get the first part of (ii). The proof of the second statement in (ii) is similar to that in (i) and we leave it to the reader. This completes the proof.

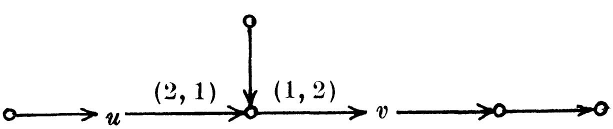
LEMMA 2.12. *If R is an l-hereditary artinian PI-ring and $(I_R, \boldsymbol{d})$ has the form ${}_3\mathbf{BF}_4$ then $\#\operatorname{mod}(R) = 41$ and every indecomposable R-module X for which $X_0 \neq 0$ and $X_3 \neq 0$ is of one of the forms:*

$$\left(F\ \begin{matrix}F & F\\ F & F\end{matrix}\right),\quad \left(F\ \begin{matrix}F & F\\ F & G\end{matrix}\right),\quad \left(F\ \begin{matrix}F & G\\ F & G\end{matrix}\right),\quad \left(F\ \begin{matrix}F & F\\ F^2 & G^2\end{matrix}\right),\quad \left(F^2\ \begin{matrix}F & F\\ F^2 & G^2\end{matrix}\right).$$

PROOF. Without loss of generality we can suppose $F_0 = F_1 = {}_FF_2 = F$, $F_3 = F_4 = G \subset F$ and $\dim_G F = \dim F_G = 2$. We will apply Proposition 2.2 to R and $a = 0$. Then the ring R_a is hereditary, its valued graph has the form $1 \to 2 \xrightarrow{(2,1)} 3 \leftarrow 4$ and $M^a = (F\ F\ F\ F)$. We know from [10] that $\#\operatorname{mod}(R) = 24$ and the indecomposable R-modules correspond to the positive roots of the Dynkin diagram $\mathbf{F}_4$. It follows from the properties of the partial Coxeter functors [28] that the indecomposable R-modules form the following diagram



where arrows mean the existence of irreducible maps [22]. Then the category $\mathbf{K}_F^a$ has exactly 5 indecomposable objects determined by the R_a-modules in the right side of the diagram. Hence we easily conclude that $\mathbf{K}_F^a$ is special Schurian, the valued poset of the ring $\boldsymbol{R}(\mathbf{K}^a)$ has the form



, $\quad d_{uv} = d_{vu} = 1$,

and we have $\#\, l\operatorname{her}(\boldsymbol{R}(\mathbf{K}^a)) = 2 + \#\, l\operatorname{her}(T') = 17$ according to Lemmas 2.1 and 2.7. Then the required equality follows from Proposition 2.2. The proof of the second statement is left to the reader.

LEMMA 2.13. *If R is an l-hereditary artinian PI-ring and $(I_R, \boldsymbol{d})$ has the form ${}_4\mathbf{FF}_4$ then $\#\operatorname{mod}(R) = 54$ and every indecomposable R-module X for which $X_a \neq 0$ and $X_c \neq 0$ is of one of the forms:*

$$\begin{pmatrix} & G\oplus G & \\ F & F\oplus F & F\oplus F \\ & F & \end{pmatrix}, \quad \begin{pmatrix} & & G\oplus G & \\ F\oplus F & & F\oplus F & F \\ & & F & \end{pmatrix}, \quad \begin{pmatrix} & G\oplus G & \\ F\oplus F & F\oplus F & F \\ & F\oplus F & \end{pmatrix},$$

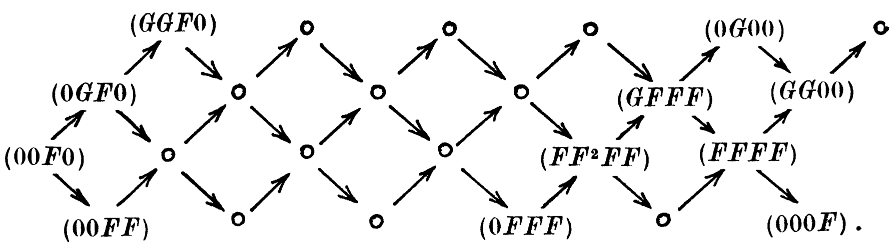
$$\begin{pmatrix} & G\oplus G & \\ F & F\oplus F & F\oplus F \\ & F\oplus F & \end{pmatrix}, \quad \begin{pmatrix} & G\oplus G & \\ F & F\oplus F & F \\ & F & \end{pmatrix}, \quad \begin{pmatrix} & & G & & \\ F & G\oplus & G\oplus & G & F \\ & & F & & \end{pmatrix},$$

$$\begin{pmatrix} & G\oplus G & & \\ F & G\oplus & G\oplus\ G & F \\ & & F & \end{pmatrix}, \quad \begin{pmatrix} & G & \\ F & F & F \\ & F & \end{pmatrix}, \quad \begin{pmatrix} & F & \\ F & F & F \\ & F & \end{pmatrix}.$$

PROOF. We suppose that $(I_R, \boldsymbol{d})$ has the form

$$\begin{tikzcd} & 1 \arrow[d] & \\ 0 \arrow[r, "{(2,1)}"] \arrow[dr] & 2 \arrow[r, "{(1,2)}"] & 3 \\ & 4 \arrow[ur] & \end{tikzcd}, \qquad d_{03} = d_{30} = 1 .$$

The general case can be reduced to this particular one by applying the partial Coxeter functors arguments we use in the proof of Lemma 2.11. By Remark 2 we can suppose $F_1 = F_2 = G \subset F = F_0 = F_3 = F_4$. In the notation of Proposition 2.2 the ring R_0 is hereditary, the valued graph of R_0 has the form $1 \to 2 \xrightarrow{(1,2)} 3 \leftarrow 4$ and $M_0 = (0\ F\ F\ F)$. The indecomposable R_0-modules form the following diagram



It follows that the vector space category $\mathbf{K}_F^0$ has exactly 7 indecomposable objects which are determined by the indecomposable R_0-modules in the right side of the diagram. It is easy to see that $\mathbf{K}_F^0$ is special Schurian and the ring $(\boldsymbol{R}(\mathrm{K}_0)'')''$ (in the notation of Lemma 2.1 (ii)) is isomorphic to the ring E' in the Lemma 2.7. It follows from Lemmas 2.1 and 2.7 that $\#\, l\,\mathrm{her}\,(\boldsymbol{R}(\mathrm{K}_0)) = 2 + \#\, l\,\mathrm{her}\,(E') = 30$. Then the required equality follows from Proposition 2.2. The proof of the remaining statement is left to the reader.

LEMMA 2.14. *If R is an l-hereditary artinian PI-ring and $(I_R, \boldsymbol{d})$ has the form ${}_3\mathbf{BC}_4$ then $\#\,\mathrm{mod}\,(R) = 41$ and every indecomposable R-module X for which $X_1 \neq 0$ and $X_4 \neq 0$ is of one of the forms*:

$$\begin{pmatrix}\begin{matrix}F & F\\ F & F\end{matrix} & F\end{pmatrix},\quad \begin{pmatrix}\begin{matrix}F & G\\ F & G\end{matrix} & G\end{pmatrix},\quad \begin{pmatrix}\begin{matrix}F & F\\ F & G\end{matrix} & G\end{pmatrix},\quad \begin{pmatrix}\begin{matrix}F & F\\ F & G\end{matrix} & F\end{pmatrix},\quad \begin{pmatrix}F & F & F\\ F\oplus F & G\oplus G & \end{pmatrix}.$$

PROOF. Without loss of generality we can suppose that $F_1 = F_5 = F$, $F_2 = F_3 = F_4 = G \subset F$ and $\dim_G F = \dim F_G = 2$. We apply Proposition 2.2 to R and $a = 1$. The ring R_a is hereditary, its valued graph has the form $2 \to 3 \to 4 \xleftarrow{(1,2)} 5$ and $M^a = (F\ F\ F\ F)$. It follows from [10, 28] that the indecomposable R_a-modules form the following diagram

$$\begin{tikzcd}[column sep=tiny]
 & & (GGG0) \arrow[dr] & & (00G0) \arrow[dr] & & (0G00) \arrow[dr] & & (G000) \\
 & (0GG0) \arrow[ur] \arrow[dr] & & (GGFF) \arrow[ur] \arrow[dr] & & (0GGF) \arrow[ur] \arrow[dr] & & (GG00) \arrow[ur] & \\
(00G0) \arrow[ur] \arrow[dr] & & (0GGF) \arrow[ur] \arrow[dr] & & (GFFF) \arrow[ur] \arrow[dr] & & (GGGF) \arrow[ur] \arrow[dr] & & \\
 & (00FF) \arrow[ur] & & (0FFF) \arrow[ur] & & (FFFF) \arrow[ur] & & (000F). &
\end{tikzcd}$$

Hence the category $\mathbf{K}_F^a$ has exactly 5 indecomposable objects determined by the modules followed the module $(FFFF)$ in the diagram. Then it is easy to check that the category $\mathbf{K}_F^a$ is special Schurian and the ring $\boldsymbol{R}(\mathrm{K}^a)''$ (in the notation of Lemma 2.1 (ii)) is isomorphic to the ring T in Lemma 2.7. Hence $\#\, l\,\mathrm{her}\,(\boldsymbol{R}(\mathrm{K}^a)) = 25$ and in view of Proposition 2.2 the required equality follows. The proof of remaining part of the lemma is left to the reader.

Proof of Theorem 2.5. (4) $\to$ (1) If the valued poset $(I_R, \boldsymbol{d})$ or its opposite is of one of the forms a) or b) then R is of finite representation type according to [10, 16]. If $(I_R, \boldsymbol{d})$ or its opposite is of one of the forms c) then R is also of finite representation type because Remark 2 and the partial Coxeter functors arguments allow us to reduce to Lemmas 2.6-2.14.

The implication (1) $\to$ (2) follows from [24, Theorem 3.1]. Since (3) $\to$ (4) is proved in [4] it remains to prove (2) $\to$ (3). We will do it by showing that any l-hereditary artinian PI-ring such that either $(I_R, \boldsymbol{d})$ or its opposite is of one of the forms $\mathbf{R}_1$— $\mathbf{R}_9$ is not right pure semisimple. We recall from [28] that this is also the case if $(I_R, \boldsymbol{d})$ is of the form a) or b) in (3).

First we note that without loss of generality we can restrict our considerations to a fixed orientation of nondirected edges in each of the valued posets $\mathbf{R}_1$— $\mathbf{R}_9$. This is an easy consequence of the partial Coxeter functors arguments in the proof of Lemma 2.11 and Proposition 2.4.

Now suppose that $(I_R, \boldsymbol{d})$ is of one of the forms $\mathbf{R}_1$— $\mathbf{R}_7$. Then there exists a source $\boldsymbol{a}$ in I_R such that the ring R_a is hereditary of finite representation type and the valued graph of R_a is homogeneous (we use the notation of Proposition 2.2). It follows that the vector space category $\mathbf{K}_F^a$ is special Schurian and the valued poset of the ring $\boldsymbol{R}(\mathbf{K}^a)$ is homogeneous. Hence the category $l\,\mathrm{her}\,(\boldsymbol{R}(\mathbf{K}^a))$ is equivalent to the category of representations of the poset $I_{R(\mathbf{K}^a)}$ in the sense of Nazarova and Rojter [19] (see [5]). Since R is of infinite representation type [16], then by Proposition 2.2 the category $l\,\mathrm{her}\,(\boldsymbol{R}(\mathbf{K}^a))$ is of infinite representation type and therefore the poset $I_{R(\mathbf{K}^a)}$ is of infinite type. Then, in view of [12, 19] (see also [7]), by applying the Nazarova-Rojter differentiation procedure to the poset $I_{R(\mathbf{K}^a)}$ in a finite number of steps we get a poset of weight $\geqslant 4$. This means that there exists a full additive subcategory $\mathcal{A}$ of $l\,\mathrm{her}\,(\boldsymbol{R}(\mathbf{K}^k))$ and a representation equivalence $\mathcal{A} \to l\,\mathrm{her}\,(S)$ where S is a hereditary ring of the type

$$\begin{tikzcd} \circ \arrow[dr] & & \circ \arrow[dl] \\ & \circ & \\ \circ \arrow[ur] & & \circ \arrow[ul] \end{tikzcd}$$

. Since $l\,\mathrm{her}\,(S)$ is obviously cofinite in $\mathrm{mod}\,(S)$ and by [28] S is not right pure semisimple then we conclude from Propositions 2.2 and 2.4 that R is not right pure semisimple, as required.

Next suppose $(I_R, \boldsymbol{d})$ has the form

$$\begin{tikzcd} & \circ \arrow[d] & \\ & \circ \arrow[d] & \\ a\ \circ \arrow[r, "{(2,1)}"] & \circ \arrow[r, "{(1,2)}"] & \circ\ c \end{tikzcd}, \qquad d_{ac} = d_{ca} = 1\,.$$

In the notation of Proposition 2.2 the ring R_a is hereditary, the valued graph of R_a has the form $\circ \to \circ \to \circ \xrightarrow{(1,2)} \circ$ and $M^a = (0\ 0\ F\ F)$. We note that there are nonzero maps from M^a into the R_a-modules $U = (0\ G\ G\ 0)$, $W = (G\ G\ F)$, whereas there is no nonzero map between U and W. Hence the full subcategory of $\mathfrak{U}(\mathbf{K}^a_F)$ consisting of objects (X, V, t) where $|X|_F$ is of the form $|U|^i \oplus |W|^j$ is cofinite in mod (S), where S a hereditary artinian PI-ring with the valued graph $\circ \xleftarrow{(1,2)} \circ \xrightarrow{(2,1)} \circ$. We know from [28, Corollary 3.4] and [10] that S is not right pure semisimple. Then Propositions 2.2 and 2.4 imply that R is not right pure semisimple.

Finally, we suppose that $(I_R, \boldsymbol{d})$ is of one of the forms

$$\begin{tikzcd} & \circ \arrow[d, dash] & & \\ a \arrow[r, "{(2,1)}"] & \circ \arrow[r, "{(1,2)}"] & c \arrow[r] & \circ \end{tikzcd}, \qquad \begin{tikzcd} & & \circ \arrow[d] & \\ b \arrow[r] & a \arrow[r, "{(2,1)}"] & \circ \arrow[r, "{(1,2)}"] & c \end{tikzcd}, \qquad d_{ac} = d_{ca} = 1\,.$$

Then $M^a = \begin{pmatrix} 0 & & \\ F & F & 0 \end{pmatrix}$, $M^b = \begin{pmatrix} & 0 & \\ F & F & F \end{pmatrix}$ and taking for U and W the R_a-modules $\begin{pmatrix} 0 & & \\ G & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} G & & \\ F & F & 0 \end{pmatrix}$ and R_b-modules $\begin{pmatrix} & 0 & \\ F & G & 0 \end{pmatrix}$, $\begin{pmatrix} & G & \\ F & F & 0 \end{pmatrix}$ respectively we are in the same position as above. Hence we conclude that R is not right pure semisimple. The proof of the theorem is complete.

REMARK 3. The Lemmas 2.6-2.14 give an explicit description of indecomposable modules over non-homogeneous l-hereditary artinian PI-rings of finite representation type. In particular it is easy to see that for every such a module $X = (X_i)$ the dimension of the F_i-space X_i is at most 6 for all i. Moreover X is uniquely determined by its composition factors.

REMARK 4. The functor Φ in Section 1 gives a simple method for reconstructing indecomposable R-modules from indecomposable modules in l her $(\boldsymbol{R}(\mathrm{K}^a))$. In order to describe it suppose that (in the notation of Proposition 2.2) the category K_F^a is special Schurian and we know the indecomposable R_a-modules as well as the indecomposable modules in l her $(\boldsymbol{R}(\mathrm{K}^a))$. This was the case in all situations we consider in Lemmas 2.6-2.14.

Let us identify the R-modules and triples (V_F, Y, t) where Y is an R_a-module and $t\colon V_F \to \mathrm{Hom}\,({}_FM^a, Y)$ is an F-linear map.

If $Y_1, \ldots, Y_n$ are all indecomposable R_a-modules such that $\mathrm{Hom}\,(M^a, Y) \neq 0$ we denote by $\mathfrak{M}_R^a$ the full subcategory of mod (R) consisting of modules having no direct summands of the forms $(F, 0, 0)$, (F, Y_j, t) where t is an isomorphism, $(0, Y, 0)$ where Y is indecomposable different from $Y_1, \ldots, Y_n$. Then the functor Φ induces a full additive functor

$$\hat{\Phi}\colon \mathfrak{M}_R^a \to l\ \mathrm{her}\ (\boldsymbol{R}(\mathrm{K}^a))$$

having the following properties:

(i) $\hat{\Phi}$ reflects isomorphisms.

(ii) Every indecomposable module in l her $(\boldsymbol{R}(\mathrm{K}^a))$ except the unique simple projective one is of the form $\hat{\Phi}(Y)$ where Y is indecomposable.

(iii) If (V_F, Y, t) is an indecomposable module in $\mathfrak{M}_R^a$,

$$Y = Y_1^{s_1} \oplus \ldots \oplus Y_n^{s_n} \quad \text{and} \quad \hat{\Phi}(V_F, Y, t) = (M_i, {}_j\varphi_i)_{1 \leqslant i, j \leqslant n+1}$$

then

$$M_{n+1} = \mathrm{Coker}\, t \quad \text{and} \quad s_i = \dim\,(M_i/\overline{M}_i)_{F_i}$$

where $\overline{M}_i$ is the F_i-subspace of M_i defined in the proof of [31, Theorem 1.5]. Furthermore, if the modules $Y_1, \ldots, Y_n$ and $(M_i, {}_j\varphi_i)$ are uniquely determined by their composition factors then also (V_F, Y, t) is uniquely determined by its composition factors.

The property (iii) gives a method for constructing the indecomposable modules in $\mathfrak{M}_R^a$ from their images under the functor $\hat{\Phi}$.

In a subsequent paper (joint with B. Klemp) we will give a diagrammatic characterization of l-hereditary right QF-2 artinian PI-

rings R for which the category $l\,\mathrm{her}\,(R)$ is of finite representation type. We do it by applying the upper and lower differentiations in the sense of [7] and few simple reduction lemmas like the Lemma 2.1. The list of critical valued posets for this class coincides with that one for the finite type structures of division rings [8].

REFERENCES

[1] M. Auslander, *Rrepresentation theory of artin algebras - I*, Comm. Algebra, **1** (1974), pp. 177-268.

[2] M. Auslander - M. I. Platzeck - I. Reiten, *Coxeter functors without diagrams*, Trans. Amer. Math. Soc., **250** (1979), pp. 1-46.

[3] R. Bautista, *On algebras close to hereditary artin algebras*, An. Inst. Mat. Univ. Nac. Autónoma México, **21**, No. 1 (1981), pp. 21-104.

[4] R. Bautista, *Classification of certain algebras of finite representation type*, Comunicacion Interna No. 51, 1978, Departamento de Matematicas, UNAM, México 20, D.F.

[5] R. Bautista - R. Martinez, *Representations of partially ordered sets and 1-Gorenstein artin algebras*, Proc. Conf. on Ring Theory (Antwerp 1978), Marcel Dekker, Inc. New York and Basel, 1979.

[6] R. Bautista - L. Colavita - L. Salmeron, *On adjoint functors in representation theory*, Proceedings ICRA III, Puebla, 1980, in Lecture Notes in Math., No. 903, Springer-Verlag, 1981, pp. 9-25.

[7] R. Bautista - D. Simson, *Torsionless modules over l-hereditary 1-Gorenstein artinian rings*, to appear in Comm. Algebra, **12** (1984).

[8] V. Dlab - C. M. Ringel, *On algebras of finite representation type*, J. Algebra, **33** (1975), pp. 306-394.

[9] V. Dlab - C. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc., **173** (1976).

[10] P. Dowbor - D. Simson, *Quasi-Artin species and rings of finite representation type*, J. Algebra, **63** (1980), pp. 435-443.

[11] Ju. A. Drozd, *Coxeter transformations and representations of partially ordered sets*, Funkc. Anal. i Priložen., **8** (1974), pp. 34-42 (in Russian).

[12] M. M. Kleiner, *Partially ordered sets of finite type*, Zap. Naučn. Sem. LOMI, **28** (1972), pp. 32-41 (in Russian).

[13] M. M. Kleiner, *Induced modules and comodules and representations of BOCS' and DGS's*, Proceedings ICRA III, Puebla, 1980.

[14] Z. Leszczyński - D. Simson, *On triangular matrix rings of finite representation type*, J. London Math. Soc., **20** (1979), pp. 396-402.

[15] M. Loupias, *Indecomposable representations of finite partially ordered sets*, Lecture Notes in Mathematics 488 (Springer-Verlag, Berlin, Heidelberg, New York, 1975), pp. 201-209.

[16] M. Loupias, *Représentations indécomposables des ensembles ordonnés finis, Thèse*, Université Francois Rabelais de Tours, 1975.

[17] S. Mac Lane, *Categories for the Working Mathematicians*, Springer-Verlag, New York, Heidelberg, Berlin, 1971.

[18] L. A. Nazarova, *Representations of quivers of infinite type*, Izv. Akad. Nauk SSSR, **37** (1973), pp. 749-792.

[19] L. A. Nazarova - A. V. Rojter, *Representations of partially ordered sets*, Zap. Naučn. Sem. LOMI, **28** (1972), pp. 4-31 (in Russian).

[20] L. A. Nazarova - A. V. Rojter, *Kategorielle Matrizen-Probleme und die Brauer-Thrall-Vermutung*, Mittelungen Math. Sem. Giessen, **115** (1975), pp. 1-153.

[21] C. M. Ringel, *Report on the Brauer-Thrall conjectures: Rojter's theorem and the theorem of Nazarova and Rojter*, Lecture Notes in Mathematics, **831** (Springer-Verlag, Berlin, Heidelberg, New York, 1980), pp. 104-136.

[22] C. M. Ringel, *Tame algebras*, Lecture Notes in Mathematics, **831** (Springer-Verlag, Berlin Heidelberg, New York, 1980), pp. 137-287.

[23] A. Rosenberg - D. Zelinsky, *Finiteness of the injective hull*, Math. Z., **70** (1959), pp. 372-380.

[24] D. Simson, *Functor categories in which every flat object is projective*, Bull., Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., **22** (1974), pp. 375-380.

[25] D. Simson, *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math., **96** (1977), pp. 91-116.

[26] D. Simson, *Categories of representations of species*, J. Pure Appl. Algebra, **14** (1979), pp. 101-114.

[27] D. Simson, *Pure semisimple categories and rings of finite representation type*, Corrigendum, J. Algebra, **67** (1980), pp. 25-256.

[28] D. Simson, *Partial Coxeter functors and right pure semisimple hereditary rings*, J. Algebra, **71** (1981), pp. 195-218.

[29] D. Simson, *On pure semisimple Grothendieck categories, II*, Fund. Math., **110** (1980), pp. 107-116.

[30] D. Simson, *Right pure semisimple structures of division rings*, Proceedings of the Eleventh Annual Iranian Math. Conf., Mashhad, March 1980, pp. 186-201.

[31] D. Simson, *Special Schurian vector space categories and l-hereditary right QF-2 rings*, to appear in Commentationes Math., **25** (1984).

Manoscritto pervenuto in redazione il 7 luglio 1982.

Note Added in Proof.

The main theorem of this paper can be extended to a class of triangular matrix rings R of the form written in the introduction with division PI-rings $F_1, \ldots, F_n$ and such that the bilinear maps $c_{ijk}\colon {}_iM_j \otimes {}_jM_k \to {}_iM_k$ are nonzero provided ${}_iM_j$ and ${}_jM_k$ are nonzero. To any such a ring R the valued poset (I_R, d) is associated and Theorem 2.5 is extended in a natural way. This is given in the author's notes *On methods for the computation of indecomposable modules over artinian rings,* published in Proc. Conf. Ring Theory and Algebraic Geometry, Chiba (Japan), 1982, pp. 143-170. A characterization of l-hereditary right QF-2 artinian PI-rings R with l-her(R) of finite type we have mentioned at the end of the paper is given by B. KLEMP and the author in the paper *A diagrammatic characterization of schurian vector space PI-categories of finite type* which will appear in Bull. Acad. Polon. Sci. in 1984.