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On the properties of oscillation and almost periodicity of certain convolutions

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On the Properties of Oscillation and Almost Periodicity of Certain Convolutions.

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Introduction.

In a paper of 1936 (cfr. [7], p. 266) Hardy and Littlewood consider the problem of the asymptotic behaviour of the functions $P(x) = \sum_{n \leqslant x} (1/n) \cos (x/n)$ and $Q(x) = \sum_{n \leqslant x} (1/n) \sin (x/n)$. To be precise, they are interested in proving that $P(x)$ and $Q(x)$ are not bounded: what they really prove (cfr. [7], p. 266) is something more, i.e.

$$(0.1) \qquad P(x) = \Omega(\lg \lg x)$$

$$(0.2) \qquad Q(x) = \Omega((\lg \lg x)^{\frac{1}{4}}) .$$

In 1949 T. M. Flett (cfr. [6]) obtained the following estimates for the functions $P(x)$ and $Q(x)$.

$$(0.3) \qquad P(x) = O((\lg x)^{\frac{2}{3}}(\lg \lg x)^{\frac{1}{3} + \varepsilon})$$

$$(0.4) \qquad Q(x) = O((\lg x)^{\frac{2}{3}}(\lg \lg x)^{\frac{1}{3} + \varepsilon}) .$$

Recently H. Delange (cfr. [4], p. 52) in a paper of 1980 proved that

$$(0.5) \qquad Q(x) = \Omega \pm ((\lg \lg x)^{\frac{1}{2}})$$

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and, by means of a result of Saffari and Vaughan (cfr. [10]),

(0.6) $Q(x) = O((\lg x)^{\frac{2}{3}})$.

Convolutions of the type $\sum_{n \leqslant x} (\alpha(n)/n) f(x/n)$, where $f(x)$ is a periodic function, arise naturally in some problems of number theory.

Besides the two examples $P(x)$ and $Q(x)$ let's consider the following: set $\sigma_0(n) = \sigma(n)/n = \sum_{d|n} 1/d$ and consider the function $S_0(x) = \sum_{n \leqslant x} \sigma_0(n)$ It is easy to see (cfr. [11], p. 100) that

(0.7) $S_0(x) = \dfrac{\pi^2}{6} x - \dfrac{1}{2} \lg x - \sum_{n \leqslant n} \dfrac{1}{n} \left(\left\{ \dfrac{x}{n} \right\} - \dfrac{1}{2} \right) + O(1)$.

Now let $\varphi(n)$ be the Euler's function: remembering that $\varphi(n)/n = \sum_{d|n} \mu(d)/d$ we have immediately

(0.8) $S_1(x) = \sum_{n \leqslant x} \dfrac{\varphi(n)}{n} = \dfrac{6}{\pi^2} x - \sum_{n=1}^{\infty} \dfrac{\mu(n)}{n} \left\{ \dfrac{x}{n} \right\}$.

In both cases the remainder terms are expressed by convolutions of the type considered. The best known estimates of the remainder terms in (0.7) and (0.8) were obtained by Walfisz (cfr. [11] p. 88 and p. 115) using Vinogradov's method: precisely they are

(0.9) $\sum_{n \leqslant x} \dfrac{1}{n} \left(\left\{ \dfrac{x}{n} \right\} - \dfrac{1}{2} \right) = O((\lg x)^{\frac{2}{3}})$

(0.10) $\sum_{n \leqslant x} \dfrac{\mu(n)}{n} \left(\left\{ \dfrac{x}{n} \right\} - \dfrac{1}{2} \right) = O\left(\lg^{\frac{2}{3}} x (\lg \lg x)^{\frac{2}{3}}\right)$.

The asymptotic mean square behaviour of the error terms in (0.7) and (0.8) has also been studied: precisely, setting $S_0(x) = (\pi^2/6)x - \frac{1}{2} \lg x + T_0(x)$ Walfisz proved that (cfr. [12])

(0.11) $\displaystyle\int_0^x T_0^2(u)\, du = \left(\left(\dfrac{\gamma + \lg 2\pi}{2} \right)^2 + \dfrac{5}{144} \pi^2 \right) x + O(x^{\frac{1}{2}} \lg x)$

and successively improved (0.11) by showing that (cfr. [13])

$$(0.12) \qquad \int_0^x T_0^2(u)\,du = \left(\left(\frac{\gamma + \lg 2\pi}{2}\right)^2 + \frac{5}{144}\,\pi^2\right)x + O(x^{\frac{1}{2}})\,.$$

Chowla, 'following Walfisz's method, obtained the following result (cfr. [2])

$$(0.13) \qquad \int_1^x H^2(u)\,du \sim \frac{1}{2\pi^2}\,x \qquad \text{where } H(x) = \sum_{n\leqslant x}\frac{\varphi(n)}{n} - \frac{6}{\pi^2}\,x\,.$$

As for the function $H(x)$, Pillai and Chowla (cfr. [9]) also proved that

$$(0.14) \qquad \sum_{n\leqslant N} H(n) \sim \frac{3}{\pi^2}\,N$$

$$(0.15) \qquad H(x) = \Omega(\lg\lg\lg x)$$

and Erdös and Shapiro (cfr. [5], p. 382) showed that

$$(0.16) \qquad \overline{\lim_{n\to+\infty}}\,H(n) = +\infty\,, \qquad \underline{\lim_{n\to+\infty}}\,H(n) = -\infty$$

disproving Sylvester's conjecture, *i.e.* $H(n)\geqslant 0$ for every $n\in \mathbf{N}$.

The aim of the present paper is to obtain, under suitable assumptions, two kinds of results concerning convolutions of the type $\sum_{n\leqslant x}(\alpha(n)/n)\,f(x/n)$ where $f(x)$ is periodic, and precisely:

a) To prove the existence of the mean value of such convolutions on every arithmetic progression, and to find and explicit formula for this mean value. This will afford us to prove easily properties of unboundedness and of oscillation for the functions considered (see Corollary 1). We will also find, among other things, the result (0.16) of Erdös and Shapiro.

b) To prove that the convolutions considered are always B^2 almost periodic functions.

We remember that a function is B^2 almost periodic if it satisfies the following pair of conditions:

α) $f(x)$ has a Fourier-Bohr series $f(x) \sim \sum a(\lambda_n) \exp [2\pi i \lambda_n x]$.

β) Parseval identity holds, i.e.

$$M(|f|^2) = \sum |a(\lambda_n)|^2 \quad \text{where} \quad M(|f|^2) = \lim_{x \to +\infty} \frac{1}{x} \int_1^x |f(u)|^2 \, du \text{ is finite} .$$

Precisely we obtain the following results:

THEOREM 1. Let $f(x)$ be a periodic function with period 1, of bounded variation on $[0, 1]$ and such that $\int_0^1 f(x)\,dx = 0$. Let $(\alpha(n))_{n \in \mathbb{N}}$ be a sequence of real numbers such that

i) $\alpha(n) = O(1)$

ii) $\sum_{n \leqslant x} \alpha(n) = cx + O(x \lg^{-\alpha} x)$ with $\alpha > 1$, $c \geqslant 0$.

Put $g(x) = \sum_{n \leqslant x} (\alpha(n)/n) f(x/n)$: then the mean value of $g(x)$ on every arithmetic progression exists and we have [1]

$$(1) \qquad M(a, b) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n \leqslant N} g(an + b) =$$

$$= \sum_{k=1}^{\infty} \frac{\alpha(k)}{k} \cdot \left(\frac{1}{k^*} \sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{b}{k} \right) \right) + c \int_1^{+\infty} \frac{f(u)}{u} \, du$$

with $k^* = k/(a, k)$, where as usual (a, k) is the greatest common divisor of a and b.

From theorem 1 we obtain easily the following

COROLLARY 1. Put

$$T_0(x) = \sum_{n \leqslant x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6} x + \frac{1}{2} \lg x = - \sum_{n \leqslant x} \frac{1}{n} \left(\left\{ \frac{x}{n} \right\} - \frac{1}{2} \right) + O(1) ,$$

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[1] The convergence of the series in formula (1) follows from Koksma inequality (4.2) and from the estimate (4.5).

$$H(x) = \sum_{n \leqslant x} \frac{\varphi(n)}{n} - \frac{6}{\pi^2} x = - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ \frac{x}{n} \right\}, \qquad Q(x) = \sum_{n \leqslant x} \frac{1}{n} \sin 2\pi \frac{x}{n}$$

then we have that $T_0(x), H(x), Q(x)$, are not bounded neither from above nor from below.

THEOREM 2. Put $g(x) = \sum_{n \leqslant x} (\alpha(n)/n) f(x/n)$ where the sequence $(\alpha(n))_{n \in \mathbf{N}}$ and the function $f(x)$ satisfy the conditions of theorem 1. Then $g(x)$ is a B^2 almost periodic function.

Although it is possible that the convolutions $g(x) = \sum_{n \leqslant x} (\alpha(n)/n) \cdot f(x/n)$ considered in the theorems 1 and 2 are unbounded (see corollary 1), we note that they are always absolutely bounded on the mean, i.e. $\int_1^x |g(t)| \, dt = O(x)$. This follows immediately from theorem 2 and Schwarz inequality.

We begin by proving some lemmata.

LEMMA 1. Let $g(x) = \sum_{n \leqslant x} (1/n) f(x/n)$ where $f(x)$ is a periodic function of period 1, of bounded variation on $[0, 1]$ and such that $\int_0^1 f(x) \, dx = 0$. If $y(x)$ is a monotonically increasing function such that $0 < y(x) < x$, $\lim_{x \to +\infty} y(x) = +\infty$ and $\lim_{x \to +\infty} (x/y(x)) = +\infty$ we have

$$(1.1) \qquad g(x) = \sum_{n \leqslant x} \frac{1}{n} f\left(\frac{x}{n}\right) = \sum_{n \leqslant y(x)} \frac{1}{n} f\left(\frac{x}{n}\right) + \int_1^{+\infty} \frac{f(u)}{u} \, du +$$

$$+ O\left(\frac{y(x)}{x} + \frac{1}{y(x)} + \frac{x}{y^2(x)}\right).$$

PROOF. The proof is based on Euler's summation formula. Let $F(t)$ be a function of bounded variation on the interval $[n, n+1]$: integrating by parts we have

$$(1.2) \qquad \int_n^{n+1} (t - n - \tfrac{1}{2}) \, dF(t) = \tfrac{1}{2} \big(F(n+1) + F(n) \big) - \int_n^{n+1} F(t) \, dt.$$

From (1.2) obviously follows

$$(1.3) \qquad \tfrac{1}{2}\big(F(n+1)+F(n)\big)=\int_n^{n+1}F(t)\,dt+O(V_{F_{[n,n+1]}})$$

Summing over n we obtain from (1.3)

$$(1.4) \qquad \sum_{n=m}^{N}F(n)=\int_m^N F(t)\,dt+\tfrac{1}{2}\big(F(m)+F(N)\big)+O(V_{F_{[m,N]}})\ .$$

If $f(x)$ is a function which satisfies the conditions of lemma 1 put $F(t)=(1/t)f(x/t)$ where x is fixed. From (1.4) with $m=[y(x)]+1$ and $N=[x]$ we have

$$(1.5) \qquad \sum_{y(x)<n\leqslant x}\frac{1}{n}f\left(\frac{x}{n}\right)=\int_{y(x)}^{x}\frac{1}{t}f\left(\frac{x}{t}\right)dt+O\left(\frac{1}{y(x)}\right)+O(V_{F_{[y(x),x]}})\ .$$

Consider now the term $V_{F_{[y(x),x]}}$: it is not difficult to see that

$$(1.6) \qquad V_{F_{[y(x),x]}}=O\left(\frac{x}{y^2(x)}+\frac{1}{y(x)}\right).$$

In fact, if $y(x)\equiv t_0<t_1<\ldots<t_n\equiv x$ is a partition of the interval $[y(x),x]$ we have

$$(1.7) \qquad \sum_{i=1}^{n}|F(t_i)-F(t_{i-1})|\leqslant\sum_{i=1}^{n}\left|\left(\frac{1}{t_i}-\frac{1}{t_{i-1}}\right)f\left(\frac{x}{t_i}\right)\right|+$$

$$+\sum_{i=1}^{n}\frac{1}{t_{i-1}}\left|f\left(\frac{x}{t_i}\right)-f\left(\frac{x}{t_{i-1}}\right)\right|\leqslant\left(\sup_{0\leqslant x\leqslant 1}|f(x)|\right)\left(\frac{1}{y(x)}-\frac{1}{x}\right)+$$

$$+\frac{1}{y(x)}\,V_{f_{[0,1]}}\left(\frac{x}{y(x)}+1\right).$$

From (1.7) follows (1.6).

As for the first term on the right of equality (1.5), with the sub-

stitution $x/t = u$ we obtain

$$(1.8) \qquad \int\limits_{y(x)}^{x} \frac{1}{t} f\left(\frac{x}{t}\right) dt = \int\limits_{1}^{x/y(x)} \frac{f(u)}{u} du = \int\limits_{1}^{+\infty} \frac{f(u)}{u} du + O\left(\frac{y(x)}{x}\right)$$

where the last equality is easily justified if we observe that

$$(1.9) \qquad \int\limits_{z}^{+\infty} \frac{f(u)}{u} du = \int\limits_{z}^{+\infty} \frac{1}{u^2} \left(\int\limits_{z}^{u} f(t)\, dt \right) du = O\left(\int\limits_{z}^{+\infty} \frac{1}{u^2} du \right) = O\left(\frac{1}{z}\right).$$

From (1.5), (1.6) and (1.8) follows (1.1).

LEMMA 2. Let us set $g(x) = \sum\limits_{n \leqslant x} (\beta(n)/n) f(x/n)$ where $f(x)$ satisfies the assumptions of lemma 1 and suppose that $\sum\limits_{n \leqslant x} \beta(n) = O(x \lg^{-\alpha} x)$ with $\alpha > 1$: if $1 < \alpha' < \alpha$ we have

$$(2.1) \qquad \sum\limits_{n \leqslant x} \frac{\beta(n)}{n} f\left(\frac{x}{n}\right) = \sum\limits_{n \leqslant x/(\lg x)^{\alpha'}} \frac{\beta(n)}{n} f\left(\frac{x}{n}\right) + o(1).$$

PROOF. Put $y = y(x) = x(\lg x)^{-\alpha'}$

$$(2.2) \qquad g(x) = \sum\limits_{n \leqslant x} \frac{\beta(n)}{n} f\left(\frac{x}{n}\right) = \sum\limits_{n \leqslant y(x)} \frac{\beta(n)}{n} f\left(\frac{x}{n}\right) + \sum\limits_{y(x) < n \leqslant x} \frac{\beta(n)}{n} f\left(\frac{x}{n}\right) =$$
$$= \Sigma_1(x) + \Sigma_2(x).$$

Let us consider the sum $\Sigma_2(x)$: setting $B(x) = \sum\limits_{n \leqslant x} \beta(n)$ we have

$$(2.3) \qquad \sum\limits_{y < n \leqslant x} \frac{\beta(n)}{n} f\left(\frac{x}{n}\right) = \sum\limits_{y < n \leqslant x} \frac{B(n) - B(n-1)}{n} f\left(\frac{x}{n}\right) =$$

$$= \sum\limits_{y < n \leqslant x-1} B(n) \left(\frac{1}{n} f\left(\frac{x}{n}\right) - \frac{1}{n+1} f\left(\frac{x}{n+1}\right) \right) + \frac{B([x])}{[x]} f\left(\frac{x}{[x]}\right) -$$

$$- \frac{B([y])}{[y]+1} f\left(\frac{x}{[y]+1}\right) = \sum\limits_{y < n \leqslant x-1} B(n) \left(\frac{1}{n} f\left(\frac{x}{n}\right) - \frac{1}{n+1} f\left(\frac{x}{n+1}\right) \right) +$$

$$+ O(\lg^{-\alpha} y).$$

But

$$(2.4) \qquad \sum_{\nu < n \leqslant x-1} B(n) \left(\frac{1}{n} f\left(\frac{x}{n}\right) - \frac{1}{n+1} f\left(\frac{x}{n+1}\right) \right) =$$

$$= \sum_{\nu < n \leqslant x-1} \frac{B(n)}{n} \left(f\left(\frac{x}{n}\right) - f\left(\frac{x}{n+1}\right) \right) + \sum_{\nu < n \leqslant x-1} \frac{B(n)}{n(n+1)} f\left(\frac{x}{n+1}\right) =$$

$$= \sigma_1(x) + \sigma_2(x) \,.$$

As for $\sigma_1(x)$ and $\sigma_2(x)$ we have

$$(2.5) \qquad \sigma_1(x) = O\left(\sum_{\nu < n \leqslant x-1} (\lg n)^{-\alpha} \left| f\left(\frac{x}{n}\right) - f\left(\frac{x}{n+1}\right) \right| \right) =$$

$$= O\left((\lg y)^{-\alpha} (V_{f_{[0,1]}}) \frac{x}{y} \right) = O(\lg^{\alpha'-\alpha} x) = o(1)$$

$$(2.6) \qquad\qquad \sigma_2(x) = O\left(\sum_{n>y}^{\infty} \frac{1}{n \lg^{\alpha} n} \right) = o(1) \,.$$

The result follows from (2.2), (2.3), (2.4), (2.5) and (2.6).

LEMMA 3. Put as usual $g(x) = \sum_{n \leqslant x} (\alpha(n)/n) f(x/n)$ where the sequence $(\alpha(n))_{n \in \mathbb{N}}$ and the function $f(x)$ satisfy the assumption of theorem 1. If $1 < \alpha' < \alpha$ we have

$$(3.1) \qquad g(x) = \sum_{n \leqslant x} \frac{\alpha(n)}{n} f\left(\frac{x}{n}\right) = \sum_{n \leqslant y(x)} \frac{\alpha(n)}{n} f\left(\frac{x}{n}\right) + c \int_{1}^{+\infty} \frac{f(u)}{u} \, du + o(1)$$

where $y(x) = x \lg^{-\alpha'} x$.

PROOF. Write

$$g(x) = \sum_{n \leqslant x} \frac{\alpha(n)}{n} f\left(\frac{x}{n}\right) = \sum_{n \leqslant x} \frac{c}{n} f\left(\frac{x}{n}\right) + \sum_{n \leqslant x} \frac{\alpha(n) - c}{n} f\left(\frac{x}{n}\right) = g_1(x) + g_2(x)$$

and apply lemma 1 to $g_1(x)$ with $y(x) = x \lg^{-\alpha'} x$, and lemma 2 to $g_2(x)$. Let us recall the definition of discrepancy [2].

[2] If A is a set with a finite number of elements, we write $\# A$ to mean the number of elements of A.

Let $x_1, x_2, ..., x_N$ be points of the unit interval $[0, 1)$: we define discrepancy D_N of the N points considered, the number

$$(4.1) \qquad D_N = \sup_{0 < \alpha \leqslant 1} \left| \frac{\mathcal{N}([0, \alpha))}{N} - \alpha \right|$$

where $\mathcal{N}([0, \alpha]) = \# \{x_1, x_2, ..., x_N\} \cap [0, \alpha)$.

In the proof of theorem 1 we will use the following well known result (cfr. [8], p. 143).

Koksma inequality. Let $f(x)$ be a function of bounded variation $V_{f_{[0,1]}}$ defined on $[0, 1)$: if $x_1, x_2, ..., x_N$ are points in $[0, 1)$ with discrepancy D_N we have

$$(4.2) \qquad \left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_0^1 f(x)\, dx \right| \leqslant V_{f_{[0,1]}} \cdot D_N .$$

We can now prove theorem 1.

PROOF OF THEOREM 1. Let $a \geqslant 1$ and $b \geqslant 0$ be integers and set $u(x) = y(ax + b)$ where $y(x) = x(\log^{-\alpha'} x)$ with $1 < \alpha'$. We write $g_1(x) = \sum_{n \leqslant y(x)} (\alpha(n)/n) f(x/n)$ and consider the sum

$$(4.3) \qquad \sum_{m \leqslant n \leqslant N} g_1(an + b) = \sum_{m \leqslant n \leqslant N} \sum_{k \leqslant u(n)} \frac{\alpha(k)}{k} f\left(\frac{an + b}{k}\right) =$$

$$= \sum_{1 < k \leqslant u(m)} \frac{\alpha(k)}{k} \sum_{m \leqslant n \leqslant N} f\left(\frac{an + b}{k}\right) + \sum_{u(m) < k \leqslant u(N)} \frac{\alpha(k)}{k} \left(\sum_{u^{-1}(k) \leqslant n \leqslant N} f\left(\frac{an + b}{k}\right) \right)$$

where m is a fixed number large enough and u^{-1} denotes the inverse functions of u. If we remember that $f(x)$ is a periodic function with period 1 from $n_1 \equiv n_2(k^*)$, where $k^* = k/(a, k)$, follows obviously

$$f\left(\frac{an_1 + b}{k}\right) = f\left(\frac{an_2 + b}{k}\right).$$

Moreover, if A indicates a complete system of residues to modulus k^* we have

$$\sum_{n \in A} f\left(\frac{an + b}{k}\right) = \sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{b}{k}\right)$$

so that we can evidently write for $N_1 < N_2$

$$(4.4) \qquad \sum_{N_1 \leqslant n \leqslant N_2} f\left(\frac{an+b}{k}\right) = \left(\sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{b}{k}\right)\right)\frac{N_2 - N_1}{k^*} + O(k^*).$$

Consider now the points $x_n = \{n/k^* + b/k\}$ with $1 \leqslant n \leqslant k^*$: it is easy, to see that

$$(4.5) \qquad\qquad\qquad D_{k^*} = O\left(\frac{1}{k^*}\right)$$

where D_{k^*} denotes the discrepancy of these points.

Remembering Koksma inequality (4.2), from (4.5) follows

$$(4.6) \qquad\qquad \frac{1}{k^*}\sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{b}{k}\right) = O\left(\frac{1}{k^*}\right).$$

From (4.3) and (4.4) we then obtain

$$(4.7) \qquad \frac{1}{N}\sum_{n \leqslant N} g_1(an+b) = \sum_{k \leqslant u(N)} \frac{\alpha(k)}{k}\left(\frac{1}{k^*}\sum_{n \leqslant k^*} f\left(\frac{n}{k^*} + \frac{b}{k}\right)\right) + O\left(\frac{u(N)}{N}\right)$$

If we observe that (4.6) implies the convergence of the series on the right of (4.7), in order to obtain theorem 1 it is enough to take the limit in (4.7) for $N \to +\infty$, and remember (3.1) of lemma 3.

We can now prove easily corollary 1.

EXAMPLE 1. Let us prove that

$$\overline{\lim_{x \to +\infty}} T_0(x) = +\infty \quad \text{and} \quad \underline{\lim_{x \to +\infty}} T_0(x) = -\infty.$$

A simple calculation gives

$$(5.1) \qquad\qquad \sum_{n \leqslant k^*}\left\{\frac{n}{k^*} + \frac{b}{k}\right\} = \frac{k^*-1}{2} + \left\{\frac{b}{(a,k)}\right\}$$

where, as usual, $k^* = k/(a,k)$.

If we remember that

$$- T_0(x) = \sum_{n \leqslant x} \frac{1}{n} \left(\left\{ \frac{x}{n} \right\} - \frac{1}{2} \right) + O(1) = T_1(x) + O(1) \,,$$

from formula (1) of theorem 1 and from (5.1) we get

$$(5.2) \qquad \lim_{N \to +\infty} \frac{1}{N} \sum_{n \leqslant N} T_1(an + b) = M(a, b) =$$

$$= \sum_{k=1}^{\infty} \frac{(a, k)}{k^2} \left(\left\{ \frac{b}{(a, k)} \right\} - \frac{1}{2} \right) + \int_{1}^{+\infty} \frac{\{u\} - \frac{1}{2}}{u} \, du \,.$$

From (5.2) follows immediately

$$\lim_{n \to +\infty} M(n!, 0) = -\infty \,, \qquad \lim_{n \to +\infty} M(n!, n! - 1) = +\infty$$

and this concludes the proof.

EXAMPLE 2. Let us now prove that

$$\overline{\lim_{x \to +\infty}} H(x) = +\infty \,, \qquad \underline{\lim_{x \to +\infty}} H(x) = -\infty \,.$$

If we remember that $\sum_{n=1}^{\infty} \mu(n)/n = 0$ (prime number theorem) it is easy to see that

$$H(x) = - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ \frac{x}{n} \right\} = - \sum_{n \leqslant x} \frac{\mu(n)}{n} \left(\left\{ \frac{x}{n} \right\} - \frac{1}{2} \right) + o(1) \,.$$

We now observe that the conditions i) and ii) of theorem 1 are evidently satisfied with $\alpha(n) = \mu(n)$ (prime number theorem with an estimate of the remainder), so that from formula (1) of theorem 1 and from (5.1) follows immediately

$$(6.1) \qquad \lim_{N \to +\infty} \frac{1}{N} \sum_{n \leqslant N} H(an + b) = M(a, b) =$$

$$= - \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} (a, k) \left(\left\{ \frac{b}{(a, k)} \right\} - \frac{1}{2} \right).$$

Let c be a fixed positive integer and set $b = a - c$: we have from (6.1)

$$(6.2) \qquad M(a, a - c) = \frac{1}{2} \sum_{\substack{k=1 \\ (a,k)|c}}^{\infty} \frac{\mu(k)}{k^2} (a, k) - \sum_{\substack{k=1 \\ (a,k)\nmid c}}^{\infty} \frac{\mu(k)}{k^2} (a, k) \left(\frac{1}{2} - \left\{ \frac{c}{(a, k)} \right\} \right) =$$

$$= \sum_{\substack{k=1 \\ (a,k)|c}}^{\infty} \frac{\mu(k)}{k^2} (a, k) - \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} (a, k) + \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} (a, k) \left\{ \frac{c}{(a, k)} \right\} = S_1 + S_2 + S_3 .$$

Choose now $a = n!$: it is easy to see that

$$\lim_{n \to +\infty} S_1 = \sum_{k|c} \frac{\mu(k)}{k} = \frac{\varphi(c)}{c} \quad \text{and} \quad \lim_{n \to +\infty} S_3 = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \left\{ \frac{c}{k} \right\} = - H(c) .$$

As for S_2, observing that $f(k) = \mu(k)(a, k)$ is multiplicative, we have

$$(6.3) \qquad S_2 = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} (a, k) = \prod_{p|a} \left(1 - \frac{1}{p} \right) \cdot \prod_{p \nmid a} \left(1 - \frac{1}{p^2} \right)$$

so that $\lim\limits_{n \to +\infty} S_2 = 0$.

From (6.2) follows

$$(6.4) \qquad \lim_{n \to +\infty} M(n!, n! - c) = \frac{\varphi(c)}{c} - H(c) .$$

If we remember that $H(x)$ is not bounded (cfr. [9]), and that the choice of c is arbitrary, from (6.4) follows that we must necessarily have

$$\varlimsup_{n \to +\infty} H(n) = +\infty \quad \text{and} \quad \varliminf_{n \to +\infty} H(n) = -\infty .$$

This is result (0.16) of Erdôs and Shapiro (cfr. [5]).

EXAMPLE 3. Let us show that

$$\varlimsup_{x \to +\infty} Q(x) = +\infty , \quad \varliminf_{x \to +\infty} Q(x) = -\infty .$$

If as usual, $k^* = k/(a, k)$ we have

(7.1) $S = \sum\limits_{n \leqslant k^*} \sin 2\pi \left(\dfrac{n}{k^*} + \dfrac{b}{k} \right) = \begin{cases} 0 & \text{if } k \nmid a \\ \sin 2\pi(b/k) & \text{if } k \mid a \end{cases}$

From theorem 1 and (7.1) follows

(7.2) $\lim\limits_{N \to +\infty} \dfrac{1}{N} \sum\limits_{n \leqslant N} Q(an + b) = M(a, b) =$

$$= \sum\limits_{k \mid a} \frac{1}{k} \sin 2\pi \frac{b}{k} + \int\limits_{1}^{+\infty} \frac{\sin 2\pi u}{u} \, du \, .$$

Now choose $a = 4b$ and let's suppose that $p \mid b \Rightarrow p \equiv 1(4)$, where p denotes a prime divisor. In this case we have

$$\sum\limits_{k \mid a} \frac{1}{k} \sin 2\pi \frac{b}{k} = \sum\limits_{b_1 \mid b} \frac{1}{4 b_1} \sin \frac{\pi}{2} \frac{b}{b_1} = \frac{1}{4} \sum\limits_{b_1 \mid b} \frac{1}{b_1} \, ,$$

and this implies that $Q(x)$ is not bounded above, since $\sum\limits_{p \equiv 1(4)} (1/p) = +\infty$.
If we choose $a = 4c$ and $b = 3c$ with c such that $p \mid c \Rightarrow p \equiv 1(4)$ we obviously obtain that $Q(x)$ is not bounded below. This concludes the proof of corollary 1.

We will now prove theorem 2. This theorem will follow rapidly from analogous results obtained by the author in a preceding paper (cfr. [3] theorems 2 and 3).

First of all let's determine the Fourier-Bohr series of our convolutions. In what follows $M(f(x))$ will always denote the mean asymptotic value of $f(x)$, i.e.

$$M(f(x)) = \lim\limits_{x \to +\infty} \frac{1}{x} \int\limits_{1}^{x} f(t) \, dt \, .$$

We first observe that theorem 2 of [3] can be written as follows:

LEMMA 4. Let $f(x)$ be a periodic function of period 1, of bounded variation on $[0, 1)$ and such that $\int\limits_{0}^{1} f(x)\, dx = 0$; let also $f(x) \sim$

$\sim \sum\limits_{n=-\infty}^{\infty} A(n) \exp\left[2\pi inx\right]$ be the Fourier series of $f(x)$. Let $(\alpha(n))_{n\in\mathbb{N}}$ be a sequence of real numbers such that

    i) $\alpha(n) = O(1)$.

Put $g(x) = \sum\limits_{n\leqslant x} (\alpha(n)/n)\, f(x/n)$, then we have

(3) $C(\lambda) = M\big(g(x) \exp\left[-2\pi i\lambda x\right]\big) =$

$$= \begin{cases} 0 & \text{if } \lambda \text{ is irrational}\,(^3) \\[2ex] \displaystyle\sum_{\substack{n=1 \\ n\equiv 0(s)}}^{\infty} \frac{\alpha(n)}{n}\, A\left(n\,\frac{r}{s}\right) & \text{if } \lambda \text{ is rational}\,, \quad \lambda = \frac{r}{s} \neq 0\,. \end{cases}$$

If, in addition to condition i) $\alpha(n) = O(1)$, we have also ii) $\sum\limits_{n\leqslant x}\alpha(n) = o(x)$ then $M\big(g(x)\big) = 0$.

For the proof see [3], theorem 2.

It is now easy to prove the following

LEMMA 5. Put $g(x) = \sum\limits_{n\leqslant x} (\alpha(n)/n)\, f(x/n)$, where the sequence $(\alpha(n))_{n\in\mathbb{N}}$ satisfies the conditions

    i) $\alpha(n) = O(1)$

    ii) $\sum\limits_{n\leqslant x}\alpha(n) = cx + o(x), \ c\geqslant 0$

and $f(x) \sim \sum\limits_{n=-\infty}^{\infty} A(n) \exp\left[2\pi inx\right]$ satisfies the same assumptions as in lemma 4. Then the Fourier-Bohr series of $g(x)$ is given by

(8.1) $g(x) \sim C(0) + \sum C(r,s) \exp\left[2\pi i(r/s)x\right]$

$(^3)$ The convergence of the series $\sum\limits_{\substack{n=1 \\ n\equiv 0(s)}}^{\infty} (\alpha(n)/n)\, A\big(n(r/s)\big)$ follows from the estimate $A(n) = O(1/n)$, since $f(x)$ is of bounded variation (cfr. [1], vol. 1, p. 71).

where the sum is taken over all the rationals r/s with $(r, s) = 1$,

$$r \in \mathbf{Z}/\{0\}, \quad s \geqslant 1 \quad \text{and} \quad C(r, s) = M\left(g(x) \exp - 2\pi i \frac{r}{s} x\right) =$$

$$= \sum_{\substack{n=1 \\ n \equiv O(s)}}^{\infty} \frac{\alpha(n)}{n} A\left(n \frac{r}{s}\right),$$

$$C(0) = M(g(x)) = c \int_{1}^{+\infty} \frac{f(u)}{u} du.$$

PROOF. Set

$$(8.2) \qquad g(x) = c \sum_{n \leqslant x} \frac{1}{n} f\left(\frac{x}{n}\right) + \sum_{n \leqslant x} \frac{\alpha(n) - c}{n} f\left(\frac{x}{n}\right) = g_1(x) + g_2(x).$$

Let us first prove that

$$(8.3) \qquad M(g_1(x)) = c \int_{1}^{+\infty} \frac{f(u)}{u} du.$$

By choosing $y(x) = x/(\lg x)^{\alpha'}$ with $\alpha' > 1$ in formula (1.1) of Lemma 1 we get

$$(8.4) \qquad g_1(x) = c \sum_{n \leqslant y(x)} \frac{1}{n} f\left(\frac{x}{n}\right) + c \int_{1}^{+\infty} \frac{f(u)}{u} du + o(1).$$

If we denote with y^{-1} the inverse function of $y(x) = x \lg^{-\alpha'} x$, we have, for x_0 fixed large enough,

$$(8.5) \quad \int_{x_0}^{x} \sum_{n \leqslant y(t)} \frac{1}{n} f\left(\frac{t}{n}\right) dt = \sum_{n \leqslant y(x_0)} \frac{1}{n} \int_{x_0}^{x} f\left(\frac{t}{n}\right) dt + \sum_{y(x_0) < n \leqslant y(x)} \frac{1}{n} \int_{y^{-1}(n)}^{x} f\left(\frac{t}{n}\right) dt =$$

$$\sum_{y(x_0) < n \leqslant y(x)} \int_{y^{-1}(n)/n}^{x/n} f(u) du + O(1) = O(y(x))$$

because $f(x)$ is periodic with a zero mean value.

From (8.4) and (8.5) follows immediatly (8.3).

In order to prove lemma 5, apply lemma 4 to $g_1(x)$ and $g_2(x)$ in (8.2).

To complete the proof of theorem 2 we remember the following result:

LEMMA 6.   Set $g_3(x) = \sum\limits_{n \leqslant \nu(x)} (\alpha(n)/n) f(x/n)$ where $y(x) = x/(\lg x)^{\alpha'}$ with $\alpha' > 1$. and suppose that the sequence $(\alpha(n))_{n \in \mathbb{N}}$ is bounded and that for the function $f(x) \sim \sum\limits_{n = -\infty}^{\infty} A(n) \exp [2\pi i n x]$ hold the same assumptions as in lemma 4: then $M(|g_3(x)|^2)$ is finite and we have (⁴)

$$(8.6) \qquad M(|g_3(x)|^2) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha(l)}{l} \frac{\alpha(m)}{m} \left(\sum_{k=-\infty}^{\infty} A\left(\frac{l}{(l, m)} k \right) \overline{A}\left(\frac{m}{(l, m)} k \right) \right).$$

For the proof see [3], pp. 242-243.

We can now prove theorem 2.

The formula (3.1) of lemma 3 allows us to write

$$(8.7) \qquad g(x) = \sum_{n \leqslant x} \frac{\alpha(n)}{n} f\left(\frac{x}{n} \right) = \sum_{n \leqslant \nu(x)} \frac{\alpha(n)}{n} f\left(\frac{x}{n} \right) + c \int_{1}^{+\infty} \frac{f(u)}{u} \, du + R(x) =$$

$$= g_3(x) + c \int_{1}^{+\infty} \frac{f(u)}{u} \, du + R(x)$$

where $y(x) = x(\lg^{-\alpha'} x)$ with $\alpha' > 1$ and $R(x) = o(1)$.

If we remember lemma 6 we obtain immediately

$$(8.8) \qquad M(|g(x)|^2) = M(|g_3(x)|^2) + \left(c \int_{1}^{+\infty} \frac{f(u)}{u} \, du \right)^2.$$

It is easy to justify (8.8) if we observe that (8.5) implies $M(g_3(x)) = 0$, and that $M(g_3(x)R(x)) = 0$ follows from Schwarz inequality.

In order to conclude the proof of theorem 2 it is enough to remember

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(⁴) The convergence of the series on the right side of equality (8.6) follows from the estimate $A(n) = O(1/n)$, since $f(x)$ is of bounded variation.

the following identity (for a proof see [3] p. 239)

$$(8.9) \qquad \sum |C(r, s)|^2 = M(|g_3(x)|^2)$$

the lemma 5 and (8.8).

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