RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 70 (1983), p. 99-108

<http://www.numdam.org/item?id=RSMUP_1983__70__99_0>

Rend. Sem. Mat. Univ. Padova, Vol. 70 (1983)

An Asymptotic Stability Theorem for Nonautonomous Functional Differential Equations (*).

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Abstract - An asymptotic stability theorem for nonautonomous functional differential equations with unbounded right-hand side is given. This result, which extends the classical Liapunov theorem, is obtained following Matrosov ideas. An example is also presented.

1. Introduction.

In this paper we give an asymptotic stability theorem for nonautonomous functional differential systems with unbounded right-hand side.

Extensions of the classical Liapunov and Matrosov theorems to functional equations with bounded r.h.s. are well known (see [3], [4], [7], [10]). It is also well known that if the boundedness condition is removed, some difficulties arise in the investigation of the asymptotic behavior of solutions.

If one follows the strategy used in o.d.e. case, (see for example [1], [2], [5], [6], [12]), a natural approach to these problems is to assume that the given equation satisfies a standard Condition (A), see sect. 2; two different methods are then available. The first one uses the limit-

(*) Work performed under the auspices of the Italian Council of Research (C.N.R.).

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ing equations techniques and the extension of La Salle invariance principle.

The second method, which follows Matrosov ideas, introduces several auxiliary functionals or functions.

The first approach is considered in [11]. The author gives there a general form for limiting equations of nonautonomous functional systems, assuming that the r.h.s. of the given equation satisfies an « uniform Condition (A) », see sect. 2, and moreover is precompact and regular. It is also shown that the invariance principle can be extended to this case and that the uniform asymptotic stability of the stationary solution is « inherited » from the same property of the limiting equations solutions. However this method works only in proving uniform asymptotic stability results.

In this paper we develop the second approach to the problem.

To be more precise, we give sufficient conditions for the asymptotic stability, not necessarely uniform, by introducing an auxiliary function, say W, with suitable properties. W « checks » the behavior of solutions close to the sets in which the upper right-hand derivate of the Liapunov functional V is no more sign definite. In this way one does not need to study the asymptotic behavior of limiting equations solutions and it is not necessary to require any precompactness and regularity hypotheses on the given equation. Moreover it is also possible to show that the uniform asymptotic stability is a consequence of the « uniform Condition (A) » (see Remark 2).

Notice that, instead of the function W, one can require the existence of a functional W which has, along the motion, the same properties as the function. However, as pointed out in the Remark 1, to test these properties by means of functionals is in general much more difficult.

In sect. 2 of this paper we give basic definitions, sect. 3 is devoted to the main theorem, an example is presented in sect. 4.

2. Definitions and notations.

Given any set $A \subset R^n$, we define $\mathcal{C}_A = \mathcal{C}([-r, 0], A)$ the set of all continuous functions mapping $[-r, 0] \to A$, $r \in R$, $r > 0$ and designate the uniform norm of an element $\varphi \in \mathcal{C}_A$ by $\|\varphi\|_r = \sup (\|\varphi(\theta)\|, \theta \in [-r, 0])$, here $\|\cdot\|$ is the usual euclidean norm.

Consider now a retarded functional differential equation

$$\dot{x} = F(t, x_t) \tag{2.1}$$

where $F\colon R \times \mathcal{C}_D \to R^n$, $D = \{x \in R^n\colon \|x\| < H\}$, is continuous.

As usual, for a given $t_0 \in R$, $\varphi \in \mathcal{C}_D$ and for some $c > 0$, a function $x(t_0, \varphi) \equiv x \in \mathcal{C}([t_0 - r, t_0 + c), R^n)$ such that $x_{t_0} = \varphi$ and such that $x(t)$ satisfies (2.1) for $t \in [t_0, t_0 + c)$ is said a solution of (2.1) through (t_0, φ). We shall denote by x_t the r-profile of x, i.e. $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

We assume through this paper that F satisfies the following condition:

Condition (A). Let I be an interval $[a, b]$ of lenght greater than r. For each $\alpha > 0$, each compact set $K \subset D$, each $\chi \in \mathcal{C}(I, K)$, there exist $T = T(\alpha, K, \chi) \in [a + r, b]$ and $\zeta = \zeta(\alpha, K) > 0$ such that if

$$\left\| \int_{t'}^{t} F(s, \chi_s)\, ds \right\| > \alpha \qquad \text{for } t, t' \in (T, b]\,,$$

then $|t - t'| > \zeta$.

If T does not depend on the particular choice of the function χ, we will refer to the previous assumption as to the « uniform Condition (A) ».

It is not difficult to show that if F satisfies the Condition (A), then for each $(t_0, \varphi) \in R \times \mathcal{C}_D$ there exists a noncontinuable solution of (2.1) through (t_0, φ) defined on a maximal interval of existence $[t_0, \omega)$; moreover if $\omega < \infty$, then $\varrho(x(t), \partial D) \to 0$ as $t \to \omega^-$. In fact the Condition (A) implies that for every $\alpha > 0$ there exist a $\zeta > 0$ and a $T > t_0 + r$ such that

$$\|x(t' + \zeta) - x(t')\| = \left\| \int_{t'}^{t'+\zeta} F(s, x_s)\, ds \right\| < \alpha \qquad \text{for all } t' > T,\ t' + \zeta < \omega\,,$$

that is x is uniformly continuous on $[T, \omega)$. From this statement, following the usual theory of functional differential equations, see for example [4], the result is proved.

Suppose from now on that $F(t, 0) = 0$ for all $t \in R$. Let us review some basic definitions of the stability theory.

DEFINITION 2.1. The solution $x = 0$ of the equation (2.1) is said to be:

(i) *stable* if for any $t_0 \in R$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that whenever $\|\varphi\|_r < \delta$, the solution through (t_0, φ) exists and $\|x_t\|_r < \varepsilon$ for all $t \in [t_0, \omega)$. If the number δ is independent on t_0, the solution $x = 0$ is said to be uniformly stable [1];

(ii) *asymptotically stable* if it is stable and attractive i.e. there exists a $\delta_1 = \delta_1(t_0) > 0$ such that whenever $\|\varphi\|_r < \delta_1$, $x(t) \to 0$ in the following manner: for each $\eta > 0$, there exists a $T = T(t_0, \varphi, \eta) > 0$ such that $\|x_t\|_r < \eta$ for all $t \geqslant t_0 + r + T$.

If the stability is uniform and $T = T(\eta)$, $x = 0$ is said to be uniformly asymptotically stable.

If $V: R \times \mathcal{C}_D \to R$ is continuous and x_{t+h}, $t \in [t_0 + r, \omega)$ is a solution of (2.1) through (t, φ), let define

$$D^+ V(t, \varphi) = \limsup_{h \to 0^+} \frac{1}{h} \left[V\big(t + h, x_{t+h}(t, \varphi)\big) - V(t, \varphi) \right]. \tag{2.2}$$

$D_+ V$ is the upper right-hand derivate of V along the solutions of (2.1).

Let us assume finally the following definition.

DEFINITION 2.2. A function $W \in \mathcal{C}^1(R \times D, R)$ is definitely divergent on some set $S \subset \mathcal{C}_D$, if for any $\nu \in (0, H)$, $\chi \in \mathcal{C}(I, D)$ there exist $A = A(\nu, H) > 0$, $B = B(\nu, H) > 0$, $\Theta = \Theta(\nu, \chi) \in [a + r, b]$ such that for all $t, \bar{t} \in [\Theta, b]$, $t > \bar{t}$, one has

$$\left| \int_{\bar{t}}^{t} \dot{W}(s, \chi_s)\, ds \right| \geqslant A(t - \bar{t}) \tag{2.3}$$

as long as $\nu < \|\chi_s\|_r < H$, $\varrho(\chi_s, S) < B$ for $s \in [\bar{t}, t]$.

Here $\dot{W}(t, x_t) = F(t, x_t) \operatorname{grad} W(t, x) + \partial W(t, x)/\partial t$.

REMARK 1. Instead of the previous definition, we can set the following assumption in terms of functionals.

[1] From the previous remark it follows that if F satisfies the Condition (A) and if the null solution is stable, then a noncontinuable solution is defined on $[t_0, \infty)$, for sufficiently small initial data φ.

DEFINITION 2.2′. $W: R \times \mathcal{C}_D \to R$ continuous is said definitely divergent along the motion on some set $S \subset \mathcal{C}_D$, if for any $\nu \in (0, H)$, x_t solution of (2.1) through (t_0, φ), there exist $A = A(\nu, H) > 0$, $B = B(\nu, H) > 0$, $\Theta = \Theta(\nu, \varphi) \in [t_0 + r, \omega)$ such that for all $t_1, t_2 \in [\Theta, \omega)$, $t_2 > t_1$

$$|W(t_2, x_{t_2}) - W(t_1, x_{t_1})| \geqslant A(t_2 - t_1) \tag{2.3'}$$

as long as $\nu < \|x_s\|_r < H$, $\varrho(x_s, S) < B$ for $s \in [t_1, t_2]$.

This assumption, stated in terms of functionals, is the natural generalization of the definition given in the o.d.e. case (see Def. 2.1 in [5]). However let us remark that (2.3′) stands in place of (2.3). This depends on the fact that, as it is well known, there are no general criteria to differentiate a functional along a solution of a functional differential equation. Then we must introduce the Dini derivate of a functional along the motion. Unfortunately we are able to obtain estimates on the growth of the functional itself along the solutions only in the case that the Dini derivate is sign semidefinite (see [8]). By the way notice that in [10] a concept analogous to that one of the previous Def. 2.2′ is introduced. D^+W is said integrally bounded along the motion on S if for an arbitrary $B > 0$ there exist a T_B and a continuous function $\eta(t)$ such that $D^+W \leqslant \eta(t)$, $\int_t^{t+T_B} \eta(s)\, ds \leqslant -B$. Though the author does not observe it explicitely, this in general implies $W(t + T_B) - W(t) \leqslant -B$ only if $\eta(t) \leqslant 0$.

3. The asymptotic stability.

In this section the asymptotic stability of the null solution of (2.1) is proved by means of an auxiliary functional, as in the Liapunov theorem, and introducing moreover a second auxiliary function.

THEOREM 3.1. Suppose that F in (2.1) is continuous and satisfies the Condition (A). Let be $u, v: R^+ \to R^+$ continuous nondecreasing functions such that $u(0) = v(0) = 0$ and $u(s), v(s) > 0$ for $s \in (0, H)$.

Moreover assume that $V: R \times \mathcal{C}_D \to R$ is continuous and such that

(i) $u(\|\varphi(0)\|) \leqslant V(t, \varphi) \leqslant v(\|\varphi\|_r)$;

(ii) $D^+ V(t, \varphi) \leqslant -V^*(\varphi) \leqslant 0$

where $V^*: C_D \to R$ is continuous and nonnegative.

If $E^* = \{\varphi \in \mathcal{C}_D\colon V^*(\varphi) = 0\}$, let $W \in C^1(R\times D, R)$ be definitely divergent on E^* and such that $|W(t, x)| < L$ for all $(t, x) \in R\times D$.

Then $x = 0$ is asymptotically stable for (2.1).

PROOF. The uniform stability of the zero solution of (2.1) is immediate from (i) and (ii). One must prove that the origin is attractive. Consider δ_1 of the Def. 2.1 as $\delta(\varepsilon)$ of the uniform stability. Let be $\eta \in (0, \varepsilon)$, it is sufficient to show that for every solution through (t_0, φ), $\|\varphi\|_r < \delta(\varepsilon)$, there exists a $t_1 \in [t_0 + r, \infty)$ such that $\|x_{t_1}\|_r < \delta(\eta)$: this will imply, because of the uniform stability, that $\|x_t(t_1, x_{t_1})\|_r < \eta$ for all $t \geqslant t_1 + r$. Then the attractivity will be proved. Suppose by contradiction that there exist a solution through (t_0, φ), $\|\varphi\|_r < \delta(\varepsilon)$ and an $\eta \in (0, \varepsilon)$ such that $\delta(\eta) \leqslant \|x_t\|_r < \varepsilon$ for all $t \geqslant t_0 + r$.

— First of all we prove that for every positive number k, there exists a divergent sequence $\{\tau_n\}$ such that $\varrho(x_{\tau_n}, E^*) < k$.

Assume in fact that there exists a $\bar{K} > 0$ such that $\varrho(x_t, E^*) \geqslant \bar{K}$ for all $t \in [t_0, \infty)$. It is easy to prove [2], that there exists a positive number $c = c(\bar{K})$ such that, along the solutions of (2.1), one has

$$V^*(x_t) \geqslant c > 0; \tag{3.1}$$

from hypothesis (ii), taking into account (3.1), $D^+ V(t, x_t) \leqslant - V^*(x_t) \leqslant -c < 0$. This will imply $V \to -\infty$ as $t \to \infty$, contradicting hypothesis (i).

— We prove now that, from a certain time on, the solution is « rejected » from the set E^*.

In fact from the hypotheses there exists a bounded function W, definitely divergent on E^*. Then as long as $\varrho(x_t, E^*) < B(\eta, \varepsilon)$ for $t \geqslant \Theta$, it follows that

$$2L > |W(t, x(t)) - W(\bar{t}, x(\bar{t}))| \geqslant A(t - \bar{t})$$

[2] To prove (3.1) it is sufficient to suppose, by contradiction, that there exists a sequence $\{x_{t_n}\}$, along the motion, such that

$$\varrho(x_{t_n}, E^*) \geqslant \bar{K}\,, \qquad \lim_{t_n\to\infty} V^*(x_{t_n}) = 0\,. \tag{3.2}$$

But the uniform stability and the Condition (A) imply that $\{x_{t_n}\}$ belongs to a compact set, thus (3.2) is impossible.

for all $t, \bar{t} \in [\Theta, \infty)$, $t > \bar{t}$. Then $\varrho(x_t, E^*) < B$ only for a finite time interval $t - \bar{t} < 2L/A$.

Consider now the previous $k = B/2$. As a consequence of the previous two steps, there exist two divergent sequences $\{s_n'\}, \{s_n''\}$, $s_n' < s_n''$, such that s_n' is the last instant for which $\varrho(x_{s_n'}, E^*) = B/2$, s_n'' is the first instant for which $\varrho(x_{s_n''}, E^*) = B$. Of course

$$B/2 < \varrho(x_s, E^*) < B \quad \text{for all } s \in (s_n', s_n'') . \tag{3.3}$$

— We are ready now for the contradiction argument.

From the hypothesis (ii), taking into account (3.3), one has for all $t \in [s_n', s_n'']$

$$D^+ V(t, x_t) \leqslant - V^*(x_t) \leqslant - \tilde{c}$$

where $\tilde{c} = \tilde{c}(B/2) > 0$. Then

$$V(s_n'', x_{s_n''}) - V(s_n', x_{s_n'}) \leqslant - \tilde{c}(s_n'' - s_n') .$$

But $\varrho(x_{s_n'}, x_{s_n''}) \geqslant B/2$. From the Condition (A) this implies that there exist an $\bar{n} = \bar{n}(B/2, \mathcal{B}_\varepsilon, \varphi)$, $\zeta = \zeta(B/2, \mathcal{B}_\varepsilon)$ such that $s_n' - s_n'' > \zeta$ for $n > \bar{n}$. Here $\mathcal{B}_\varepsilon$ denotes the set of all $\varphi \in \mathcal{C}_D$ such that $\|\varphi\|_r < \varepsilon$.

Then $V \to -\infty$ as $n \to \infty$ and this contradicts hypothesis (i). The result is then proved.

REMARK 2. If F in the previous theorem satisfies the « uniform Condition (A) » and Θ of Def. 2.2 is independent on χ, then $x = 0$ is uniformly asymptotically stable.

4. An example.

Consider the following functional differential equation:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = - h(t, x(t), y(t))\, y(t) - f(x(t)) + \\ \qquad + \displaystyle\int_{-r(t)}^{0} k(t + \theta, x(t + \theta))\, y(t + \theta)\, d\theta \end{cases} \tag{4.1}$$

where we assume h, k and f continuous, $f(0) = 0$ and $h(t, x, y) \geqslant b > 0$, $f(x)/x \geqslant a > 0$ for $x \neq 0$, $|k(t, x)| \leqslant K$ and $0 \leqslant r(t) \leqslant r < b/K$. We suppose moreover that $h(t, x, y)$ satisfies the Condition (A) of sect. 2.

This example is considered also in [11], but there the author makes the additional assumption that the r.h.s. of (4.1) is positively pre-compact and regular.

We are interested in the asymptotic stability of the zero solution. Consider the Liapunov functional defined by:

$$V(\varphi, \psi) = 2\int_0^{\varphi(0)} f(s)\, ds + \psi^2(0) + K\int_{-r}^{0}\int_{\tau}^{0} \psi^2(\theta)\, d\theta\, d\tau$$

V satisfies the hypothesis (i) of the Theorem 3.1. Moreover one has:

$$D^+ V(\varphi, \psi) \leqslant -(b/r - K) r \psi^2(0) - K\int_{-r}^{0} (|\psi(0)| - |\psi(\theta)|)^2\, d\theta \leqslant 0$$

this yields the uniform stability of the null solution of (4.1).

In this particular case one has moreover

$$E^* = \{(\varphi, \psi)\colon \psi(\theta) = 0 \text{ for all } \theta \in [-r, 0]\}.$$

Consider now the auxiliary function $W = xy$ and let set

$$C_{\delta(\eta),\varepsilon} = \{(\varphi, \psi)\colon \delta^2(\eta) \leqslant \|\varphi\|_r^2 + \|\psi\|_r^2 < \varepsilon^2\}$$

where $\delta(\eta)$ and ε are the same as in the Theorem 3.1.

We prove that W is definitely divergent on E^*. Consider in fact the arbitrary continuous function $\chi = (\chi', \chi'')$, $\chi_t \in C_{\delta(\eta),\varepsilon}$, and suppose that $\|\chi_t''\|_r < \delta(\eta)/2$. One has

$$\int_{\bar t}^{t} \dot W(\chi_\tau', \chi_\tau'')\, d\tau = \int_{\bar t}^{t} [\chi''^2(\tau) - \chi'(\tau)\chi''(\tau) h(\tau, \chi'(\tau), \chi''(\tau)) -$$

$$- \chi'(\tau) f(\chi'(\tau)) + \chi'(\tau)\int_{-r(\tau)}^{0} k(\tau + \theta, \chi'(\tau + \theta))\, \chi''(\tau + \theta)\, d\theta]\, d\tau \leqslant$$

$$\leqslant \int_{\bar{t}}^{t} \Big[\chi''^2(\tau) + |\chi'(\tau)||\chi''(\tau)| \cdot h(\tau, \chi'(\tau), \chi''(\tau)) + $$
$$+ |\chi'(\tau)| K \int_{-r}^{0} |\chi''(\tau+\theta)|\, d\theta \Big] d\tau - m(t-\bar{t})$$

where $m = m(\eta, \varepsilon) = \min \{\chi'(\tau) f(\chi'(\tau)),\ |\chi'(\tau)| \in [\delta(\eta)/2, \varepsilon]\}$.

Moreover from the Condition (A) it follows that for every $\alpha > 0$ there exist T, ζ such that for all $t, \bar{t} \geqslant T$, $t > \bar{t}$ one has

$$\int_{\bar{t}}^{t} h(\tau, \chi'(\tau), \chi''(\tau))\, d\tau = \sum_{K=0}^{n} \int_{\bar{t}+K\zeta'}^{\bar{t}+(K+1)\zeta'} h(\tau, \chi'(\tau), \chi''(\tau))\, d\tau \leqslant$$
$$\leqslant \sum_{K=0}^{n} \int_{\bar{t}+K\zeta}^{\bar{t}+(K+1)\zeta} h(\tau, \chi'(\tau), \chi''(\tau))\, d\tau \leqslant n\alpha$$

where $\zeta' \leqslant \zeta$ is such that $n\zeta' = t - \bar{t}$.

Fix now α such that $g(\alpha) = \zeta'^2/\varepsilon^2 + Kr\zeta' + \alpha < m/2$ and suppose that $\|\chi''_t\|_r < (m/c)^{1/2} \leqslant \delta(\eta)/2$ where $c \geqslant \max\, (\varepsilon^2 m/\zeta'^2,\ 4m/\delta^2(\eta))$.

Then

$$\int_{\bar{t}}^{t} \dot{W}\, d\tau \leqslant \big(m/c - m + Kr\varepsilon(m/c)^{1/2}\big) \cdot (t - \bar{t}) + \varepsilon (m/c)^{1/2} n\alpha \leqslant$$
$$\leqslant \big(-m + g(\alpha)\big) \cdot (t - \bar{t}) \leqslant -m/2(t - \bar{t})\,.$$

Then W is definitely divergent on E^*. All the hypotheses of Theorem 3.1 are then satisfied and this implies the asymptotic stability of the null solution of (4.1).

Notice that if $h(t, x, y)$ satisfies the « uniform Condition (A) », then the null solution of (4.1) is uniformly asymptotically stable.

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Manoscritto pervenuto in redazione l'11 maggio 1982.