RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

LADISLAV BICAN

Factor-splitting length of torsionfree abelian groups of rank two

Rendiconti del Seminario Matematico della Università di Padova, tome 70 (1983), p. 77-88

<http://www.numdam.org/item?id=RSMUP_1983__70__77_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 70 (1983)

Factor-Splitting Length of Torsionfree Abelian Groups of Rank Two.

LADISLAV BICAN (*)

The factor-splitting length of a torsionfree abelian group is defined as the supremum of the splitting lengths of all its factor-groups. Describing the splitting length of a given factor-group we present a characterization of the splitting length of a rank two torsionfree group in terms of its bases. For every positive integer n there is a rank two torsionfree group G having the factor-splitting length n and for each $k \leqslant n$ there is a factor-group of G with the splitting length k. If the factor-splitting length of a given rank two group G is infinite then two cases can occur: Either G has a factor-group of infinite splitting length and factor-groups of all finite splitting lengths, or G has a factor-group of infinite splitting length and factor-groups of finite splitting lengths up to some n, only.

By the word « group » we shall always mean an additively written abelian group. As in [1], we use the notions « characteristic » and « type » in the broad meaning, i.e. we deal with these notions in mixed groups. The symbols $h_p^G(g)$, $\tau^G(g)$ and $\hat{\tau}^G(g)$ denote respectively the p-height, the characteristic and the type of the element g in the group G. π will denote the set of all primes. If $\pi' \subseteq \pi$ and if G is a torsionfree group then for each subset $M \subseteq G$ the symbol $\langle M \rangle_{\pi'}^G$ denotes the π'-pure closure of M in G. Any maximal linearly independent set of elements of a torsionfree group G is called a basis. The set

(*) Indirizzo dell'A.: MFF UK, Sokolovská 83, 18600 Praha 8-Karlín, Cecoslovacchia.

This work was written while the author was a visiting professor supported by Italian C.N.R.

of all positive integers is denoted by $\mathbf{N}$, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Other notation will be essentially that as in [7].

For a mixed group G with the torsion part $T = T(G)$ we denote by $\bar{G}$ the factor-group G/T and for $g \in G$ $\bar{g}$ is the element $g + T$ of $\bar{G}$. The rank of a mixed group G is that of $\bar{G}$.

LEMMA 1 ([1; Theorem 2]): *A mixed group G of rank one splits if and only if each element $g \in G \setminus T(G)$ has a non-zero multiple mg such that $\hat{\tau}^G(mg) = \hat{\tau}^{\bar{G}}(\bar{g})$ and mg has p-sequence in G whenever $h_p^{\bar{G}}(\bar{g}) = \infty$ (i.e. there exist elements $h_0^{(p)} = mg$, $h_1^{(p)}, \ldots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, \ldots$).*

Recall [5], that the p-height sequence of an element g of a mixed group G is the double sequence $\{k_i, l_i\}_{i=0}^{\infty}$ of elements of $\mathbf{N}_0 \cup \{\infty\}$ defined inductively in the following way: Put $k_1 = k_0 = l_0 = 0$ and $l_1 = h_p^G(g)$. If k_i, l_i are defined and either $h_p^G(p^{k_i} g) = l_i = \infty$, or $l_i < \infty$ and $h_p^G(p^{k_i+k} g) = l_i + k$ for all $k \in \mathbf{N}$ then put $k_{i+1} = k_i$ and $l_{i+1} = l_i$. If $l_i < \infty$ and there are $k \in \mathbf{N}$ with $h_p(p^{k_i+k} g) > l_i + k$ then let k_{i+1} be the smallest positive integer for which $h_p^G(p^{k_{i+1}} g) = l_{i+1} > l_i + k_{i+1} - k_i$.

Let p be a prime and n an integer, $n > 1$. We say that an element g of a mixed group G has the (p, n)-property if for its p-height sequence $\{k_i, l_i\}_{i=0}^{\infty}$ the sequence $\{(n-1)(l_i - k_i) - k_{i+1}\}_{i=0}^{\infty}$ has non-negative elements and $\lim\limits_{i \to \infty} \{(n-1)(l_i - k_i) - k_{i+1}\} = nh_p^{\bar{G}}(\bar{g}) - \lim\limits_{i \to \infty} l_i$, where we put $\infty - m = \infty$ for every $m \in \mathbf{N}_0 \cup \{\infty\}$.

Recall [10], that the splitting length of a mixed group G is the infimum of the set of all positive integers n such that the n-th tensor power $G^n = G \otimes G \otimes \ldots \otimes G$ splits.

LEMMA 2 ([5; Theorem]): *A mixed group G of rank one has the splitting length $n > 1$ if and only if G does not split and $n > 1$ is the smallest integer such that every element $g \in G \setminus T(G)$ has a non-zero multiple mg which has the (p, n)-property for every prime p.*

DEFINITION 3: Let H be a subgroup of a torsionfree group G. For each prime p we define the p-height $h_p^G(H)$ of H in G as the minimum of the p-heights of all elements of H in G, i.e. $h_p^G(H) = \min \{h_p^G(h) \mid h \in H\}$.

LEMMA 4: *Let H be a rank one subgroup of a torsionfree group G, $S = \langle H \rangle_\pi^G$, $g \in G \setminus S$. If $h_p^G(g) \geq h_p^G(H)$ then $h_p^{G/H}(g + H) = h_p^{G/S}(g + S)$.*

PROOF: Without loss of generality we can assume that $h_p^G(g) = l < \infty$. Denoting $h_p^G(H) = r$ we can choose an element $h \in H$ with $h_p^G(h) = r$. If the equation $p^k x = g + s$, $k > l$, is solvable in G, then $ah = bs$ for some relatively prime integers a, b. From $p|b$ we get $h_p^G(s) < h_p^G(ah) = h_p^G(h) = r \leqslant h_p^G(g)$ — a contradiction. Hence there are integers u, v with $p^k u + bv = 1$ and $p^k(bvx + ug) = bvg + bvs + p^k ug = g + avh$, from which the assertion follows.

LEMMA 5: *Let H be a rank one subgroup of a torsionfree group G, $S = \langle H \rangle_\pi^G$, $g \in G \setminus S$. If $h_p^{G/H}(p^k g + H) = h_p^G(g) + k$ for every $k \in \mathbb{N}_0$ then $h_p^{G/H}(g + H) = h_p^{G/S}(g + S)$. The converse holds provided $h_p^G(g) < h_p^G(H)$.*

PROOF: Without loss of generality we can assume that $h_p^G(g) = l < \infty$. If the equation $p^r x = g + s$, $s \in S$, $r > l$, is solvable in G, then $mp^k s = h \in H$ for some $k \in \mathbb{N}_0$ and $(m, p) = 1$. Then $mp^{k+r} x = mp^k g + h$ and $k + r \leqslant h_p^G(mp^k g + h) \leqslant h_p^{G/H}(mp^k g + H) = l + k$ which contradicts the inequality $l < r$.

Conversely, by the hypothesis we have $h_p^{G/H}(g + H) = h_p^G(g) = l$. Let the equation $p^{r+l+1} x = p^r g + h$, $h \in H$, be solvable in G. Then $p^r(p^{l+1} x - g) = h$, hence $p^{l+1} x - g = s \in S$ and $h_p^{G/S}(g + S) \geqslant l + 1$, which contradicts the hypothesis.

LEMMA 6: *Let H be a rank one subgroup of a torsionfree group G of rank two, $S = \langle H \rangle_\pi^G$, $g \in G \setminus S$. If $(\langle g \rangle_\pi^G \oplus S) \otimes Z_p = G \otimes Z_p$ then $h_p^{G/H}(g + H) = h_p^{G/S}(g + S)$. The converse holds provided $h_p^G(g) < h_p^G(H)$.*

PROOF: We can suppose that $h_p^G(g) = l < \infty$. If the equation $p^k x = g + s$ is solvable in G then $p^k(x \otimes 1) = g \otimes 1 + s \otimes 1$. If $p^l g' = g$ then from the hypothesis it easily follows that $p^k(g' \otimes \alpha) = g \otimes 1$ for some $\alpha \in Z_p$. From this and from $p^l(g' \otimes 1) = g \otimes 1$ we get $g' \otimes (p^k \alpha - p^l) = 0$, hence $p^k \alpha = p^l$; $k \leqslant l$, from which the direct part follows immediately.

Suppose that $h_p^G(g) < h_p^G(H) < \infty$, the case $h_p^G(H) = \infty$ being trivial. Choose $s \in S$ with $h_p^G(s) = 0$ and take an arbitrary element $x \otimes m/n \in G \otimes Z_p$. Then $kp^r x = ug + vs$ for some $r \in \mathbb{N}_0$, $(k, p) = (kp^r, u, v) = 1$. For $r = 0$ we have $x \otimes m/n = g \otimes um/kn + s \otimes vm/kn \in (\langle g \rangle_\pi^G \otimes S) \otimes Z_p$. If $r > 0$ then $p|u$ yields $p|vs$, hence $p|v$ which is impossible. Thus $ua + p^r b = 1$ for some a, $b \in \mathbb{N}$ and consequently $p^r(kax + bg) = g + vas$. So, $r \leqslant h_p^G(g + vas) \leqslant h_p^{G/S}(g + S) = h_p^G(g)$ and from $p^r g' = g$, $kp^r x = up^r g' + vs$ we get $v = p^r v'$ and consequently $lx = ug' + v' s$, from which the assertion follows.

LEMMA 7: *Let H be a rank one subgroup of a torsionfree group G, $S = \langle H \rangle_\pi^G$, $g \in G \setminus S$, and let $\{k_i, l_i\}_{i=0}^\infty$ be the p-height sequence of the element $g + H$ in G/H. Then:*

(i) *If $h_p^{G/H}(g + H) < h_p^{G/S}(g + S)$ then $h_p^G(H) - h_p^G(g) = k_2 = k_3 = \dots$, $l_1 = h_p^G(g)$ and $h_p^G(H) + \alpha = l_2 = l_3 = \dots$, $\alpha \in \mathbf{N} \cup \{\infty\}$;*

(ii) *If $h_p^{G/H}(g + H) = h_p^{G/S}(g + S)$ then $0 = k_0 = k_1 = \dots$ and $h_p^{G/H}(g + H) = l_1 = l_2 = \dots$.*

PROOF: (i) By Lemma 4 we have $h_p^G(g) < h_p^G(H) < \infty$. Putting $k_2 = h_p^G(H) - h_p^G(g)$ we obviously get $h_p^{G/H}(p^r + H) = h_p^G(g) + r$ for each $r \in \mathbf{N}_0$, $r < k_2$. Further, again by Lemma 4, $h_p^{G/H}(p^{k_2} g + H) = h_p^{G/S}(p^{k_2} g + S) = h_p^{G/S}(g + S) + k_2 > h_p^{G/H}(g + H) + k_2$ and finally

$$h_p^{G/H}(p^{k_2+r} g + H) = h_p^{G/S}(p^{k_2+r} g + S) = h_p^{G/S}(p^{k_2} g + S) + r = h_p^{G/H}(p^{k_2} g + H) + r\,.$$

(ii) For each $r \in \mathbf{N}$ we have

$$h_p^{G/H}(p^r g + H) \leqslant h_p^{G/S}(p^r g + S) = h_p^{G/S}(g + S) + r = h_p^{G/H}(g + H) + r \leqslant h_p^{G/H}(p^r g + H)\,.$$

PROPOSITION 8: *Let H be a rank one subgroup of a torsionfree group G of rank two, $S = \langle H \rangle_\pi^G$. The following conditions are equivalent:*

(i) *The factor-group G/H splits;*

(ii) *For every $g \in G \setminus S$ and for almost all primes p with $h_p^G(g) < h_p^G(H) < \infty$ it is $h_p^{G/H}(p^k g + H) = h_p^G(g) + k$ for each $k \in \mathbf{N}_0$;*

(iii) *There is $g \in G \setminus S$ such that for all primes p with $h_p^G(g) < h_p^G(H) < \infty$ it is $h_p^{G/H}(p^k g + H) = h_p^G(g) + k$ for each $k \in \mathbf{N}_0$;*

(iv) *There is $g \in G \setminus S$ such that for all primes p with $h_p^G(g) < h_p^G(H) < \infty$ it is $h_p^{G/H}(p^k g + H) = h_p^G(g) + k$ for $k = h_p^G(H) - h_p^G(g)$.*

PROOF: (i) $\Rightarrow$ (ii). Since S/H is the torsion part of G/H, it follows from Lemma 1 that for almost all primes p it is $h_p^{G/H}(g + H) = h_p^{G/S}(g + S)$ and it suffices to use Lemma 5.

(ii) $\Rightarrow$ (iii). If $p_1, p_2, \dots, p_n$ are all primes with $h_p^G(g) < h_p^G(H) < \infty$ for which the equality in question do not hold, consider the element $p_1^{l_1}, p_2^{l_2}, \dots, p_n^{l_n} g$ where $l_i = h_{p_i}^G(H) - h_{p_i}^G(g)$, $i = 1, 2, \dots, n$.

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i). For each prime p with $h_p^G(g) < h_p^G(H) < \infty$ and for $k = h_G^p(H) - h_p^G(g)$ we have by Lemma 4 and the hypothesis

$$h_p^{G/H}(g + H) = h_p^G(g) + k - k = h_p^{G/H}(p^k g + H) - k =$$
$$= h_p^{G/S}(p^k g + S) - k = h_p^{G/S}(g + S) .$$

Since the same equality holds by Lemma 4 for all other primes, it remains with respect to Lemma 1 to show that for each prime p with $h_p^{G/S}(g + S) = \infty$ the element $g + H$ has a p-sequence in G/H.

The p-primary component of S/H is obviously either a cyclic group $C(p^r)$ or a quasicyclic group $C(p^\infty)$. In the either case put $g_0 = g$ and assume that $g_0, g_1, \ldots, g_n$ are such elements that

$$p(g_{i+1} + H) = g_i + H\ , \qquad i = 0, 1, \ldots, n-1\ ,$$

and that each $g_i + H$ is of infinite p-height in G/H. Thus for each $s \in \mathrm{N}$ there is $g^{(s)} \in G$ and $h_s \in H$ with $p^{r+s} g^{(s)} = g_n + h_s$. Now $p^{r+1}(p^{s-1} g^{(s)} - g^{(1)}) = h_s - h_1$ and so

$$p^{s-1} g^{(s)} - g^{(1)} \in S \qquad \text{and} \qquad p^r(p^{s-1} g^{(s)} - g^{(1)}) \in H\ .$$

Setting $g_{n+1} = p^r g^{(1)}$ we get $p(g_{n+1} + H) = p^{r+1} g^{(1)} + H = g_n + H$ and $p^{r+s-1}(g^{(s)} + H) = p^r(p^{s-1} g^{(s)} - g^{(1)} + g^{(1)} + H) = p^r g^{(1)} + H = g_{n+1} + H$.

In the respective case the factor-group S/H is p-divisible. Again, put $g_0 = g$ and assume that $g_0, g_1, \ldots, g_n$ are such elements that

$$p(g_{i+1} + H) = g_i + H\ , \qquad i = 0, 1, \ldots, n-1\ .$$

Since G/S is p-divisible, there are elements $g'_{n+1} \in G$ and $s \in S$ such that $pg'_{n+1} = g_n - s$. Further, $ps' = s + h$ for suitable elements $s' \in S$, $h \in H$. Setting $g_{n+1} = g'_{n+1} + s'$ we get $p(g_{n+1} + H) = g_n + H$ and the proof is complete.

THEOREM 9: *Let H be a rank one subgroup of a torsionfree group G of rank two, $S = \langle H \rangle_\pi^G$. The following conditions are equivalent*:

(i) *G/H has the splitting length $n \geq 1$;*

(ii) *n is the smallest integer such that for each element $g \in G \setminus S$ the inequality $n h_p^G(g) \geq h_p^G(H)$ holds for almost all primes p with $h_p^{G/H}(g + H) < h_p^{G/S}(g + S)$;*

(iii) *$G\setminus S$ contains an element g such that n is the smallest integer satisfying $nh_p^G(g)\geqslant h_p^G(H)$ for all primes p with $h_p^{G/H}(g+H) < h_p^{G/S}(g+S)$.*

Proof: (i) $\Rightarrow$ (ii). Let $g\in G\setminus S$ be arbitrary, $n\geqslant 2$. It follows from Lemma 2 that $g+H$ has the (p,n)-property for almost all primes p. If p is such a prime with $h_p^{G/H}(g+H) < h_p^{G/S}(g+S)$ then Lemma 7(i) yields $(n-1)(l_1-k_1)-k_2\geqslant 0$, i.e. $nh_p^G(g)\geqslant h_p^G(H)$. If $n=1$ then $h_p^{G/H}(g+H)=h_p^{G/S}(g+S)$ for almost all primes p by Lemma 1 and condition (ii) is trivially satisfied.

(ii) $\Rightarrow$ (iii). If $p_1, p_2, \ldots, p_n$ are all primes with $h_p^{G/H}(g+H) < h_p^{G/S}(g+S)$ and $nh_p^G(g) < h_p^G(H)$ consider the element $g' = p_1^{l_1}p_2^{l_2}\ldots p_n^{l_n}g$ where $l_i = h_{p_i}^G(H) - h_{p_i}^G(g)$, $i=1,2,\ldots,n$. Then for each p_i $h_{p_i}^{G'}(g') = h_{p_i}^G(H)$ so that $h_{p_i}^{G/H}(g'+H) = h_{p_i}^{G/S}(g'+S)$ by Lemma 4, while for the remaining primes in question the inequality considered remains valid.

(iii) $\Rightarrow$ (i). For $n=1$ the inequality $h_p^G(g)\geqslant h_p^G(H)$ yields the equality $h_p^{G/H}(g+H)=h_p^{G/S}(g+S)$ so that this equality holds for each prime p. If $h_p^G(g) < h_p^G(H) < \infty$ then Lemma 5 gives $h_p^{G/H}(p^kg+H) = h_p^G(g)+k$ for each $k\in\mathbf{N}_0$ and G/H splits by Proposition 8.

Let $n>1$. If we put $\pi_1 = \{p\in\pi \mid h_p^{G/H}(g+H) < h_p^{G/S}(g+S)\}$ and $\pi_2 = \pi\setminus\pi_1$ then the set π_1 is infinite (for otherwise G/H splits and $n=1$ by the first part of this proof). For each prime $p\in\pi_1$ the p-height sequence of the element $g+H$ is given by Lemma 7(i). Then

$$(n-1)(l_1-k_1)-k_2 = nh_p^G(g)-h_p^G(H)\geqslant 0\,,$$

$$(n-1)(l_2-k_2)-k_3 = (n-1)\big(h_p^G(g)+\alpha\big)-h_p^G(H)+h_p^G(g) = nh_p^G(g)-h_p^G(H)+(n-1)\alpha\geqslant 0$$

and by the hypothesis n is the smallest integer for which the element $g+H$ has the (p,n)-property for all primes $p\in\pi_1$. Concerning the primes $p\in\pi_2$ the p-height sequence of $g+H$ is given by Lemma 7(ii) and $g+H$ has the (p,n)-property trivially. Thus G/H is of the splitting length n by Lemma 2.

For the sake of brevity we shall denote the splitting length of a group G by $\mathrm{sl}(G)$.

Corollary 10: *Let G be a torsionfree group of rank two and $H_1\subseteq H_2$ be rank one subgroups. Then $\mathrm{sl}(G/H_2)\leqslant \mathrm{sl}(G/H_1)$.*

PROOF: Since $H_1 \subseteq H_2$, $h_p^G(H_2) \leqslant h_p^G(H_1)$ and for $\mathrm{sl}(G/H_1) < \infty$ it suffices to use Theorem 9, while for $\mathrm{sl}(G/H_1) = \infty$ the assertion is trivial.

THEOREM 11: *Let G be a torsionfree group of rank two and H be its rank one subgroup such that $\mathrm{sl}(G/H) = n < \infty$. Then for each $k \in \mathbb{N}$, $k \leqslant n$, there is a rank one subgroup K of G with $\mathrm{sl}(G/K) = k$. Moreover, K can be selected such that $H \subseteq K$.*

PROOF: Put $S = \langle H \rangle_\pi^G$ and $\pi' = \{p \in \pi | h_p^{G/H}(g + H) < h_p^{G/S}(g + S)\}$. By Theorem 9 we can choose an element $g \in G \setminus S$ in such a way that $nh_G^p(g) \geqslant h_p^G(H)$ for each $p \in \pi'$. For $k = 1$ it suffices to put $K = S$ (or take some subgroup of finite index in S). So, suppose $n > k \geqslant 2$ and let $p \in \pi'$ be arbitrary. Denoting $h_p^G(H) = s_p$ define $l_p \in \mathbb{N}_0$ to be the smallest integer with $l_p + kh_p^G(g) \geqslant s_p$.

First show, that $h_p^G(g) < s_p - l_p$. By the choice of l_p we have $l_p - 1 + kh_p^G(g) < s_p$. Now the assumption $h_p^G(g) \geqslant s_p - l_p$ yields $(k-1) \cdot h_p^G(g) < 1$ — a contradiction.

By hypothesis, for each prime $p \in \pi'$ there is $y_p \in H$ with $h_p^G(y_p) = s_p$ and with respect to the preceding item there is $x_p \in G$ with $p^{l_p} x_p = y_p$. Now consider the group $K = \langle H \cup \{x_p | p \in \pi'\} \rangle$.

Let $p \in \pi'$ be arbitrary and let $y \in K$ be such that $h_p^G(y) = r < s_p - l_p$. Then

$$y = h + \alpha_p x_p + \sum_{i=1}^{n} \alpha_{p_i} x_{p_i}, \qquad h \in H,\ p_i \in \pi'$$

pairwise different $p_i \neq p$, and $\varrho h = \sigma y_p$ for some integers $(\varrho, \sigma) = 1$. For $m = p^{l_p} p_1^{l_1} p_2^{l_2} \dots p_n^{l_n} = p^{l_p} \overline{m} = p_i^{l_i} m_i$, where $l_i = l_{p_i}$, we have

$$\varrho m y = m \sigma y_p + \overline{m} \varrho \alpha_p y_p + \sum_{i=1}^{n} m_i \varrho \alpha_{p_i} y_{p_i} \in H .$$

Hence $h_p(\varrho) = h_p^G(\varrho m y) - l_p - r \geqslant s_p - l_p - r > 0$ and so $h_p(\sigma) = 0$. On the other hand,

$$h_p^G(h) = h_p^G(\varrho h) - h_p(\varrho) = h_p^G(\sigma y_p) - h_p(\varrho) = s_p - h_p(\varrho) < s_p .$$

This contradiction shows that $h_p^G(K) \geqslant s_p - l_p$ and, in fact, $h_p(K) = s_p - l_p$ since $h_p^G(x_k) = s_p - l_p$.

For each $p \in \pi \setminus \pi'$ it is $h_p^{G/H}(g + H) \leqslant h_p^{G/K}(g + K) \leqslant h_p^{G/S}(g + S) = h_p^{G/H}(g + H)$ and for each prime $p \in \pi'$ the preceding parts give

$h_p^G(g) < h_p^G(K)$, so $h_p^{G/K}(g+K) = h_p^G(g) = h_p^{G/H}(g+H) < h_p^{G/S}(g+S)$, and $h_p^G(g) \geqslant h_p^G(K)$. To finish the proof it suffices now to show that k is the smallest such integer and to apply Theorem 9.

By hypothesis and Theorem 9 there are infinitely many primes $p \in \pi'$ with $(n-1)\,h_p^G(g) < h_p^G(H) = s_p$. Suppose now that for such a prime p it is $(k-1)\,h_p^G(g) \geqslant h_p^G(K)$. Then

$$l_p + (k-1)\,h_p^G(g) \geqslant s_p > (n-1)\,h_p^G(g)\,,$$

hence $l_p > (n-k)\,h_p^G(g)$ and consequently

$$l_p - 1 + k h_p^G(g) \geqslant (n-k)\,h_p^G(g) + k h_p^G(g) = n h_p^G(g) \geqslant h_p^G(H) = s_p$$

which contradicts the choice of l_p.

Definition 12: The factor-splitting length, $\mathrm{fsl}(G)$, of a torsionfree group G is defined to be $\sup\{\mathrm{sl}(G/H) | H \text{ a subgroup of } G\}$.

Corollary 13: *If a torsionfree group G of rank two has the factor-splitting length $n < \infty$ then for each $k \in \mathbb{N}$, $k \leqslant n$, there is a homomorphic image of G having the splitting length k.*

Proof: By the definition there is a (rank one) subgroup H of G with $\mathrm{sl}(G/H) = n$ and it suffices to use Theorem 11.

Remark 14: If G is a torsionfree group of rank two then it follows from Corollary 10 that $\mathrm{fsl}(G) = \sup\{\mathrm{sl}(G/\langle g\rangle) | 0 \neq g \in G\}$.

Theorem 15: *A torsionfree group G of rank two has the factor-splitting length $n \geqslant 1$ if and only if n is the smallest integer such that for each basis $\{g, h\}$ of G and for almost all primes p with $h_p^G(g) \neq h_p^G(h)$ and $(\langle g\rangle_\pi^G \oplus \langle h\rangle_\pi^G) \otimes Z_p \subsetneqq G \otimes Z_p$ it is $1/n \leqslant h_p^G(h)/h_p^G(g) \leqslant n$.*

Proof: Assume first that $\mathrm{fsl}(G) = n$. If $\{g, h\}$ is an arbitrary basis of G we put

$$\pi_1 = \{p \in \pi | h_p^G(g) < h_p^G(h),\ (\langle g\rangle_\pi^G \oplus \langle h\rangle_\pi^G) \otimes Z_p \subsetneqq G \otimes Z_p\}$$

and

$$\pi_2 = \{p \in \pi | h_p^G(h) < h_p^G(g),\ (\langle g\rangle_\pi^G \oplus \langle h\rangle_\pi^G) \otimes Z_p \subsetneqq G \otimes Z_p\}.$$

By the hypothesis we have $\mathrm{sl}(G/\langle h\rangle) = k \leqslant n$. For each prime $p \in \pi_1$ Lemma 6 yields the inequality $h_p^{G/H}(g + H) < h_p^{G/S}(g + S)$, where $H = \langle h\rangle$ and $S = \langle h\rangle_\pi^G$, and consequently Theorem 9 gives the inequality $1 < h_p^G(h)/h_p^G(g) \leqslant k \leqslant n$ for almost all primes $p \in \pi_1$: Considering the factor-group $G/\langle g\rangle$ we similarly obtain the inequality $1/n \leqslant h_p^G(h)/h_p^G(g) < 1$ for almost all primes $p \in \pi_2$.

Conversely, assume that the condition of Theorem is satisfied. Let $0 \neq h \in G$ be an arbitrary element and $g \in G \setminus \langle h\rangle_\pi^G$ be arbitrary. Denote $H = \langle h\rangle$, $S = \langle h\rangle_\pi^G$ and set

$$\pi' = \{p \in \pi \mid h_p^{G/H}(g + H) < h_p^{G/S}(g + S)\} .$$

Then for each $p \in \pi'$ $h_p^G(g) < h_p^G(H)$ by Lemma 4 and

$$(\langle g\rangle_\pi^G \oplus \langle h\rangle_\pi^G) \otimes Z_p \subsetneqq G \otimes Z_p$$

by Lemma 6. By hypothesis, for almost all primes $p \in \pi'$ it is $n h_p^G(g) \geqslant h_p^G(H)$ and, consequently, $\mathrm{sl}(G/\langle h\rangle) \leqslant n$ by Theorem 9. So with respect to Remark 14 we have $\mathrm{fsl}(G) \leqslant n$ and it remains to show the existence of $h \in G$ with $\mathrm{sl}(G/\langle h\rangle) = n$.

By hypothesis, there si a basis $\{g, h\}$ of G such that for no $k < n$, $k \in \mathrm{N}$, the inequalities $1/k \leqslant h_p^G(h)/h_p^G(g) \leqslant k$ hold for almost all primes

$$p \in \pi_1 \cup \pi_2 ,$$

$$\pi_1 = \{p \in \pi \mid h_p^G(g) < h_p^G(h),\ (\langle g\rangle_\pi^G \oplus \langle h\rangle_\pi^G) \otimes Z_p \subsetneqq G \otimes Z_p\} ,$$

$$\pi_2 = \{p \in \pi \mid h_p^G(h) < h_p^G(g),\ (\langle g\rangle_\pi^G \oplus \langle h\rangle_\pi^G) \otimes Z_p \subsetneqq G \otimes Z_p\} .$$

Then either π_1 is infinite and n is the exact upper bound of $h_p^G(h)/h_p^G(g)$ for almost all primes $p \in \pi_1$, or π_2 is infinite and $1/n$ is the exact lower bound of $h_p^G(h)/h_p^G(g)$ for almost all primes $p \in \pi_2$. In the first case we easily get from Lemma 6, Lemma 4 and Theorem 9 that $\mathrm{sl}(G/\langle h\rangle) = n$ and the same arguments in the second case yields $\mathrm{sl}(G/\langle g\rangle) = n$.

EXAMPLE 16: Consider the group G generated by the elements $a, b, a_p, b_p, c_p, p \in \pi$, with respect to the relations $pa_p = a$, $p^n b_p = b$, $pc_p = a_p + b_p$, $p \in \pi$, $n \geqslant 2$. The group G is torsionfree of rank two, the element b has the characteristic $(n, n, \ldots)$ and all the elements outside of $\langle b\rangle_\pi^G$ are of the type $(1, 1, \ldots)$. Hence for each $h \in \langle b\rangle_\pi^G$,

$H = \langle h \rangle$, $S = \langle h \rangle_\pi^G$, we have $h_p^{G/H}(a + H) < h_p^{G/S}(a + S)$, $nh_p^G(a) = h_p^G(H)$ for almost all primes p and $\mathrm{sl}(G/\langle h \rangle) = n$ by Theorem 9. Moreover, for each $h \in G \setminus \langle b \rangle_\pi^G$ Proposition 8 yields the splitting of $G/\langle h \rangle$ and G has the factor-splitting length n.

EXAMPLE 17. Consider the group G generated by the elements $a, b, b_p, c_p, p \in \pi$, with respect to the relations $pb_p = b$, $pc_p = a + b_p$, $p \in \pi$. The group G is torsionfree of rank two, the element b has the characteristic $(1, 1, \ldots)$ and all the elements outside of $\langle b \rangle_\pi^G$ are of the type Z (the additive group of integers). For each $h \in G \setminus \langle b \rangle_\pi^G$ Proposition 8 yields the splitting of $G/\langle h \rangle$. Consider now the subgroup H of G with $\langle b \rangle \subseteq H \subseteq \langle b \rangle_\pi^G = S$. If $\hat{\tau}(H) = \hat{\tau}^G(b)$ then the torsion part of G/H is finite and G/H therefore splits. In the opposite case there is an infinite set of primes with $h_p^G(H) = 1$. For all these primes we have $h_p^{G/H}(g + H) < h_p^{G/S}(g + S)$ but there is no $n \in \mathbb{N}$ with $nh_p^G(a) \geqslant h_p^G(H)$. Hence, by Theorem 9, G/H has no finite splitting length. From these observations it immediately follows that $\mathrm{fsl}(G) = \infty$ and for each factor-group G/H of G it is either $\mathrm{sl}(G/H) = \infty$ or $\mathrm{sl}(G/H) = 1$.

THEOREM 18: (i) *For each positive integer n there exists a torsionfree group G of rank two having the factor-splitting length n;*

(ii) *There exists a torsionfree group G of rank two and of infinite factor-splitting length such that G has a homomorphic image of splitting length n for each $n \in \mathbb{N} \cup \{\infty\}$;*

(iii) *For each $n \in \mathbb{N}$ there exists a torsionfree group G of rank two and of infinite factor-splitting length such that the splitting length of any homomorphic image of G is either infinite or at most n.*

PROOF: (i) See Example 16.

(ii) Decompose the set π into disjoint infinite subset $\pi_1, \pi_2, \ldots$ and consider the group G (« composition » of groups from Example 16 for all $n \geqslant 2$) generated by the elements a, b, a_{p_i}, b_{p_i}, c_{p_i}, $p_i \in \pi_i$, $i = 2, 3, \ldots$, with respect to the relations

$$p_i a_{p_i} = a, \quad p_i^i b_{p_i} = b, \quad p_i c_{p_i} = a_{p_i} + b_{p_i}, \qquad p_i \in \pi_i, \ i = 2, 3, \ldots.$$

For $h = b$, $H = \langle h \rangle$, $S = \langle h \rangle_\pi^G$ we obviously have $h_p^{G/H}(a + H) < h_p^{G/S}(a + S)$ for each prime $p \in \pi$ and $h_p^G(H) = ih_p^G(a)$ for each $p \in \pi_i$.

Hence $\mathrm{sl}(G/H) = \infty$ by Theorem 9. Further, for $H = \langle b\rangle^G_{\pi\setminus\pi_n}$ we have $h^G_p(H) = n$ for each $p \in \pi_n$ and, consequently, $\mathrm{sl}(G/H) = n$.

(iii) For $n = 1$ see Example 17. If $n \geqslant 2$ then decompose the set π into two disjoint infinite parts π_1 and π_2 and consider the group G (« composition » of groups from Examples 16 and 17) generated by the elements a, b, a_p, b_p, c_p, b_q, c_q, $p \in \pi_1$, $q \in \pi_2$, with respect to the relations $pa_p = a$, $p^n b_p = b$, $pc_p = a_p + b_p$, $qb_q = b$, $qc_q = a + b_q$, $p \in \pi_1$, $q \in \pi_2$. Using similar arguments as in Examples 16 and 17 we see that $G/\langle h\rangle$ splits for each $h \in G\setminus\langle b\rangle^G_\pi$. Further, for $\langle b\rangle \subseteq H \subseteq \langle b\rangle^G_\pi$ it is $\mathrm{sl}(G/H = \infty)$ whenever $|\{p \in \pi_2 | h^G_p(H) = 1\}| = \infty$. In the opposite case, using Corollary 10, we easily get $\mathrm{sl}(G/H) = k \leqslant n$. Especially, $\mathrm{sl}(G/\langle b\rangle^G_{\pi_2}) = n$.

Remark 19: Theorem 15 and Example 16, 17, as well as the proof of Theorem 18 show that the factor-splitting length of a torsionfree group G of rank two depends on the type set $\hat{\tau}(G)$ of G, only. It is not too hard to show that the factor-splitting length of G can be found in the following way: Let $M = \{\hat{\tau}, \hat{\sigma} | \hat{\tau} \neq \hat{\sigma}, \hat{\tau}, \hat{\sigma} \in \hat{\tau}(G)\}$ and let $g, h \in G$ be such that $\hat{\tau}^G(g) = \hat{\tau}$, $\hat{\tau}^G(h) = \hat{\sigma}$. Set $\pi_M = \{p \in \pi | \tau(p) \neq \sigma(p)$ are finite and $\langle g\rangle^G_\pi \oplus \langle h\rangle^G_\pi$ is not p-pure in $G\}$ and

$$\varrho_M = \limsup_{p \in \pi_M} \sigma(p)/\varrho(p)\,, \qquad \sigma_M = \liminf_{p \in \pi_M} \sigma(p)/\varrho(p)\,,$$

where we put $m/0 = \infty$. If $\varrho = \sup \varrho_M$, $\sigma = \inf \sigma_M$, where M ranges over all pairs of different types from $\hat{\tau}(G)$, then $\mathrm{fsl}(G) = \infty$ if either $\varrho = \infty$ or $\sigma = 0$ and $\mathrm{fsl}(G) = n$ if n is the smallest integer with $1/n \leqslant \sigma$ and $\varrho \leqslant n$.

REFERENCES

[1] L. Bican, *Mixed abelian groups of torsionfree rank one*, Czech. Math. J., **20** (1970), pp. 232-242.

[2] L. Bican, *Factor-splitting abelian groups of rank two*, Comment. Math. Univ. Carolinae, **11** (1970), pp. 1-8.

[3] L. Bican, *Factor-splitting abelian groups of finite rank*, Comment. Math Univ. Carolinae, **17** (1976), pp. 473-480.

[4] L. Bican, *Factor-splitting abelian groups of arbitrary rank*, Comment. Math. Univ. Carolinae, **19** (1978), pp. 653-672.

[5] L. Bican: *The splitting length of mixed abelian groups of rank one*, Czech. Math. J., **27** (1977), pp. 144-154.
[6] L. Bican: *The splitting of the tensor product of two mixed abelian groups of rank one*, to appear.
[7] L. Fuchs, *Abelian groups*, Budapest, 1958.
[8] L. Fuchs, *Infinite abelian groups I*, Academic Press, New York and London, 1970.
[9] L. Fuchs, *Infinite abelian groups II*, Academic Press, New York and London, 1973.
[10] I. M. Irwin - S. A. Khabbaz - G. Rayna, *The rôle of the tensor product in the splitting of abelian groups*, J. Algebra, **14** (1970), pp. 423-442.
[11] L. Procházka, *O rasčepljajemosti faktorgrupp abelevych grupp bez kručenija konečnogo ranga (Russian)*, Czech. Math. J., **11** (1961), pp. 521-557.
[12] L. Procházka, *Zametka o faktorno rasčepljajemych abelevych gruppach (Russian)*, Čas. pěst. mat., **12** (1962), pp. 404-414.
[13] A. I. Stratton, *A splitting theorem for mixed abelian groups*, Symposia Mathematica, Vol. XIII, Academic Press, London, 1974, pp. 109-125.

Manoscritto pervenuto in redazione il 16 aprile 1982.