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The q-Analogue of Hölder's Theorem for the Gamma Function.

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Introduction.

F. H. Jackson (see [1]) defined a q-analogue of the classical Euler gamma function:

$$\Gamma_q(x) = (1-q)^{1-x}(q; q)_\infty/(q^x; q)_\infty$$

where $0 < q < 1$, and the product $(a; q)_\infty$ is defined by

$$(a; q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n) .$$

In (2.3) R. Askey has proved that the functions $\Gamma_q(x)$ are uniquely determined by the conditions:

i) $\Gamma_q(x+1)(1-q) = \Gamma_q(x)(1-q^x)$;

ii) $\log \Gamma_q(x)$ is convex for positive x;

iii) $\Gamma_q(1) = 1$.

This is analogous to the celebrated theorem of Bohr-Mollerup for the gamma function. Condition i) can be considered a differential equation of infinite order if one writes $e^D f(x) = f(x+1)$. In analogy

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with the classical theorem of Hölder (see [4]) which asserts that $\Gamma(x)$ satisfies no algebraic differential equation over $C(x)$, one is naturally led to conjecture the same result for the functions $\Gamma_q(x)$, which for fixed x tend to $\Gamma(x)$ as $q \to 1^-$. The principal result of this note is to establish that conjecture.

1. – In what follows $C[x, q^x]$ and $C(x, q^x)$ denote the ring and the field generated by x and q^x over the field C of the complex numbers respectively.

PROPOSITION 1. If $y(x)$ is a meromorphic function not identically 0 satisfying an algebraic equation of $C(x, q^x)$, then $y(x)$ has at most a finite number of zeroes and poles on the real axis.

The proposition follows from the next

LEMMA. Let $g(x, q^x) \in C[x, q^x]$ not identically 0. Then g has a finite number of real zeroes.

PROOF. Since:

$$g(x, q^x) = (q^x)^r Q(x, q^x) = (q^x)^r \big(p_0(x) + \dots + q^{nx} p_n(x)\big)$$

where $p_0(x), p_n(x) \in C[x] \setminus (0)$; $p_i(x) \in C[x]$, $i = 1, \dots, n-1$; it follows that $Z(g(x, q^x)) = Z(Q(x, q^x))$ where, as usual and in what follows, $Z(f)$ denotes the zero-set of the function f. If $n = 0$ the lemma follows easily, since $Z(Q) = Z(p_0)$. If $n \neq 0$ one supposes that $0 < q < 1$ and considers x a real variable. One then has

$$\lim_{x \to -\infty} |Q(x, q^x)| = \infty \quad \text{and} \quad \lim_{x \to +\infty} |Q(x, q^x)| = c\,,$$

where c denotes a positive real number (possibly ∞). Consequently, there is a real neighborhood of infinity in which $|Q(x, q^x)|$ is strictly positive. Hence the real zeroes of Q are bounded, and thus finite in number. This proves the lemma.

It follows immediately that each non-zero element of $C(x, q^x)$ has at most a finite number of real zeroes and poles.

Proof of proposition 1.

By hypothesis, there exists a polynomial $f(t)$ in one indeterminate t over $C(x, q^x)$ such that for every complex number x, one has $f(y(x)) = 0$, that is,

$$\sum_{n=0}^{k} a_n(y(x))^n = 0\ , \qquad a_n \in C(x, q^x) \qquad \text{and} \qquad a_0 \neq 0\ .$$

If y had infinitely many real zeroes, then a_0 would necessarily vanish at all these zeroes, except possibly at the poles of the other a_n, $n \neq 0$, which are finite in number.

This would then contradict the lemma. Q.E.D.

Corollary 1. The functions $\Gamma_q(x)$ and $g(x) = (d/dx)(\log \Gamma_q(x))$ are trascendental over $C(x, q^x)$.

Corollary 2. If $R(x) \in C(x, q^x)$ has a non-zero real period, then $R(x)$ is a constant.

2. – In the proof of the q-analogue of the Hölder's theorem the functional equation $g(x+1) = g(x) - (q^x \log q)/(1 - q^x)$, which follows immediately from condition i) of the introduction, and the following two propositions will be used extensively.

Proposition 2. For every positive integer n

$$\frac{d^n}{dx^n}\big(g(x+1)\big) = \frac{d^n}{dx^n}\big(g(x)\big) + P_n(q^x)/(1 - q^x)^{n+1}\ ,$$

where $P_n(q^x) \in C[q^x] \setminus (0)$ is not divisible by $(1 - q^x)$.

Proof. The result is an easy induction on n.

Proposition 3. Let $P(q^x) \in C[q^x]$, $P(1) \neq 0$. Then there exists no $R(x) \in C(x, q^x)$ such that for some positive integer k

$$(1 - q^x)^k\big(R(x) - R(x+1)\big) = P(q^x)\ .$$

Proof. The result follows easily from proposition 1 when one observes that such an $R(x)$ would necessarily have poles at all the negative integers, or at all the positive integers.

In what follows $T = (t_1, \ldots, t_n)$, t_i indeterminates; and $T = G(x)$ denotes the substitution $t_{i+1} = (d^i/dx^i)(g(x))$, $i = 0, 1, \ldots, n-1$.

THEOREM. The function $g(x)$ satisfies no algebraic differential equation over $C(x, q^x)$, that is there does not exist any non-zero polynomial $F(T)$ in the indeterminates T over $C(x, q^x)$ such that $F(G(x)) \equiv 0$.

PROOF. Define the weight of $(t_1)^{m_1} \dots (t_n)^{m_n}$ to be the integer $\sum_{k=1}^{n} k m_k$, define the weight of a polynomial in the indeterminates T over $C(x, q^x)$ as the maximum of the weights of the monomials which appear in it.

We reason by contradiction. Suppose that there exists a polinomial $E_x(T)$ such that $E_x(G(x)) \equiv 0$. One can write

$$E_x(T) = \sum_{i=0}^{k_s} R_{s,i}(x) Q_{s,i}(T) + \Delta_0$$

where $s > 1$ is the weight of E; $R_{s,i}(x) \in C(x, q^x)$ and $R_{s,0}(x) = 1$; $Q_{s,i}(T)$ is a term of the type $(t_1)^{m_1} \dots (t_n)^{m_n}$ of weight s, and $Q_{s,i} \neq Q_{s,j}$ for $i \neq j$, and Δ_a represents « a generic sum of terms of weight strictly less than $s - a$ ». (The case $s = 1$ is excluded by Corollary 1). One notes that $E_{x+1}(T)$ has weight s, and that $E_{x+1}(G(x+1)) \equiv 0$. Consider

$$E_x^{[1]}(G(x)) = E_x(G(x)) - E_{x+1}(G(x+1)) .$$

Then

$$E_x^{[1]}(G(x)) = \sum_{i=0}^{k_s} (R_{s,i}(x) - R_{s,i}(x+1)) Q_{s,i}(G(x)) + \Delta_0$$

since

$$Q_{s,i}(G(x+1)) = Q_{s,i}(G(x)) + \Delta_0 .$$

Clearly $E_x^{[1]}(G(x)) \equiv 0$. Furthermore $E_x^{[1]}$ has at most k_s terms of weight s, ($R_{s,0}(x) \equiv 1$, by assumption). In the case that $E_x^{[1]}(T) \neq 0$ this would lead to a contradiction if one started from an E_x with s and k_s minimal. There remains only to treat the case $E_x^{[1]}(T) \equiv 0$.

Since $Q_{l,i} = Q_{v,j}$ if and only if $l = v$ and $i = j$, it follows that $R_{s,i}(x) \equiv R_{s,i}(x+1)$ and hence by corollary 2 $R_{s,i}(x) \equiv a_{s,i} \in C$.

Therefore

$$E_x^{[1]}(G(x)) = \sum_{i=0}^{k_s} a_{s,i} \Big(Q_{s,i}(G(x)) - Q_{s,i}(G(x+1)) \Big) + \Delta_0 .$$

By the definition of the weight function, one finds that

$$Q_{s,i}(G(x)) - Q_{s,i}(G(x+1)) = \delta_{s,i} v_{s,i} (1-q^x)^{-1} P_0(q^x) Q_{s,i}(G(x))/g(x) + \Delta_1 ,$$

where $\delta_{s,i}$ is 0 or 1 according as the difference lacks or includes a term of weight $s-1$, and $v_{s,i}$ is a complex number. Furthermore, it follows that every class of weight $s-1$ can be represented in a unique way as a difference of the form $Q_{s,i}(G(x)) - Q_{s,i}(G(x+1))$.

Hence

$$E_x^{[1]}(G(x)) = \sum_{i=0}^{k_{s-1}} \big(\delta_{s,i}^{[1]} v_{s,i}(1-q^x)^{-1} P_0(q^x) + R_{s-1,i}(x) - \\ - R_{s-1,i}(x+1)\big) Q_{s-1,i}(G(x)) + \Delta_1$$

where $Q_{s-1,i}$ varies over all possible terms of weight $s-1$, and $\delta_{s,j}^{[i]}$ is 1 or 0 according as $Q_{s,j}(G(x)) - Q_{s,j}(G(x+1)) = Q_{s-1,i}(G(x)) + \Delta_1$ or not.

The assumption that $E_x^{[1]}(G(x)) \equiv 0$ implies that

$$R_{s-1,i}(x) - R_{s-1,i}(x+1) \equiv - \delta_{s,j}^{[i]} v_{s,j} (1-q^x)^{-1} P_0(q^x)$$

(j is uniquely determinated).

By Corollary 2 and proposition 3 one deduces that

i) $R_{s-1,i}(x) = a_{s-1,i} \in C,\ i = 0, \ldots, k_{s-1}$;

ii) $\delta_{s,i}^{[i]} = 0, j = 0, \ldots, k_s, i = 0, \ldots, k_{s-1}$, or equivalently, if $Q_{r,v}(T) = t_1^{m(1,r,v)} \ldots t_n^{m(n,r,v)}$ where $m(l,r,v)$ is an integer that depends upon l, r, v, then $m(1,s,i) = 0,\ i = 0, \ldots, k_s$.

Therefore

$$E_x^{[1]}(G(x)) = \sum_{i=0}^{k_s} a_{s,i}\Big(Q_{s,i}(G(x) - Q_{s,i}(G(x+1))\Big) + \\ + \sum_{i=0}^{k_{s-1}} a_{s-1,i}\Big(Q_{s-1,i}(G(x)) - Q_{s-1,i}(G(x+1))\Big) + \Delta_1 .$$

Repeating that same argument just made, one obtains

i) $R_{s-2,i}(x) = a_{s-2,i} \in C,\ i = 0, \ldots, k_{s-2}$;

ii) $m(2,s,i) = 0,\ m(1,s-1,j) = 0,\ i = 0, \ldots, k_s,\ j = 0, \ldots, k_{s-1}$.

Continuing recursively, one finds that $m(l, j, i) = 0$ for $j = 1, \ldots, s$; $i = 0, \ldots, k_j$; and $l = 1, \ldots, n$.

Thus one concludes that E_x is really a polynomial of weight 1, which contradicts our initial hypothesis, $s > 1$, this completes the proof that $\varphi(x)$ satisfies no algebraic differential equation over $C(x, q^x)$.

One has the same result for q-gamma function, since if a function $y(x)$ satisfies an algebraic differential equation then its logarithmic derivative $z(x)$ also satisfies such an equation.

One sees this assertion in the following way.

If $F(T) = 0$ is the equation for $Y(x)$, then substituting $d/dx(y(x)) = z(x)y(x)$, one obtains the identity

$$F\left(y(x), z(x), \ldots, \frac{d^{n-1}}{dx^{n-1}} z(x)\right) \equiv 0 .$$

Taking the derivative with respect to x, and substituting once again one finds

$$E\left(y(x), z(x), \ldots, \frac{d^{n}}{dx^{n}} z(x)\right) \equiv 0 .$$

$F(t_1, z(x), \ldots)$ and $E(t_1, z(x), \ldots)$ have a common zero $y(x)$ as polynomials in the first variable only. Hence their resultant with respect to t_1, the first variable, which depends only on $z(x)$ is identically 0. Thus $z(x)$ satisfies an algebraic differential equation. Q.E.D.

Addendum.

D. Moak [5] has recently characterized the q-gamma functions for $q > 1$. They are defined by:

$$\Gamma_q(x) = \frac{q^{\binom{x}{2}}(q^{-1}; q^{-1})_\infty (q-1)^{1-x}}{(q^{-x}; q^{-1})_\infty}$$

and satisfy the same functional equations as the $\Gamma_q(x)$ with $q < 1$:

$$\Gamma_q(x+1) = \frac{q^x - 1}{q - 1}\, \Gamma_q(x) .$$

Clearly these new $\Gamma_q(x)$ have poles at all the negative integers. Hence, the results of the preceding article apply also to these functions. In fact, the only place in which the condition $q < 1$ is used in section 1. is in the proof of the lemma, and there the same proof works for $q > 1$. In section 2 all results follow from the functional equation of $g(x) = d/dx(\log \Gamma_q(x))$, and this equation also holds for $\Gamma_q(x)$ when $q > 1$.

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