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On horospheres and holomorphic endomorfisms of the Siegel disc

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On Horospheres and Holomorphic Endomorfisms of the Siegel Disc.

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RIASSUNTO - Introdotte le nozioni di orosfera e di orociclo nel cerchio d Siegel $\mathcal{E}$ si estende ad $\mathcal{E}$ il classico lemma di Julia. Si prova, inoltre, che se F è un endomorfismo olomorfo di $\mathcal{E}$ con comportamento « regolare » su un orociclo e vicino ad un punto del bordo, allora F è un automorfismo.

Introduction.

The concept of horocycle and horosphere in the unit disc of $\mathbf{C}$ have been introduced by Poincaré with an immediate and suggestive interpretation: « the [h]orocycles may be regarded [...] as the loci of points having the same distance from a non euclidean line that lies at infinity » ([3], § 82). Similar notions of horospheres can be defined in the unit ball B_n (for the euclidean norm) of $\mathbf{C}^n$. The horospheres of B_n are characterised in terms of the Kobayashi distance, which plays, in this case, the role of the Poincaré distance (see [12]). One of the most important results about horospheres is the classical Julia's lemma.

P. C. Yang (see [12] and [8]) has extended these concepts and Julia's lemma to strictly pseudo-convex domains of $\mathbf{C}^n$, with smooth boundary.

In this paper, we shall introduce the notions of horosphere and horocycle in the Siegel disc $\mathcal{E}$. We characterise the Šilov boundary of horospheres in terms of the Kobayashi distance (Theorem 1.6)

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and we establish an extension of Julia's lemma (Theorem 2.5). In the last part of the article it is proved that if F is a holomorphic endomorphism of $\mathcal{E}$, which behaves « regularly » on a horocycle and near a boundary point of $\mathcal{E}$, then $F \in \mathrm{Aut}(\mathcal{E})$ (Theorem 5.3). Comparison with an analogous theorem concerning the endomorphisms of B_n suggests that similar results might hold for other classical domains.

§ 1. – This section is devoted to the proof of Theorem 1.6 (which establishes a connection between the Kobayashi distance and Šilov horocycles) and of Theorem 1.7 about the behaviour of automorphisms on horocycles and horospheres.

For the points of $\mathbf{C}^m$ we use the notation $\xi = (\xi_1, \xi_2, \dots, \xi_m)$ and we set $\|\xi\| = (\sum_j |\xi_j|^2)^{\frac{1}{2}}$. As usual

$$e_j = (0, \dots, 0, \underset{j}{1}, 0, \dots, 0)\,, \qquad j = 1, 2, \dots, m\,,$$

will denote the canonical base of $\mathbf{C}^m$. For any $m \times m$, complex matrix Z, $\|Z\|$ will be the operator-norm $\|Z\| = \sup\limits_{\substack{\xi \in \mathbf{C}^m \\ \|\xi\| = 1}} \|Z\xi\|$.

Let $N \geqslant 1$ be a natural number; we denote by $S(N; \mathbf{C})$ (respectively $S(N; \mathbb{R})$) the set of $N \times N$, complex (resp. real) symmetric matrices; by $U(N; \mathbf{C})$ the set of $N \times N$, complex unitary matrices.

The Siegel dise is the set

$$\mathcal{E} = \{Z \in S(N; \mathbf{C});\ I - \bar{Z}Z > 0\} = \{Z \in S(N; \mathbf{C});\ \|Z\| < 1\}$$

where I is the identity matrix of order N, and $I - \bar{Z}Z > 0$ means that $I - \bar{Z}Z$ is positive definite. The Šilov boundary of $\mathcal{E}$ is

$$\partial_S \mathcal{E} = \{Z \in S(N; \mathbf{C});\ I - \bar{Z}Z = 0\} = S(N; \mathbf{C}) \cap U(N; \mathbf{C})\,.$$

The group $\mathrm{Aut}(\mathcal{E})$ has been determined by C. L. Siegel in [11]. For any $Z_0 \in \mathcal{E}$ the map Φ_{Z_0} defined by

$$(1.1) \qquad \Phi_{Z_0}(Z) = (I - Z_0\bar{Z}_0)^{-\frac{1}{2}}(Z - Z_0)(I - \bar{Z}_0 Z)^{-1}(I - \bar{Z}_0 Z_0)^{\frac{1}{2}}$$

belongs to $\mathrm{Aut}(\mathcal{E})$. The set of all Φ_{Z_0} when Z_0 varies on $\mathcal{E}$ is a subgroup acting transitively on $\mathcal{E}$. For any $\Psi \in \mathrm{Aut}(\mathcal{E})$ there exists $U \in$

$\in U(N;\mathbf{C})$ and $Z_0 \in \mathcal{E}$ such that $\Psi(Z) = U\Phi_{Z_0}(Z)\,{}^tU$ $(Z \in \mathcal{E})$. This formula and (1.1) show that every automorphism Ψ is defined in a neighbourhood of $\bar{\mathcal{E}}$ and $\Psi(\partial_S \mathcal{E}) = \partial_S \mathcal{E}$.

1.1. Definition. *Let* $W \in \partial_S \mathcal{E}$, $k \in \mathbb{R}^+_*$. *The set*

$$H(k, W) = \{Z \in S(N;\mathbf{C});\ 0 < k(I - \bar{Z}Z) - (I - \bar{Z}W)(I - \bar{W}Z)\}$$

is called horosphere; $\partial H(k, W)$ *is called horocycle, and the Šilov horocycle is, by definition*

$$\partial_S H(k, W) = \{Z \in S(N;\mathbf{C});\ 0 = k(I - \bar{Z}Z) - (I - \bar{Z}W)(I - \bar{W}Z)\}\,.$$

1.2. Remark.

$$H(k, W) = \left\{Z \in S(N;\mathbf{C});\ \left\| Z - \frac{1}{k+1}\,W \right\| < \frac{k}{k+1}\right\}.$$

The Carathéodory and Kobayashi metrics and distances on $\mathcal{E}$ coincide (see [4], Theorem IV.1.8 and Lemma V.1.5) and we can state the following

1.3. Definition. *Let* $Z, W \in \mathcal{E}$; *the distance between* Z *and* W *is*

$$d(Z, W) = \frac{1}{2}\log\frac{1 + \|\Phi_Z(W)\|}{1 - \|\Phi_Z(W)\|}\,.$$

1.4. Lemma. *Let* $Z_0 \in \mathcal{E}$ *and* $Z, W \in S(N;\mathbf{C})$. *If* $\Phi_{Z_0}(Z)$ *and* $\Phi_{Z_0}(W)$ *are defined then*

$$(1.2)\qquad I - \overline{\Phi_{Z_0}(Z)}\,\Phi_{Z_0}(W) = (I - \bar{Z}_0 Z_0)^{\frac{1}{2}}(I - \bar{Z}Z_0)^{-1}(I - \bar{Z}W)\cdot \\ \cdot (I - \bar{Z}_0 W)^{-1}(I - \bar{Z}_0 Z_0)^{\frac{1}{2}}\,.$$

Proof. See [9], p. 145, formula (2) ■

For $s_1, s_2, \ldots, s_N \in \mathbb{R}^+$, $[s_1, s_2, \ldots, s_N]$ will stand for the diagonal matrix

$$\begin{bmatrix} s_1 & & & & \\ & s_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & s_N \end{bmatrix}.$$

1.5. LEMMA. *For* $Z \in S(N;\mathbf{C})$, *there exist* $s_1, s_2, \ldots, s_N \in \mathbf{R}^+$ *and* $U \in U(N;\mathbf{C})$ *such that* $Z = U[s_1, s_2, \ldots, s_N]\,{}^tU$ *and* $s_1^2, s_2^2, \ldots, s_N^2$ *are the characteristic roots of* $\bar{Z}Z$.

PROOF. See [11], Lemma 1, p. 12 ■

1.6. THEOREM. *Let* $Z \in \mathcal{E}$, $W \in \partial_S \mathcal{E}$, $k \in \mathbf{R}_*^+$. *Then* $Z \in \partial_S H(k, W)$ *if and only if*

$$\lim_{\substack{A\to W\\ A\in\mathcal{E}}} d(A, Z) - d(A, O) = \tfrac{1}{2}\log k\,. \tag{1.3}$$

PROOF. Let $Z \in U[s_1, s_2, \ldots, s_N]\,{}^tU \in \mathcal{E}$, $U \in U(N;\mathbf{C})$; then

$$\|Z\| = \max_j s_j \qquad \text{and} \qquad \|(I - \bar{Z}Z)^{-1}\| = \frac{1}{1 - \|Z\|^2}\,.$$

(1.3) is equivalent to

$$k = \lim_{\substack{A\to W\\ A\in\mathcal{E}}} \frac{1 - \|A\|^2}{1 - \|\Phi_A(Z)\|^2} = \lim_{\substack{A\to W\\ A\in\mathcal{E}}} \frac{\|(I - \overline{\Phi_A(Z)}\,\Phi_A(Z))^{-1}\|}{\|(I - \bar{A}A)^{-1}\|}\,,$$

i.e. (by (1.2)) to

$$k = \lim_{\substack{A\to W\\ A\in\mathcal{E}}} \cdot \tag{1.4}$$

$$\cdot \left\| \frac{(I - \bar{A}A)^{-\frac12}}{\|(I - \bar{A}A)^{-\frac12}\|}\,(I - \bar{A}Z)(I - \bar{Z}Z)^{-1}(I - \bar{Z}A)\,\frac{(I - \bar{A}A)^{-\frac12}}{\|(I - \bar{A}A)^{-\frac12}\|} \right\|.$$

Assume that (1.4) holds. Let $C = (I - \bar{W}Z)(I - \bar{Z}Z)^{-1}(I - \bar{Z}W)$, and $A = A(t) = tW$, $t \in (0, 1)$. Then, for each t,

$$\frac{(I - \bar{A}A)^{-\frac12}}{\|(I - \bar{A}A)^{-\frac12}\|} = I$$

and, in view of (1.4), as $t \nearrow 1$

$$k = \|C\|\,. \tag{1.5}$$

Let $q(t) = \sqrt{1-\sqrt{1-t}}$, $t \in (0,1)$; by Lemma 1.5 there exists $V \in U(N; \mathbf{C})$ such that $W = V\,{}^tV$. Let $A_j(t) = V[q(t), \dots, q(t), \sqrt{t}, q(t), \dots, q(t)]\,{}^tV$, $j = 1, 2, \dots, N$; then $A_j(t) \in \mathcal{E}$, $\lim_{t \nearrow 1} A_j(t) = W$ and

$$\frac{(I - \overline{A_j(t)}\,A_j(t))^{-\frac{1}{2}}}{\|(I - \overline{A_j(t)}\,A_j(t))^{-\frac{1}{2}}\|} = \bar{V}[(1-t)^{\frac{1}{2}}, \dots, (1-t)^{\frac{1}{2}}, \underset{j}{1}, (1-t)^{\frac{1}{2}}, \dots, (1-t)^{\frac{1}{2}}]\,{}^tV,$$

therefore

$$\lim_{t \nearrow 1} \frac{(I - \overline{A_j(t)}\,A_j(t))^{-\frac{1}{2}}}{\|(I - \overline{A_j(t)}\,A_j(t))^{-\frac{1}{2}}\|} = \bar{V}[0, \dots, 0, \underset{j}{1}, 0, \dots, 0]\,{}^tV.$$

Let $B_j = [0, \dots, 0, \underset{j}{1}, 0, \dots, 0]$. Condition (1.4) yields

$$k = \|\bar{V}B_j\,{}^tVC\bar{V}B_j\,{}^tV\| = \|B_j\,{}^tVC\bar{V}B_j\| = |d_{jj}|,$$

with ${}^tVC\bar{V} = (d_{ij})$; then $d_{jj} = ke^{i\theta_j}$, $\theta_j \in \mathbf{R}$. In view of (1.5)

$$k^2 \geqslant \|{}^tVC\bar{V}e_j\|^2 = \sum_{i \neq j} |d_{ij}|^2 + k^2.$$

Hence ${}^tVC\bar{V} = k[e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N}]$. Moreover

$$ke^{i\theta_j} = {}^t\bar{e}_j\,{}^tVC\bar{V}e_j = {}^t\bar{e}_j\,{}^tV(I - \bar{W}Z)(I - \bar{Z}Z)^{-1}(I - \bar{Z}W)\bar{V}e_j;$$

putting

$$\xi = (I - \bar{Z}Z)^{-1}(I - \bar{Z}W)\bar{V}e_j, \qquad ke^{i\theta_j} = {}^t\bar{\xi}(I - \bar{Z}Z)\xi = \|\xi\|^2 - \|Z\xi\|^2;$$

since $\|Z\| < 1$, $C = kI$. This proves that (1.4) implies $Z \in \partial_S H(k, W)$.

To prove the converse, let $C_A = (I - \bar{A}Z)(I - \bar{Z}Z)^{-1}(I - \bar{Z}A)$ and

$$Q_A = \frac{(I - \bar{A}A)^{-\frac{1}{2}}}{\|(I - \bar{A}A)^{-\frac{1}{2}}\|}.$$

If $Z \in \partial_S H(k, W)$, then $\lim_{\substack{A \to W \\ A \in \mathcal{E}}} C_A = kI$. Hence

$$\big|\|Q_A C_A Q_A\| - k\big| = \big|\|Q_A C_A Q_A\| - \|Q_A kIQ_A\|\big| \leqslant$$
$$\leqslant \|Q_A(C_A - kI)Q_A\| \leqslant \|C_A - kI\| \to 0$$

as $A \to W$ ∎

We investigate now how automorphism transforms horospheres.

1.7. THEOREM. *Let* $\Psi \in \mathrm{Aut}(\mathcal{E})$ *be such that* $\Psi^{-1}(0) \in \partial_S H(1/a, W)$, *with* $a \in \mathbb{R}^+_*$ *and* $W \in \partial_S \mathcal{E}$. *For every* $k \in \mathbb{R}^+_*$,

(i) $\Psi(H(k, W)) = H(ak, \Psi(W))$;

(ii) $\Psi(\partial H(k, W)) = \partial H(ak, \Psi(W))$;

(iii) $\Psi(\partial_S H(k, W)) = \partial_S H(ak, \Psi(W))$.

PROOF. There exist $U \in U(N; \mathbb{C})$ and $Z_0 \in \mathcal{E}$ such that $\Psi = U\Phi_{Z_0}{}^tU$. Since

$$Z_0 = \Psi^{-1}(0) \in \partial_S H(a^{-1}, W),$$
$$a^{-1}(I - \bar{Z}_0 Z_0) - (I - \bar{Z}_0 W)(I - \overline{W} Z_0) = 0 .$$

Hence, using (1.2) we have

$$ak(I - \overline{\Psi(Z)}\Psi(Z)) - (I - \overline{\Psi(Z)}\Psi(W))(I - \overline{\Psi(W)}\Psi(Z)) =$$
$$= \bar{U}(I - \bar{Z}_0 Z_0)^{\frac12}(I - \bar{Z}Z_0)^{-1} a[k(I - \bar{Z}Z) - (I - \bar{Z}W)(I - \overline{W}Z)] \cdot$$
$$\cdot (I - \bar{Z}_0 Z)^{-1}(I - \bar{Z}_0 Z_0)^{\frac12}\, {}^tU .$$

Then $\Psi(Z) \in H(ak, \Psi(W)) \Leftrightarrow k(I - \bar{Z}Z) - (I - \bar{Z}W)(I - \overline{W}Z) > 0 \Leftrightarrow$ $\Leftrightarrow Z \in H(k, W)$ ■

§ 2. We show that each horosphere is, in some way, the limit of a sequence of ball for the distance d (Lemma 2.2 and 2.3). This result and the fact that holomorphic endomorphisms contract d enable us to prove an analogous of Julia's lemma (Theorem 2.5).

We begin by establishing these preliminary lemmas.

2.1. LEMMA. *Let* $Z, Z_0 \in \mathcal{E}$. *For every* $r \in (0, 1)$, *the following conditions are equivalent*:

(i) $d(Z, Z_0) < \frac12 \log(1 + r)/(1 - r)$;

(ii) $0 < I - \bar{Z}Z - (1 - r^2)(I - \bar{Z}Z_0)(I - \bar{Z}_0 Z_0)^{-1}(I - \bar{Z}_0 Z)$.

PROOF.

$$d(Z, Z_0) < \frac{1}{2} \log \frac{1+r}{1-r} \Leftrightarrow \|\Phi_{Z_0}(Z)\| < r \Leftrightarrow 0 < I - r^{-2}\,\overline{\Phi_{Z_0}(Z)}\,\Phi_{Z_0}(Z) =$$
$$= I - r^{-2}(I - \bar{Z}_0 Z_0)^{\frac12}(I - \bar{Z}Z_0)^{-1}(\bar{Z} - \bar{Z}_0)(I - Z_0\bar{Z}_0)^{-1}(Z - Z_0) \cdot$$
$$\cdot (I - \bar{Z}_0 Z)^{-1}(I - \bar{Z}_0 Z_0)^{\frac12},$$

in view of (1.1). Multiplying on the left by the matrix $r(I-\bar{Z}Z_0)\cdot$ $\cdot(I-\bar{Z}_0Z_0)^{-\frac{1}{2}}$ and on the right by its adjoint we get

$$0<r^2(I-\bar{Z}Z_0)(I-\bar{Z}_0Z_0)^{-1}(I-\bar{Z}_0Z)-(\bar{Z}-\bar{Z}_0)(I-Z_0\bar{Z}_0)^{-1}(Z-Z_0)=$$
$$=-(1-r^2)(I-\bar{Z}Z_0)(I-\bar{Z}_0Z_0)^{-1}(I-\bar{Z}_0Z)+$$
$$+(I-\bar{Z}Z_0)(I-\bar{Z}_0Z_0)^{-1}(I-\bar{Z}_0Z)-(\bar{Z}-\bar{Z}_0)(I-Z_0\bar{Z}_0)^{-1}(Z-Z_0)=$$
$$=I-\bar{Z}Z-(1-r^2)(I-\bar{Z}Z_0)(I-\bar{Z}_0Z_0)^{-1}(I-\bar{Z}_0Z) \quad \blacksquare$$

2.2. Lemma. *Let $W\in\partial_S\mathcal{E}$ and let $(Z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ converging to W. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in* $(0,1)$ *such that* $\lim(1-r_n^2)\cdot$ $\cdot(I-\bar{Z}_nZ_n)^{-1}=S\neq 0$. *Put* $k=\|S\|^{-1}$. *If $Z\in H(k,W)$, there exists $n_0\in\mathbb{N}$ such that* $d(Z,Z_n)<\frac{1}{2}\log(1+r_n)/(1-r_n)$, *for every* $n\geqslant n_0$.

Proof. For $Z\in H(k,W)$, $0<k(I-\bar{Z}Z)-(I-\bar{Z}W)(I-\overline{W}Z)$. Since $k=\|S\|^{-1}$, then $I-kS\geqslant 0$ and $0\leqslant(I-\bar{Z}W)(I-kS)(I-\overline{W}Z)$. It follows that

$$0<k[I-\bar{Z}Z-(I-\bar{Z}W)S(I-\overline{W}Z)]=$$
$$=k[I-\bar{Z}Z-\lim(I-\bar{Z}Z_n)(1-r_n^2)(I-\bar{Z}_nZ_n)^{-1}(I-\bar{Z}_nZ)]$$

and, if n is sufficiently large, we must have

$$0<I-\bar{Z}Z-(I-\bar{Z}Z_n)(1-r_n^2)(I-\bar{Z}_nZ_n)^{-1}(I-\bar{Z}_nZ)\,.$$

The conclusion follows from Lemma 2.1 $\blacksquare$

2.3. Lemma. *Let $W\in\partial_S\mathcal{E}$ and let $(Z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ converging to W. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in* $(0,1)$ *such that the limit* $\lim(1-r_n^2)(I-\bar{Z}_nZ_n)^{-1}$ *exists and is* $\geqslant(1/k)I$ *for a suitable* $k\in\mathbb{R}_*^+$. *If $Z\in\mathcal{E}$ and if* $d(Z,Z_n)<\frac{1}{2}\log(1+r_n)/(1-r_n)$ *for infinitely many* $n\in\mathbb{N}$, *then* $Z\in\overline{H(k,W)}$.

Proof. In view of Lemma 2.1, $0<I-\bar{Z}Z-(1-r_n^2)(I-\bar{Z}Z_n)\cdot$ $\cdot(I-\bar{Z}_nZ_n)^{-1}(I-\bar{Z}_nZ)$ for infinitely many $n\in\mathbb{N}$. Then, as $n\to\infty$,

$$0\leqslant I-\bar{Z}Z-(I-\bar{Z}W)\lim(1-r_n^2)(I-\bar{Z}_nZ_n)^{-1}(I-\overline{W}Z)\leqslant$$
$$\leqslant I-\bar{Z}Z-\frac{1}{k}(I-\bar{Z}W)(I-\overline{W}Z) \quad \blacksquare$$

2.4. Remark. *Let $W \in \partial_S \mathcal{E}$ and let $(Z_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{E}$ converging to W. Then* $\lim \|(I - \bar{Z}_n Z_n)^{-1}\| = +\infty$.

Proof. Let $M > 0$. Since $\lim I - \bar{Z}_n Z_n = 0$, there is $\bar{n} \in \mathbf{N}$ such that $\|(I - \bar{Z}_n Z_n)\xi\| < M^{-1}$ for each $n \geqslant \bar{n}$ and for each $\xi \in \mathbf{C}^N$ with $\|\xi\| = 1$. Therefore

$$\left\| (I - \bar{Z}_n Z_n)^{-1} \frac{1}{\|(I - \bar{Z}_n Z_n)\xi\|} (I - \bar{Z}_n Z_n)\xi \right\| > M \quad \blacksquare$$

2.5. Lemma. (Julia's lemma). *Let $F\colon \mathcal{E} \to \mathcal{E}$ be a holomorphic endomorphism. Suppose there is a sequence $(Z_n)_{n\in\mathbf{N}}$ in $\mathcal{E}$ such that*

$$\lim Z_n = W \in \partial_S \mathcal{E}, \qquad \lim F(Z_n) = V \in \partial_S \mathcal{E}$$

and there exists $a \in \mathbf{R}^+_*$ such that

$$\lim \frac{(I - \overline{F(Z_n)}\, F(Z_n))^{-1}}{\|(I - \bar{Z}_n Z_n)^{-1}\|} \geqslant \frac{1}{a} I . \tag{2.1}$$

Then $F(H(k, W)) \subseteq \overline{H(ak, V)}$ for all $k \in \mathbf{R}^+_*$.

Proof. Let $Z \in H(k, W)$. We can assume, without any restriction that

$$\lim \frac{(I - \bar{Z}_n Z_n)^{-1}}{\|(I - \bar{Z}_n Z_n)^{-1}\|} = Q \quad \text{with} \quad \|Q\| = 1 .$$

By previous Remark we can define, for n sufficiently large,

$$r_n = \left(1 - \frac{1}{k\|(I - \bar{Z}_n Z_n)^{-1}\|}\right)^{\frac{1}{2}},$$

therefore

$$1 - r_n^2 = \frac{1}{k\|(I - \bar{Z}_n Z_n)^{-1}\|} .$$

It follows, from Lemma 2.2, that there exists $n_0 \in \mathbf{N}$ such that

$$d(Z, Z_n) < \frac{1}{2} \log \frac{1 + r_n}{1 - r_n}, \qquad \text{for all } n \geqslant n_0 .$$

Since F is a contraction for d,

$$d(F(Z), F(Z_n)) < \frac{1}{2}\log\frac{1+r_n}{1-r_n};$$

but

$$\frac{1}{a} I \leqslant \lim \frac{(I-\overline{F(Z_n)}\,F(Z_n))^{-1}}{\|(I-\bar{Z}_n\bar{Z}_n)^{-1}\|}\,\frac{1-r_n^2}{1-r_n^2} = k \lim\,(1-r_n^2)(I-\overline{F(Z_n)}\,F(Z_n))^{-1}.$$

Lemma 2.3 yields $F(Z)\in\overline{H(k, W)}$ ■

§ 3. H. Alexander has proved in [1] that if Ω is a domain of $\mathbf{C}^n$ $(n>1)$ with $\Omega\cap\partial B_n\neq\emptyset$ and if $F\colon\Omega\to\mathbf{C}^n$ is a holomorphic map such that $F(\Omega\cap\partial B_n)\subset\partial B_n$, then F is constant or F extends to an automorphism of B_n. Replacing $\mathbf{C}^n$ by $S(N;\mathbf{C})$ $(N>1)$ and B_n by $\mathcal{E}$ some of the machinary involved in the proof of H. Alexander cannot be adapted because $\partial\mathcal{E}$ is not a smooth hypersurface. Then we can establish only some first consequences of previous hypothesies (see Theorem 3.6).

The Siegel upper half-plane is the set $\mathcal{H}=\{X+iY;\ X,\ Y\in S(N;\mathbf{R})$ and $Y>0\}$. It is well known (see [7], p. 5) that the Cayley transformation $Z\mapsto\sigma(Z)=i(I+Z)(I-Z)^{-1}$ maps $\mathcal{E}$ bi-holomorphically onto $\mathcal{H}$. Moreover the Šilov boundary $\partial_S\mathcal{H}$ of $\mathcal{H}$ is defined by the two equivalent conditions

$$(3.1)\qquad \partial_S\mathcal{E}^*=\{Z\in S(N;\mathbf{C});\ \det(I-Z)\neq 0\}\overset{\sigma}{\underset{\simeq}{\to}}\partial_S\mathcal{H}= \\ =\{X+iY;\ X,\ Y\in S(N;\mathbf{R}) \text{ and } Y=0\}.$$

3.1. Lemma. *Let S be a smooth real submanifold of $S(N;\mathbf{C})$ such that $S\subset\partial\mathcal{E}$ and $S\cap\partial_S\mathcal{E}\neq\emptyset$. Then* $\dim_{\mathbf{R}}S\leqslant N(N+1)/2$.

Proof. Replacing $\mathcal{E}$ by $\mathcal{H}$ we can assume $0\in S\subset\partial\mathcal{H}$. Let $n=\dim_{\mathbf{R}}S$, then there is a C^∞ map $X+iY\colon\mathbf{R}^n\to S(N;\mathbf{R})+iS(N;\mathbf{R})$ such that $X(0)=Y(0)=0$ and $Y(s)\geqslant 0$ for all $s\in\mathbf{R}^n$. Moreover the jacobian matrix $[\partial X/\partial s,\ \partial Y/\partial s]$ has rank n.

Let $Y=(y_{ij})$. It is enough to prove that $(\partial y_{ij}/\partial s_\alpha)(0)=0$ for $1\leqslant i\leqslant j\leqslant N$, $\alpha=1,2,\ldots,n$. Since $y_{ii}(0)=0$ and $y_{ii}(s)\geqslant 0$, for all s, then $(\partial y_{ii}/\partial s_\alpha)(0)=0$. For all $1\leqslant i\leqslant j\leqslant N$,

$$\begin{bmatrix} y_{ii}(s) & y_{ij}(s)\\ y_{ij}(s) & y_{jj}(s)\end{bmatrix}\geqslant 0,$$

and therefore $(y_{ii}y_{jj})^{\frac{1}{2}} - y_{ij} \geqslant 0$. For $t \in \mathbb{R}$,

$$y_{ii}(te_\alpha) = \frac{1}{p!}\frac{\partial^p y_{ii}}{\partial s_\alpha^p}(0)\,t^p + o(t^p)$$

and

$$y_{jj}(te_\alpha) = \frac{1}{q!}\frac{\partial^q y_{jj}}{\partial s_\alpha^q}(0)\,t^q + o(t^q) \qquad \text{with } \frac{\partial^p y_{ii}}{\partial s_\alpha^p}(0)\,,$$

$$\frac{\partial^q y_{jj}}{\partial s_\alpha^q}(0) \neq 0 \qquad \text{and} \qquad p, q > 1\,.$$

It follows that

$$\frac{\partial}{\partial s_\alpha}(y_{ii}y_{jj})^{\frac{1}{2}}(0) =$$

$$= \lim_{t\to 0}\frac{1}{t}\left[\left(\frac{1}{p!}\frac{\partial^p y_{ii}}{\partial s_\alpha^p}(0)\,t^p + o(t^p)\right)\left(\frac{1}{q!}\frac{\partial^q y_{jj}}{\partial s_\alpha^q}(0)\,t^q + o(t^q)\right)\right]^{\frac{1}{2}} = 0\,,$$

then $(\partial y_{ij}/\partial s_\alpha)(0) = 0$ ■

3.2. Lemma. *Let $U \in \partial\mathcal{E}\setminus\partial_S\mathcal{E}$. There exists a smooth real submanifold S of $S(N; \mathbf{C})$ such that $U \in S \subset \partial\mathcal{E}$ and* $\dim_{\mathbb{R}} S > N(N+1)/2$

Proof. Replacing $\mathcal{E}$ by $\mathcal{H}$ we can assume

$$U = X_0 + iA[d_1, d_2, \ldots, d_N]\,{}^tA$$

with $X_0 \in S(N; \mathbb{R})$, where A is an orthogonal matrix of order N, $d_1, d_2, \ldots, d_N \geqslant 0$ and $d_1 > 0$, $d_2 = 0$. Then $(X, t) \equiv X + iA[t, d_2, \ldots, d_N]\,{}^tA$ with $X \in S(N; \mathbb{R})$ and $t \in \mathbb{R}^+$ is the required parametrization ■

3.3. Lemma. *Let Ω be a domain in $S(N; \mathbf{C})$ such that $\Omega \cap \partial_S\mathcal{E} \neq \emptyset$. Let F be a diffeomorphism of Ω onto an open subset $F(\Omega)$ of $S(N; \mathbf{C})$. If $F(\Omega \cap \partial\mathcal{E}) \subset \partial\mathcal{E}$, then $F(\Omega \cap \partial_S\mathcal{E}) \subset \partial_S\mathcal{E}$.*

Proof. We begin by showing that $F(\Omega \cap \partial\mathcal{E})$ is open in $\partial\mathcal{E}$. Since is the unit ball for a norm in $S(N; \mathbf{C}) = \mathbb{R}^{N(N+1)}$ equivalent to the euclidean norm, then $\partial\mathcal{E}$ is homeomorphic to $S^{N(N+1)-1}$. Then, by Theorem 6.6 in [6] Ch. III, it is enough to notice that $F_{|\Omega\cap\partial\mathcal{E}}\colon \Omega \cap \partial\mathcal{E} \to F(\Omega \cap \partial\mathcal{E})$ is a homeomorphism and $\Omega \cap \partial\mathcal{E}$ is open in $\partial\mathcal{E}$.

Let $Z \in \Omega \cap \partial_S\mathcal{E}$ and suppose $F(Z) \in \partial\mathcal{E}\setminus\partial_S\mathcal{E}$. By Lemma 3.2 there is smooth real submanifold S such that $F(Z) \in S \subset F(\Omega \cap \partial \mathcal{E})$ and $\dim_{\mathbf{R}} S > N(N+1)/2$. Thus, by Lemma 3.1, $\dim_{\mathbf{R}} F^{-1}(S) \leqslant N(N+1)/2$. This is a contradiction ∎

For $r \in \mathbf{R}_*^+$ and $Z_0 \in S(N; \mathbf{C})$, let

$$B_r(Z_0) = \{Z \in S(N; \mathbf{C});\ \|Z - Z_0\| < r\}\,.$$

3.4. Lemma. *Let* $0 < r < 1$. *There exist an open neighbourhood* Σ *of the identity matrix* I, *a continous function* f *on* $B_r(I)$ and $\varepsilon > 0$ *such that*

(i) $\Sigma \cap \mathcal{E} \subset\subset B_r(I)$;

(ii) f *is plurisubharmonic on* $B_r(I)$;

(iii) *For all* $Z_0 \in \Sigma \cap \mathcal{E} \cap B_\varepsilon(I)$, $Z_1 \in \partial\Sigma \cap \overline{\mathcal{E}}$, $f(Z_0) > f(Z_1)$.

Proof. Let $0 < \varrho < 1$ and $\Sigma = \{Z \in S(N; \mathbf{C});\ \|Ze_j + e_j\| > 2\varrho,\ j = 1, 2, \dots, N\}$ Let $Z \in \Sigma \cap \mathcal{E}$, therefore

$$\|Ze_j + e_j\|^2 + 4\varrho^2 < \|Ze_j - e_j\|^2 + \|Ze_j + e_j\|^2 = \\ = 2(\|Ze_j\|^2 + \|e_j\|^2) \leqslant 4 \tag{3.2}$$

hence $\|Ze_j - e_j\| < 2(1-\varrho^2)^{\frac12}$. For any $\xi \in \mathbf{C}^N$, with $\|\xi\| = 1$, we have

$$\|(Z - I)\xi\| \leqslant \sum_j |\xi_j|\,\|Ze_j - e_j\| < 2(1-\varrho^2)^{\frac12} N^{\frac12}\,.$$

If ϱ is such that $(1 - r^2/4N)^{\frac12} < \varrho < 1$, then (i) is satisfied.

The function $f(Z) = \sum_j \log \frac12 \|Ze_j + e_j\|$ is plurisubharmonic (see [4], Lemma II.6.2).

Let $0 < \varepsilon < 2(1 - \sqrt[N]{\varrho})$ and $\varepsilon < r$. If $Z_0 \in \Sigma \cap \mathcal{E} \cap B_\varepsilon(I)$, then

$$\|Z_0 e_j + e_j\| \geqslant \|2e_j\| - \|Z_0 e_j - e_j\| > 2 - \varepsilon\,.$$

If follows that $f(Z_0) > \log((2-\varepsilon)/2)^N > \log \varrho$.

If $Z_1 \in \partial\Sigma \cap \overline{\mathcal{E}}$, there is j_0 such that $\|Z_1 e_{j_0} + e_{j_0}\| = 2\varrho$. By (3.2), $\|Z_1 e_j + e_j\| \leqslant (4 - \|Z_1 e_j - e_j\|^2)^{\frac12}$ $(j = 1, 2, \dots, N)$; therefore

$$\prod_j \left(\frac12 \|Z_1 e_j + e_j\|\right) \leqslant \varrho \prod_{j \neq j_0} \left(1 - \frac{\|Z_1 e_j - e_j\|^2}{4}\right)^{\frac12} \leqslant \varrho\,.$$

It follows $f(Z_0) > \log \varrho \geqslant f(Z_1)$ ∎

3.5. Remark. *Let Ω be a domain in $S(N;\mathbf{C})$ such that $\Omega \cap \partial_S \mathcal{E} \neq \emptyset$. If $f\colon \Omega \to \mathbf{C}$ is a holomorphic function such that $f(Z) = 0$, for every $Z \in \Omega \cap \partial_S \mathcal{E}$, then $f \equiv 0$ in Ω.*

Proof. In view of (3.1) $\Omega \cap \partial_S \mathcal{E}$ is bi-holomorphically equivalent to an open subset of

$$\partial_s \mathcal{H} \simeq \left\{ \xi \in \mathbf{C}^{N(N+1)/2};\ \operatorname{Im} \xi_j = 0,\ j = 1, 2, \ldots, \frac{N(N+1)}{2} \right\} \qquad \blacksquare$$

3.6. Theorem. *Let Ω be a domain in $S(N;\mathbf{C})$ such that $\Omega \cap \partial_S \mathcal{E} \neq \emptyset$. If $F\colon \Omega \to S(N;\mathbf{C})$ is a holomorphic map such that $F(\Omega \cap \partial \mathcal{E}) \subset \partial \mathcal{E}$, then one of the following statements holds:*

(i) *There exists $\xi \in \mathbf{C}^N \setminus \{0\}$ such that the map $Z \mapsto F(Z)\xi$ is constant;*

(ii) *There is $\tilde{Z}$ e $\Omega \cap \partial_S \mathcal{E}$ such that $dF(\tilde{Z})$ is invertible and there is an open neighbourhood Ω_1 of $\tilde{Z}$ such that $F(\Omega_1 \cap \partial_S \mathcal{E}) \subset \partial_S \mathcal{E}$.*

Proof. If there is $\tilde{Z} \in \Omega \cap \partial_S \mathcal{E}$ such that $dF(\tilde{Z})$ is invertible, then the theorem follows from Lemma 3.3. Therefore it is enough assume $\det dF(Z) = 0$ on $\Omega \cap \partial_S \mathcal{E}$ and prove (i). Hence by previous Remark, $\det dF(Z) = 0$ on Ω. If $N = 1$, then F is a constant map. Let $N > 1$. Since $\max\limits_{Z \in \Omega} \operatorname{rank} dF(Z) = n < N(N+1)/2$, there exists a minor $M = M(Z)$ of $dF(Z)$ of order n such that $\det M(A) \neq 0$ for a suitable A. Still by Remark 3.5 there is $B \in \Omega \cap \partial \mathcal{E}_S$ such that $\det M(B) \neq 0$.

Replacing F by $F \circ \Phi$, for a suitable $\Phi \in \operatorname{Aut}(\mathcal{E})$, there is no restriction in assuming $B = I$. Let

$$M(I) = \begin{bmatrix} \dfrac{F_{j_1 h_1}}{z_{a_1 b_1}}(\mathrm{I}) \ldots \dfrac{F_{j_1 h_1}}{Z_{a_n b_n}}(\mathrm{I}) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{F_{j_n h_n}}{z_{a_1 b_1}}(\mathrm{I}) \ldots \dfrac{F_{j_n h_n}}{z_{a_n b_n}}(\mathrm{I}) \end{bmatrix}.$$

Since $\operatorname{rank} dF(I) \geqslant \operatorname{rank} dF(Z)$ $(Z \in \Omega)$, by the implicit function theorem there exists r, $0 < r < 1$, such that all the F_{jh}, $1 \leqslant j \leqslant h \leqslant N$, are functionally dependent on $B_r(I) \subset \Omega$, on $F_{j_1 h_1}, F_{j_2 h_2}, \ldots, F_{j_n h_n}$.

With the same notations as in Lemma 3.4 let $Z_0 \in \Sigma \cap \mathcal{E} \cap B_\varepsilon(I)$

and let Γ be the set defined by

$$\begin{cases} F_{j_1h_1}(Z) = F_{j_1h_1}(Z_0)\,, \\ \dots\dots\dots\dots\dots\dots \\ F_{j_nh_n}(Z) = F_{j_nh_n}(Z_0)\,. \end{cases}$$

Then $\Gamma \cap \partial(\Sigma \cap \mathcal{E}) \neq \emptyset$. (In fact in a neighbourhood of I Γ is biholomorphically equivalent to an affine subspace). The function f attains its maximum on $\Gamma \cap \overline{\Sigma \cap \mathcal{E}}$ in a suitable matrix $Z_1 \in \partial(\Sigma \cap \mathcal{E})$ (see [5], p. 272). In view of Lemma 3.4, $Z_1 \in \Sigma \cap \mathcal{E} \subset B_r(I)$. Then $F(Z_0) = F(Z_1) \in \partial\mathcal{E}$. It follows that there exists $\xi_0 \in \mathbb{C}^N$, with $\|\xi_0\| = 1$, such that $\|F(Z_0)\xi_0\| = 1$.

Let Z_2 be a matrix near Z_0. Then $Z_0 + \lambda(Z_2 - Z_0) \in \Sigma \cap \mathcal{E} \cap B_\varepsilon(I)$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Therefore, by the previous argument $F(Z_0 + \lambda(Z_2 - Z_0)) \in \partial\mathcal{E}$. It follows $\|F(Z_0 + \lambda(Z_2 - Z_0)\xi\| \leqslant 1$. Hence, by the strong maximum principle $F(Z_0 + \lambda(Z_2 - Z_0))\xi_0 = F(Z_0)\xi_0$ for each λ. Thus $F(Z)\xi_0$ is constant ■

§ 4. In this section we shall show that if F is a holomorphic endomorphism of $\mathcal{E}$ which maps a piece of the Šilov horocycle $\partial_S H(1, I)$ into the same Šilov horocycle and behaves « regularly » near a point of $\partial_S \mathcal{E}$, then F maps $\mathcal{E} \cap \overline{H(k, I)}$ into $\overline{H(k, I)}$ for every $k \in \mathbb{R}^+_*$ (Theorem 4.4).

The group $\{W \mapsto W + A;\ A \in S(N; \mathbb{R})\}$ is a group of automorphism of $\mathcal{H}$ which acts transitively on $\partial_S\mathcal{H}$. For $A \in S(N; \mathbb{R})$, let $\Psi_A(Z) = \sigma^{-1}(\sigma(Z) + A)$ $(Z \in \mathcal{E})$. Since $\Psi_A \in \operatorname{Aut}(\mathcal{E})$, we can extend Ψ_A in a neighbourhood of $\overline{\mathcal{E}}$.

4.1. Proposition. *The set* $\{\Psi_A;\ A \in S(N; \mathbb{R})\}$ *is a subgroup of* $\operatorname{Aut}(\mathcal{E})$ *which acts transitively on* $\partial_S \mathcal{E}^*$. *Moreover, for all* $A \in S(N; \mathbb{R})$,

(i) $\Psi_A(I) = I$;

(ii) $\Psi_A^{-1}(0) \in \partial_S H(1, I)$.

Proof. To prove (i) it is enough to verify that

$$\lim \Psi_A((1 - 1/n)I) = I \quad ■$$

4.2. Lemma. *Let* $U, V \in \partial_S \mathcal{E}$ *and* $k \in \mathbb{R}^+_*$. *If* $\det(U - V) \neq \emptyset$ *then there exists* $\Psi \in \operatorname{Aut}(\mathcal{E})$ *such that* $\Psi(U) = I$, $\Psi(V) = -I$, $\Psi(\partial_S H(k, U)) = \partial_S H(1, I)$ *and* $\Psi(\partial H(k, U)) = \partial H(1, I)$.

PROOF. In view of Lemma 1.5, there exists $T \in U(N; \mathbb{C})$ such that $U = T\,{}^tT$. Let $W = {}^t\overline{T}V\overline{T}$. Since $\det(I - W) = \det(U - V) \neq 0$, by Proposition 4.1, there exists $A \in S(N; \mathbb{R})$ such that $\Psi_A(W) = -I$. Let $\Psi(Z) = (\Phi_{((1-k)/(1+k))I} \circ \Psi_A)({}^t\overline{T}Z\overline{T})$. Since $\Phi^{-1}_{((1-k)/(1+k))I}(0) \in \partial_S H(k, I)$ and $\Phi_{((1-k)/(1+k))I}(I) = I$, the lemma follows from Theorem 1.7 and Proposition 4.1 ■

4.3. REMARK. *Let* $Z \in \partial_S H(2, I) \setminus \{-\frac{1}{3}I\}$. *Then* $d(0, Z) > d(0, -\frac{1}{3}I)$.

PROOF. It follows, from Definition 1.1, that $\frac{1}{2}(3Z - I) \in U(N; \mathbb{C})$ and $\bar{Z}Z = Z\bar{Z}$. Therefore there exists $U \in U(N; \mathbb{C})$ such that $Z = U[\lambda_1, \lambda_2, \ldots, \lambda_N]\,{}^t\overline{U}$. Hence $\frac{1}{2}|3\lambda_j - 1| = 1$, $j = 1, 2, \ldots, N$; thus $|\lambda_j| > \frac{1}{3}$ or $\lambda_j = -\frac{1}{3}$. Therefore $\|Z\| = \max_j |\lambda_j| > \frac{1}{3} = \|-\frac{1}{3}I\|$ ■

4.4. THEOREM. *Let* $F\colon \mathcal{E} \to \mathcal{E}$ *be a holomorphic endomorphism for which the following conditions hold*:

(i) *There is a domain* $\Omega \subset \mathcal{E}$ *such that*

(*A*) $\Omega \cap \partial_S H(1, I) \neq \emptyset$,

(*B*) $F(\Omega \cap \partial_S H(1, I)) \subset \partial_S H(1, I)$;

(ii) *There is a sequence* $(Z_n)_{n \in \mathbb{N}}$ *in* $\mathcal{E}$ *such that*

(*C*) $\lim Z_n = -I$,

(*D*) $\lim F(Z_n) = W$ *for a suitable* $W \in \partial_S \mathcal{E}$.

Then, for every $k \in \mathbb{R}^+_*$,

$$F(\mathcal{E} \cap \overline{H(k, I)}) \subset \overline{H(k, I)}\,. \tag{4.1}$$

PROOF. Setting $\beta(Z) = 2Z - I$, β is an isomorphism of $S(N; \mathbb{C})$ and $\beta(\partial_S H(1, I)) = \partial_S \mathcal{E}$, $\beta(\mathcal{E}) = \{Z \in S(N; \mathbb{C});\ \|Z + I\| < 2\}$. Let

$G = \beta \circ F \circ \beta^{-1}\colon \beta(\mathcal{E}) \to \beta(\mathcal{E})$. It follows from (*B*) that $G(\partial_S \mathcal{E} \cap \beta(\Omega)) \subset \partial_S \mathcal{E}$.

The map L defined by $L(Z) = \overline{G(\bar{Z}^{-1})}$ is holomorphic on

$$R = \{Z \in \beta(\mathcal{E});\ \det Z \neq 0 \text{ and } \bar{Z}^{-1} \in \beta(\mathcal{E})\}\,.$$

If $Z \in \beta(\Omega) \cap \partial_S \mathcal{E}$, then Z, $G(Z) \in \partial_S \mathcal{E}$ and $L(Z)\,G(Z) = I$. We denote by R_1 the union of these connected components of R which intersect

$\beta(\Omega) \cap \partial_S \mathcal{E}$. In view of (3.1),

$$\beta(\mathcal{E}) \cap \partial_S \mathcal{E} = \partial_S \mathcal{E}^* \simeq \partial_S \mathcal{H} = \\ = \{\xi \in \mathbf{C}^{N(N+1)/2};\ \operatorname{Im}\xi_j = 0;\ j = 1, 2, \ldots, N(N+1)/2\},$$

which is connected; therefore $\partial_S \mathcal{E}^* \subset R_1$. Hence by Remark 3.5, we have

$$L(Z)\,G(Z) = I\,, \qquad \text{for all } Z \in R_1 \tag{4.2}$$

and a fortiori for every $Z \in \beta(\mathcal{E}) \cap \partial_S \mathcal{E}$, i.e.

$$F(\mathcal{E} \cap \partial_S H(1, I)) \subset \partial_S H(1, I)\,. \tag{4.3}$$

For every $t \in (-3, -1)$, $tI \in R$; moreover, since $-I \in \partial_S \mathcal{E}^* \cap R_1$, $tI \in R_1$. Then $-3I \in \partial R_1$. From (C) it follows that $\lim \beta(Z_n) = -3I$; thus, for every n sufficiently large, $\beta(Z_n) \in R_1$, and (4.2) yields

$$\overline{G(\overline{\beta(Z_n)}^{-1})}\, G(\beta(Z_n)) = I \tag{4.4}$$

for every n sufficiently large. By (D), $\lim G(\beta(Z_n)) = \beta(W) = 2W - I$ and $\lim \overline{G(\overline{\beta(Z_n)}^{-1}} = \overline{G(-\frac{1}{3}I)}$. By (4.4),

$$\overline{G(-\tfrac{1}{3}I)}(2W - I) = I\,. \tag{4.5}$$

If $\det(I - W) = 0$, then, for a suitable $\xi \in \mathbf{C}^N \setminus \{0\}$,

$$\overline{G(-\tfrac{1}{3}I)}\,\xi = \xi \qquad \text{and} \qquad \|\overline{G(-\tfrac{1}{3}I)} + I\| \geqslant 2\,,$$

contradicting $G(-\frac{1}{3}I) \in \beta(\mathcal{E})$. Thus $\det(I - W) \neq 0$. We can apply Lemma 4.2 to I and W, and replace F by $\Psi \circ F$, so there is not restriction assuming $W = -I$. Therefore (4.5) becomes $\overline{G(-\frac{1}{3}I)} = -\frac{1}{3}I$, i.e.

$$F(\tfrac{1}{3}I) = \tfrac{1}{3}I\,. \tag{4.6}$$

Let $T = \Phi_{\frac{1}{3}I} \circ F \circ \Phi_{\frac{1}{3}I}^{-1} \colon \mathcal{E} \to \mathcal{E}$. It follows, from Theorem 1.7, (4.3) and (4.6) that $T(0) = 0$ and $T(\mathcal{E} \cap \partial_S H(2, I)) \subset \partial_S H(2, I)$. Since T is a contraction for d, then $d(0, -\frac{1}{3}I) \geqslant d(0, T(-\frac{1}{3}I))$.

By remark 4.3, $T(-\frac{1}{3}I) = -\frac{1}{3}I$. Thus F is a holomorphic map with $F(0) = 0$. Therefore we can apply the Schwarz lemma (see [4], Theorem III.2.3): by (4.6), $\|F(\frac{1}{3}I)\| = \|\frac{1}{3}I\|$ and I is a complex extremal point of $\mathcal{E}$, then $F(\mu\frac{1}{3}I) = \mu F(\frac{1}{3}I)$ for every $\mu \in \mathbf{C}$, $|\mu| < 3$. It follows that $F(\lambda I) = \lambda I$ for every $\lambda \in \mathbf{C}$, $|\lambda| < 1$. Since the sequence $((1-1/n)I)_{n\in\mathbf{N}}$ satisfies (2.1), then the theorem follows from Julia's lemma ■

§ 5. We come now to the proof of our main theorem (Theorem 5.3).

5.1. LEMMA. *Let $K\colon \mathcal{E} \to \mathcal{E}$ be a holomorphic endomorphism with $K(0) = 0$. Then the sequence $(K^n)_{n\in\mathbf{N}}$ of the iterates $K^n = K\circ\ldots\circ K$ of K contains a subsequence convergent on all compact subsets of $\mathcal{E}$ to a holomorphic endomorphism L of $\mathcal{E}$. Moreover*

$$d(L(A), L(B)) = d(L^2(A), L^2(B)) \tag{5.1}$$

for every $A, B \in \mathcal{E}$.

PROOF. For all i, j such that $1 \leqslant i \leqslant j \leqslant N$, the sequence $(K^n_{ij})_{n\in\mathbf{N}}$ is equibounded, because $\|K^n_{ij}(Z)\| \leqslant \|K^n(Z)\| \leqslant 1$. Thus there exists a subsequence $(K^{n_k})_{k\in\mathbf{N}}$ uniformely convergent on all compact subsets of $\mathcal{E}$ to a holomorphic map L. Moreover, by the Schwarz lemma,

$$\|Z\| \geqslant \lim_{k\to\infty} \|K^{n_k}(Z)\| = \|L(Z)\|, \text{ then } L\colon \mathcal{E} \to \mathcal{E}.$$

The holomorphic endomorphisms contract the distance d, hence

$$d(L(A), L(B)) \geqslant d(L^2(A), L^2(B))$$

and

$$\lim_{n\to\infty} d(K^n(A), K^n(B)) = \inf_{n\in\mathbf{N}} d(K^n(A), K^n(B)),$$

for every $A, B \in \mathcal{E}$. Therefore

$$d(L(A), L(B)) = \lim_{k\to\infty} d(K^{n_k}(A), K^{n_k}(B)) \leqslant$$

$$\leqslant \lim_{k\to\infty} \lim_{h\to\infty} d(K^{n_k+n_h}(A), K^{n_k+n_h}(B)) = d(L^2(A), L^2(B)) \quad ■$$

5.2. Lemma. *Let* $K\colon \mathcal{E} \to \mathcal{E}$ *be a holomorphic endomorphism. Suppose there is a domain* $\Lambda \subset \mathcal{E}$ *such that*

$$d(Z_1, Z_2) = d\big(K(Z_1), K(Z_2)\big)$$

for all $Z_1, Z_2 \in \Lambda$. *Then* $K \in \mathrm{Aut}(\mathcal{E})$.

Proof. Let $C \in \Lambda$. Replacing K by $\Phi_{K(C)} \circ K \circ \Phi_{-C}$ we may assume $C = 0$, and $K(0) = 0$.

Let $s \in (0, 1)$ be such that $B_s(0) \subset \Lambda$ therefore $d(0, Z) = d\big(0, K(Z)\big)$, i.e.—by Definition 1.3—$\|Z\| = \|K(Z)\|$, for every $Z \in B_s(0)$. Let $Z \in \mathcal{E}$, then $sZ \in B_s(0)$, $\|K(sZ)\| = \|sZ\|$ and by the Schwarz lemma $\|K(Z)\| = \|Z\|$. Moreover $K(Z) = dK(0)Z + \omega(Z)\|Z\|$, with $\lim\limits_{Z\to 0} \omega(Z) = 0$. Let $W \in \partial\mathcal{E}$, then, for $0 < \varrho < 1$, $\varrho = \|K(\varrho W)\| = \|dK(0)\,\varrho W + \omega(\varrho W)\,\varrho\|$; therefore $1 = \lim\limits_{\varrho \searrow 0} \|dK(0)\,W + \omega(\varrho W)\| = \|dK(0)\,W\|$. The lemma follows from Theorem III.2.4 in [4] ∎

5.3. Theorem. *Let* $F\colon \mathcal{E} \to \mathcal{E}$ *be a holomorphic endomorphism. Let* $V_1, V_2, W_1, W_2 \in \partial_S \mathcal{E}$, $k_1, k_2 \in \mathbb{R}^+_*$ *be such that the following conditions hold*:

(i) $\det(V_1 - W_1) \neq 0$;

(ii) *There exists a domain* $\Omega \subset \mathcal{E}$ *such that*

  (A) $\Omega \cap \partial_S H(k_1, V_1) \neq \emptyset$,

  (B) $F\big(\Omega \cap \partial H(k_1, V_1)\big) \subset \partial H(k_2, V_2)$;

(iii) *There is a sequence* $(Z_n)_{n\in\mathbb{N}}$ *in* $\mathcal{E}$ *such that* $\lim Z_n = W_1$ *and* $\lim F(Z_n) = W_2$.

Then $F \in \mathrm{Aut}(\mathcal{E})$.

Proof. Replacing F by $\Psi \circ F \circ \Phi$ by a suitable choice of $\Psi, \Phi \in \mathrm{Aut}(\mathcal{E})$ (see Lemma 4.2) we can assume $V_1 = V_2 = I$, $W_1 = -I$, $k_1 = k_2 = 1$.

As in the proof of Theorem 4.4, replace F by $G = \beta \circ F \circ \beta^{-1}$. Since $G(\beta(\Omega) \cap \partial\mathcal{E}) \subset \partial\mathcal{E}$, then we can appply Theorem 3.6. But $\lim G(\beta(Z_n)) = \beta(W_2)$ with $\|G(\beta(Z_n)) + I\| < 2$ and $\frac{1}{2}(\beta(W_2) + I) \in U(N; \mathbf{C})$, therefore there is no $\xi \in \mathbf{C}^N \setminus \{0\}$ such that $T \mapsto G(T)\xi$ is constant. Hence all hypothesies of Theorem 4.4 are satisfied.

Let $Z \in \Omega \cap \partial_S H(1, I)$ and let $\Psi_1, \Phi_1 \in \mathrm{Aut}(\mathcal{E})$ be such that $\Phi_1(0) = Z$, $\Phi_1(I) = I$, $\Psi_1(F(Z)) = 0$, $\Psi_1(I) = I$. Setting $K = \Psi_1 \circ F \circ \Phi_1$, it follows, from Theorem 1.7 and from (4.1), that

$$K(\mathcal{E} \cap \overline{H(k, I)}) \subset \overline{H(k, I)} \tag{5.3}$$

for all $k \in \mathbb{R}_*^+$. Moreover

$$K(0) = 0 \,. \tag{5.4}$$

Let $t > 0$ be such that $B_t(0) \subset \Phi_1^{-1}(0)$; then by (ii, B) and by the Schwarz lemma

$$K(B_t(0) \cap \partial H(1, I)) \subset B_t(0) \cap \partial H(1, I) \,. \tag{5.5}$$

In view of (5.4), Lemma 5.1 can be applied. Thus (by (5.5)) $\beta \circ L \circ \beta^{-1}$ satisfied the hypothesies of Theorem 3.6, and (by (5.3), (5.4))

$$L(\mathcal{E} \cap \overline{H(k, I)}) \subset \overline{H(k, I)} \,,$$

for every $k \in \mathbb{R}_*^+$, and $L(0) = 0$. Therefore there is no $\xi \in \mathbb{C}^N \setminus \{0\}$ such that $(\beta \circ L \circ \beta^{-1})(Z)\xi$ is a constant map. It follows, from Theorem 3.6, that there is $\tilde{Z} \in \mathcal{E}$ such that $dL(\tilde{Z})$ is invertible. Hence, for a suitable open neighbourhood Ω_1 of $\tilde{Z}$, $\Lambda = L(\Omega_1)$ is an open neighbourhood of $\tilde{W} = L(\tilde{Z})$ It follows, from (5.1),

$$d(W, \tilde{W}) = d(L(W), L(\tilde{W}))$$

for all $W \in \Lambda$. Since

$$d(W, \tilde{W}) \geqslant d(K(W), K(\tilde{W})) \geqslant d(L(W), L(\tilde{W})) \,,$$

then $d(W, \tilde{W}) = d(K(W), K(\tilde{W}))$. Therefore, by Lemma 5.2,

$$\Psi_1 \circ F \circ \Phi_1 = K \in \mathrm{Aut}(\mathcal{E}) \qquad \blacksquare$$

5.4. REMARK. Hypothesis (iii) in Theorem 5.3 can not be dropped (see Remark 1 in [2]).

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