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On a synonymy relation for extensional 1st order theories. Part II. A sufficient criterion for non synonymy. Applications

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On a Synonymy Relation for Extensional 1st Order Theories.

PART II
A Sufficient Criterion for Non Synonymy. Applications.

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7. Admissible generalized interpretations for the extension $\overline{\mathscr{T}}$ of $\mathscr{T}$ having primitive implication and equivalence (1).

Let $\overline{\mathscr{L}}$ be the language obtained from $\mathscr{L}$ by adding the new logical symbols $\supset_p$ and $\equiv_p$, to be called primitive implication and equivalence (signs) respectively. Obviously $\overline{\mathscr{L}}$'s formation rules are those of $\mathscr{L}$—see [1], § 2—and the following

(i) *if* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *are wffs of* $\overline{\mathscr{L}}$, *then* $\mathscr{A}_1 \equiv_p \mathscr{A}_2$ *and* $\mathscr{A}_1 \supset_p \mathscr{A}_2$ *also are.*

In connection with the above theory $\mathscr{T}$—see [1], § 6—we denote $D'_v \equiv_p D''_v$ by D^p_v and the wff (or wff-scheme) obtained from A3.r by replacing the occurrences of $\supset$ [$\equiv$] with $\supset_p$ [$\equiv_p$], by $A^p3.r$ ($r = 7, 8$).

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(1) The present paper is the second part of a work whose first part is [1]. Therefore the numbering of its sections follows the one for [1].

Furthermore we consider the (barred) extension

$$(7.1)\qquad \overline{\mathscr{T}} = (S \cup \{\equiv_p, \supset_p\}, \text{ wfe}_{\overline{\mathscr{T}}}, LA, \overline{PA}, R, \{D_\nu^p\}_{0<\nu<\omega})$$

of $\mathscr{T}$, where $\text{wfe}_{\overline{\mathscr{T}}}$ is the class of wfes of $\overline{\mathscr{L}}$ in which only symbols in $S \cup \{\equiv_p, \supset_p\}$ occur and

$$(7.2)\qquad \overline{PA} = PA \cup \{A^p 3.7, A^p 3.8\} - \{A3.7, A3.8\}.$$

Let $\mathscr{I} = (\mathfrak{D}, \mathscr{I}, \alpha)$ be an interpretation of $\mathscr{T}$. Then the v-valuations (of $\mathscr{T}$) on $\mathfrak{D}$, or $\mathscr{I}$-valuations, are called the *v-valuations* of $\overline{\mathscr{T}}$ on $\mathfrak{D}$.

DEFINITION 7.1. *We say that* $\overline{\mathscr{I}} = (\mathfrak{D}, \overline{\mathscr{I}}, \alpha)$ *is a generalized* (2) *interpretation of* $\overline{\mathscr{T}}$ *if* $\mathfrak{D}$ *is the non-empty set and* $\overline{\mathscr{I}}$ *is a function defined on the constants of* $\overline{\mathscr{T}}$ *(which are those of* $\mathscr{T}$*) and on* $\sim$, $\supset$, *such that, first, the restriction of* $\overline{\mathscr{I}}$ *on the constants of* $\overline{\mathscr{T}}$ *is a c-valuation of* $\mathscr{T}$ *(on* $\mathfrak{D}$*), and second,*

$$(7.3)\qquad \sim^* = \overline{\mathscr{I}}(\sim) \in \{0, 1\}^{\{0,1\}}, \qquad \supset^* = \overline{\mathscr{I}}(\supset) \in \{0, 1\}^{\{0,1\}^2}.$$

Let us now fix a generalized interpretation $\overline{\mathscr{I}} = (\mathfrak{D}, \overline{\mathscr{I}}, \alpha)$ of $\overline{\mathscr{T}}$ and a v-valuation V on $\mathfrak{D}$ — to be called $\mathscr{I}$-valuation. Then the (*generalized*) *designatum* $\Delta^* = \text{des}_{\overline{\mathscr{I}},V}(\Delta)$ of the wfe Δ (of $\overline{\mathscr{T}}$) at $\overline{\mathscr{I}}$ and V, and the *function* $\Psi_{\mathscr{A}; y_1, \ldots, y_n; \overline{\mathscr{I}}, V}$ (where $\overline{\mathscr{I}}$ and perhaps also V can be dropped) *associated* with the wff $\mathscr{A}$ and the n variables y_1 to y_n (with respect to $\overline{\mathscr{I}}$ and V) are defined recursively and simultaneously by clauses (1) to (9) below, where n and i run over Z^+ and $\mathscr{A}$ and $\mathscr{B}$ are arbitrary wffs of $\overline{\mathscr{T}}$.

(1) *If* Δ *is* x_i $[c_i]$, *then* Δ^* *is* V_i $[\mathscr{I}(c_i)]$.

(2) [(3)] *If* τ_1 *to* τ_n *are terms and* Δ *is* $f_i^n(\tau_1, \ldots, \tau_n)$ $[R_i^n(\tau_1, \ldots, \tau_n)]$, *then* Δ^* *is* $f_i^{n*}(\tau_1^*, \ldots, \tau_n^*)$ [0 *or* 1 *according to whether or not* $(\tau_1^*, \ldots, \tau_n^*) \in R_i^{n*}$], *where* $f_i^{n*} = \overline{\mathscr{I}}(f_i^n)$ *and* $R_i^{n*} = \overline{\mathscr{I}}(R_i^n)$.

(4) *If* Δ *is* $\mathscr{A} \equiv_p \mathscr{B}$ *and* $\mathscr{A}^* = [\neq]\ \mathscr{B}^*$, *then* $\Delta^* = 0$ [1].

(2) Ordinary interpretations are special generalized interpretations.

(5) *If* Δ *is* $\mathscr{A} \supset_p \mathscr{B}$ *and* $\mathscr{A}^* = 0$ *or* $\mathscr{B}^* = 1$, *then* $\Delta^* = 1$; *otherwise* $\Delta^* = 0$.

(6) $\Psi_{\mathscr{A};y_1,\ldots,y_n;\overline{\mathscr{I}},V}$ *is the function* $g \in \{0,1\}^{\mathfrak{D}^n}$ *such that*

$$g(\xi_1, \ldots, \xi_n) = \mathrm{des}_{\overline{\mathscr{I}},W}(\mathscr{A}) \quad \text{where } W = \begin{pmatrix} y_1 \ldots y_n \\ \xi_1 \ldots \xi_n \end{pmatrix} V \tag{7.4}$$

$$(\forall \xi_1, \ldots, \xi_n \in \mathscr{D}) .$$

(7) [(8)] *If* Δ *is* $\sim \mathscr{A}$ $[\mathscr{A} \supset \mathscr{B}]$, *then* Δ^* *is* $\sim^* (\mathscr{A}^*)$ $[\supset^* (\mathscr{A}^*, \mathscr{B}^*)]$—see (7.3).

(9) *If* Δ *is* $(x_i)\mathscr{A}$, *then* Δ^* *is* 0 *if* $\Psi_{\mathscr{A};x_i;\overline{\mathscr{I}},V}(\xi) = 0 \ \forall \xi \in \mathfrak{D}$, *and* 1 *otherwise.*

DEFINITION 7.2. We shall say that the generalized interpretation $\overline{\mathscr{I}} = (\mathfrak{D}, \overline{\mathscr{I}}, \alpha)$ of $\overline{\mathscr{T}}$ is admissible if $\overline{\mathscr{I}}$ satisfies D_ν^p $(\nu = 1, 2, \ldots)$, A^p 3.7-8, and A3.6.

8. A criterium for non-synonymy. An application of it to logic.

THEOREM. 8.1. *If* Δ_1 *and* Δ_2 *are wfes of* $\mathscr{T}$ *and* $\Delta_1 \asymp \Delta_2$, *then*

$$\mathrm{des}_{\overline{\mathscr{I}},V}\, \Delta_1 = \mathrm{des}_{\overline{\mathscr{I}},V}\, \Delta_2 \tag{8.1}$$

for every admissible generalized interpretation $\overline{\mathscr{I}}$ *of* $\overline{\mathscr{T}}$ *and all* $\mathscr{I}$*-valuations* V.

Note that admissible interpretations, unlike models (of $\overline{\mathscr{T}}$) are considered in the theorem above.

PROOF OF THEOR. 8.1. Let $\mathscr{S}$ be the equivalence relation among wfes of $\mathscr{T}$ such that $\Delta_1 \mathscr{S} \Delta_2$ *iff* $\mathrm{des}_{\overline{\mathscr{I}},V}\, \Delta_1 = \mathrm{des}_{\overline{\mathscr{I}},V}\, \Delta_2$ *for every admissible generalized interpretation* $\overline{\mathscr{I}}$ *of (the barred extension)* $\overline{\mathscr{T}}$ *(of* $\mathscr{T}$*) and every* $\mathscr{I}$*-valuation* V.

We now show that $\mathscr{S}$ fulfils conditions C_1) to C_7), which define $\asymp$ in [1]. To this end we consider an arbitrary choice of $\mathscr{I}$ and V above.

1) Since $\overline{\mathscr{I}}$ is admissible $\mathrm{des}_{\overline{\mathscr{I}},V}\,(D'_\nu \equiv_p D''_\nu) = \mathrm{des}_{\overline{\mathscr{I}},V}\, D_\nu^p = 0$ $(\nu = 1, 2, \ldots)$. Hence, by clause (4) in § 7, $\mathrm{des}_{\overline{\mathscr{I}},V}\, D'_\nu = \mathrm{des}_{\overline{\mathscr{I}},V}\, D''_\nu$. Then, (by the above arbitrariness of $\overline{\mathscr{I}}$ and V) $D'_\nu \mathscr{S} D''_\nu$.

2) Assume that $f \mathscr{S} f'$ and $\Delta_i \mathscr{S} \Delta_i'$, where f and f' are some f_i^n's while Δ_i and Δ_i' are terms; hence $f^* = f'^*$ and $\Delta_i^* = \Delta_i'^*$ $(i = 1, 2, \ldots, n)$. Then, by clause (2) in § 7, $\text{des}_{\bar{\mathscr{I}},V}(f(\Delta_1, \ldots, \Delta_n)) = \text{des}_{\bar{\mathscr{I}},V}(f'(\Delta_1', \ldots, \Delta_n'))$. Then $f(\Delta_1, \ldots, \Delta_n) \mathscr{S} f'(\Delta_1', \ldots, \Delta_n')$.

3) Assume $R \mathscr{S} R'$ and $\Delta_i \mathscr{S} \Delta_i'$, where R and R' are some R_i^n's while Δ_i and Δ_i' are terms, so that $R^* = R'^*$ and $\Delta_i^* = \Delta_i'^*$ $(i = 1, \ldots, n)$. Then, by clause (3) in § 7, $\text{des}_{\bar{\mathscr{I}},V}(R(\Delta_1, \ldots, \Delta_n)) = \text{des}_{\bar{\mathscr{I}},V}(R'(\Delta_1', \ldots, \Delta_n'))$. Hence $R(\Delta_1, \ldots, \Delta_n) \mathscr{S} R'(\Delta_1', \ldots, \Delta_n')$.

4)—6) Assume $p \mathscr{S} p'$ and $q \mathscr{S} q'$, where p, p', q, and q' are wffs. Then $p^* = p'^*$ and $q^* = q'^*$ for every $\mathscr{I}$-valuation V. Hence, by clauses (7) and (8) in § 7, $\text{des}_{\bar{\mathscr{I}},V}(\sim p) = \sim^*(p^*) = \sim^*(p'^*) = \text{des}_{\bar{\mathscr{I}},V}(\sim p')$ and (similarly) $\text{des}_{\bar{\mathscr{I}},V}(p \supset q) = \text{des}_{\bar{\mathscr{I}},V}(p' \supset q')$. Lastly, by the above arbitrariness of V,

$$\Psi_{p;x_i;\bar{\mathscr{I}},V}(\xi) = \Psi_{p';x_i;\bar{\mathscr{I}},V}(\xi) \qquad (\forall \xi \in \mathfrak{D}) .$$

Then, by clause (9) in § 7, $\text{des}_{\bar{\mathscr{I}},V}((x_i)p) = \text{des}_{\bar{\mathscr{I}},V}((x_i)p')$.

7) Let $\mathscr{A}(x_i)$ and $\mathscr{A}(x_j)$ be (x_i, x_j)-similar wffs. Then, by induction one can prove (the same way as in connection with ordinary interpretations) that

$$\text{des}_{\bar{\mathscr{I}},V}((x_i)\mathscr{A}(x_i)) = \text{des}_{\bar{\mathscr{I}},V}((x_j)\mathscr{A}(x_j)) .$$

Hence

$$(x_i)\mathscr{A}(x_i) \mathscr{S} (x_j)\mathscr{A}(x_j) .$$

We have shown that $\mathscr{S}$ is a relation that fulfils conditions C_1) to C_7). Since $\asymp$ is the smallest among these relations, $\asymp \subseteq \mathscr{S}$ q.e.d.

Note that Theor. 8.1 affords a criterium to recognize when two wfes Δ and Δ' of $\mathscr{T}$ are not synonymous: *it suffices to find an admissible generalized interpretation* $\bar{\mathscr{I}}$ *of* $\bar{\mathscr{T}}$ *and an* $\mathscr{I}$*-valuation* V *for which* $\text{des}_{\bar{\mathscr{I}},V}(\Delta) \neq \text{des}_{\bar{\mathscr{I}},V}(\Delta')$.

As an example of application of the criterium above we show $(8.2)_1$ below

$$(8.2) \qquad p \not\asymp \sim\sim p , \quad \sim p \mathscr{S} \sim\sim\sim p , \quad \sim p \not\asymp \sim\sim\sim p ,$$

where p is any atomic wff (3).

(3) Relation $\mathscr{S}$ is defined at the outset of the proof of Theor. 8.1.

To this end, we assume that the atomic wff p has the truth value 0 ($p^* = 0$) in the admissible interpretation $\overline{\mathscr{I}}$. In fact it is independent of the choice of $\sim^*$. By stipulating that $\sim^*(0) = 1$ and $\sim^*(1) = 1$, we have $p^* = 0$ and $(\sim\sim p)^* = 1$. Hence the relation $\mathscr{S}$ fails to hold between p and $\sim\sim p$. If $p^* = 1$ the same conclusion is reached by interchanging the roles of 0 and 1 in the reasoning above. Thus $(8.2)_1$ holds.

REMARK. Relations $(8.2)_{2,3}$ imply that Theor. 8.1 affords a condition that is sufficient for two wfes of $\mathscr{T}$ to be non-synonymous, but is not necessary for this.

We now prove $(8.2)_2$; $(8.2)_3$ will be proved in [2], § 14. Assume that $\overline{\mathscr{I}}$ is any generalized interpretation of $\overline{\mathscr{T}}$, V is any $\mathscr{I}$-valuation, and $f = \sim^* = \overline{\mathscr{I}}(\sim)$. Then $f \in A = \{0, 1\}^{\{0,1\}} = \{I, \underline{0}, \underline{1}, \underline{1} - I\}$ where $I(x) = x$, $\underline{0}(x) = 0$, $\underline{1}(x) = 1$ $(x = 0, 1)$. Then $f \circ f \circ f = f$ $(\forall f \in A)$. Hence, by clause (7) in § 7, $\text{des}_{\overline{\mathscr{I}},V}(\sim p) = \text{des}_{\overline{\mathscr{I}},V}(\sim\sim\sim p)$ q.e.d.

9. Application of the preceding criterion of non-synonymy to an example with arithmetics.

The 1st order theory S introduced in [3], Chap. 3, to treat natural numbers (using our notations) has, besides the logical symbols and the equality attribute R_1^2, the individual constant c_2 and the function letters f_1^1, f_1^2, and f_2^2 to denote zero, successor, sum, and product respectively. In order to construct a variant, Σ, of S, fit for our purposes we add S with the attribute R_1^1 and the function letters f_3^2 and f_4^2, to express natural numbers, exponentiation and logarithm respectively. Furthermore, partly in harmony with [3], we write: 0 for c_2, $t = s$ for $R_1^2(t, s)$, $t \in \mathscr{N}$—to be read as «t is a natural number»—for $R_1^1(t)$, t' for $f_1^1(t)$, $t + s$, $t \cdot s$, t^s, and $\lg_t s$ for $f_r^2(t, s)$ with $r = 1$ to 4 respectively, and x to z for x_1 to x_3 respectively.

We define the *numeral* $\overline{n}$ recursively: $\overline{0} = c_2$, $\overline{n + 1} = \overline{n}'$.

The non-logical axioms of Σ are axioms Σ_1 to Σ_{13} below. Among them $\Sigma_{1\text{-}2}$—i.e. Σ_1 to Σ_2—concern identity, $\Sigma_{3\text{-}6}$ and Σ_{13} are Peano's axioms (in a weak version), and $\Sigma_{7\text{-}8}$, $\Sigma_{9\text{-}10}$, and $\Sigma_{11\text{-}12}$ afford the inductive definitions of sum, product and exponentiation respectively, where e.g. $(\forall x, y \in \mathscr{N})p$ means $(x)(y)$ $(x \in \mathscr{N} \wedge y \in \mathscr{N} \supset p)$.

Σ_1 $(\forall x, y, z \in \mathcal{N}) \wedge x = y \supset (x = z \supset y = z)$

Σ_2 $(\forall x, y \in \mathcal{N}) \supset x = y \supset x' = y'$

$\Sigma_{3\text{-}4}$ $0 \in \mathcal{N}, \quad x \in \mathcal{N} \supset x' \in \mathcal{N}$

$\Sigma_{5\text{-}6}$ $x \in \mathcal{N} \supset 0 \neq x', \quad (\forall x, y \in \mathcal{N}) \supset x' = y' \supset x = y$

$\Sigma_{7\text{-}8}$ $x \in \mathcal{N} \supset x + 0 = x, \quad (\forall x, y \in \mathcal{N}) \supset x + y' = (x + y)'$

$\Sigma_{9,10}$ $x \in \mathcal{N} \supset x \cdot 0 = 0, \quad (\forall x, y \in \mathcal{N}) \supset x \cdot y' = x \cdot y + x$

$\Sigma_{11\text{-}12}$ $x \in \mathcal{N} \supset x^0 = \bar{1}, \quad (\forall x, y \in \mathcal{N}) \supset x^{y'} = x^y \cdot x$

Σ_{13} $\mathcal{A}(0) \wedge (\forall x \in \mathcal{N})[\mathcal{A}(x) \supset \mathcal{A}(x')] \supset (\forall x \in \mathcal{N}) \mathcal{A}(x)$ *for every fbf* $\mathcal{A}(x)$ *of* Σ.

The definition system $\{D_\alpha\}_{0<\alpha<\omega}$ of Σ contains only the following (non-recursive) definition

$$y = \lg_x z \equiv_D x^y = z \wedge (E_1 y) x^y = z \vee y = c_1 \wedge \sim (E_1 y) x^y = z . \tag{9.1}$$

Incidentally, for any wff $\mathcal{A}$ of the above-mentioned theory S —see [3]—, let $\mathcal{A}^{\mathcal{N}}$ be the wff of Σ obtained from the universal closure of $\mathcal{A}$ by replacing every quantifier (x_i) with its restriction to $\mathcal{N}$, i.e. $(\forall x_i \in \mathcal{N})$.

The axioms (S_1) to (S_9) of S are practically included in axioms Σ_1 to Σ_{13}, in that so are $(S_1)^{\mathcal{N}}$ to $(S_9)^{\mathcal{N}}$; and the axioms of Σ that have no counterparts in S are only $\Sigma_{3\text{-}4}$ and $\Sigma_{11\text{-}12}$. Let us incidentally add that

$$\vdash_S \mathcal{A} \Leftrightarrow \vdash_\Sigma \mathcal{A}^{\mathcal{N}} \quad \textit{for every wff } \mathcal{A} \textit{ of } S . \tag{9.2}$$

We now consider the barred extension $\bar{\Sigma}$ of Σ—see (7.1)—and the following ordinary interpretation $\mathcal{J} = (\mathfrak{D}, \mathcal{J}, \alpha)$ of it, which describes the case when only 4 natural numbers exist and hence only 3 proper (or existing) individuals exist.

It is assumed that, for some $\alpha \notin \mathcal{N}$, $\mathfrak{D} = \{0, 1, 2, 3, \alpha\}$, and that $\mathcal{J}$ is the c-valuation that fulfils conditions $\mathscr{C}_{1\text{-}9}$ below where $\Delta^* = \mathcal{J}(\Delta)$ for every wfe Δ of Σ, and where ξ and η are arbitrary elements of $\mathfrak{D}$.

$\mathscr{C}_{1\text{-}4}$ $c_1^* = \alpha$, $(c_2^* =) \, 0^* = 0$, $R_1^{1*} = \mathcal{N}^* = \{0, 1, 2, 3\}$, and $R_1^{2*} = (=^*) =$ identity in $\mathfrak{D}$.

$\mathscr{C}_5$ $f_1^1(\xi)$ is $\xi + 1$ if $\xi \in \{0, 1, 2\}$, and α otherwise.

$\mathscr{C}_{6\text{-}9}$ If both $\xi, \eta \in \{0, 1, 2, 3\}$ and some (unique) number n in $\{0, 1, 2, 3\}$ fulfils the r-th of the equalities $n = \xi + \eta$, $n = \xi \cdot \eta$, $n = \xi^\eta$, and $\xi^n = \eta$, then $f_r^2(\xi, \eta) = n$; otherwise $f_r^2(\xi, \eta) = \alpha$ $(r = 1, \ldots, 4)$.

The interpretation $\mathscr{J}$ *is admissible* in that is satisfies definition (9.1). Incidentally condition $\mathscr{C}_9$ is a consequence of ($\mathscr{C}_{1\text{-}8}$ and) the requirement that (9.1) should be true in $\mathscr{J}$; furthermore by $\mathscr{C}_{5\text{-}9}$ f_1^1 and f_1^2 to f_4^2 express proper functions.

Let us add that $\mathscr{J}$ is not a model of Σ only in that it fails to satisfy axioms Σ_4 and Σ_6, which are essential to assert that natural numbers are infinitely many. In particular $\mathscr{J}$ satisfies $\Sigma_{7\text{-}12}$, which can be regarded as inductive definitions. Thus $\mathscr{J}$ can be considered as admissible in the strong sense, so that the application below of our criterion of non-synonymy can be accepted also when inductive definitions are required to have the same role—in connection with synonymy—as the other definitions.

We are now ready to show that in Σ

$$\lg_{\bar{2}} \bar{8} \not\asymp \bar{3}\,, \qquad y = \lg_{\bar{2}} \bar{8} \not\asymp y = \bar{3}\,. \tag{9.3}$$

Indeed, referring to $\mathscr{J}$, $\bar{2}^* = 2$, $\bar{3}^* = 3$, $\bar{8}^* = \alpha$; hence, by $\mathscr{C}_9$), $\lg_{\bar{2}} \bar{8} = \alpha \neq 3 = \bar{3}^*$. Thus, by Theor. 8.1, $(9.3)_1$ holds. At this point it is clear that for $V(y) = 3$

$$\operatorname{des}_{\mathscr{J},V}(y = \bar{3}) = 0 \neq 1 = \operatorname{des}_{\mathscr{J},V}(y = \lg_{\bar{2}} \bar{8})\,.$$

Hence $(9.3)_2$ holds.

REFERENCES

[1] C. Bonotto - A. Bressan, *On a synonymy relation for extensional* 1^{st} *order theories. Part I: A notion of synonymy*, Rend. Sem. Mat. Univ. Padova, **69** (1982).

[2] C. Bonotto - A. Bressan, *On a synonymy relation for extensional* 1^{st} *order theories. Part III: A necessary and sufficient condition for synonymy*, to be printed on Rend. Sem. Mat. Univ. Padova.

[3] E. Mendelson, *Introduction to mathematical logic*, Van Nostrand-Reinhold Co., New York, 1964.

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