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Seq-Consistency Property and Interpolation Theorems.

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SUMMARY - In this paper the new notion of seq-consistency property is introduced, which, besides allowing for large supplies of witnessing constants, permits to prove the model existence theorem, its inverse, Cunningham's improvement of Karp's interpolation theorem, and opens the way to the improvement of the Maehara Takeuti type interpolation theorem for $L^{2+}_{k,k}$.

1. Introduction.

In [5] it was pointed out that the notion of k consistency property of Karp [7] was not adequate to improve Karp's interpolation theorem.

E. Cunningham in [2] already remarked this point and proposed a new notion, called chain consistency property, that allowed her to improve Karp's interpolation theorem. But the notion of chain consistency property had the drawback, remarked by Cunningham herself, that there were no sufficient supply of witnessing constants avaiable.

To overcome this problem and, at the same time, to go in the direction of a consistency property more closely related to the notion of ω-satisfiability, as Cunningham did, in [6] it was introduced the notion of ω-consistency property. But even the notion of ω-consistency property is not the final point since we have no direct proof of the improvement of Karp's interpolation theorem using it.

In this paper we introduce a further notion of consistency property that we call seq-consistency property since the elements of it are sequences of sets of sentences.

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Seq-consistency properties will allow for sufficient supplies of witnessing constants, still remaining very close to the notion of ω-satisfiability as to permit to obtain the model existence theorem, its inverse and Cunningham's improvement of Karp's interpolation theorem for $L_{k,k}$.

Seq-consistency properties can be naturally extended to $L^{2-}_{k,k}$ languages and this will eventually yield the improvement of the Maehara Takeuti type interpolation theorem for $L^{2+}_{k,k}$ as to read ω-validity also in the definition of an interpolant [1].

2. ω-satisfiability of good ω-sequences of sets of sentences.

For the notation that we are going to use we refer to [3] from which we depart in that there are no second order variables in what we are doing now.

For the notions of ω-chain of models and of ω-satisfiability we refer to [7] and [3].

Stmt(L) will denote the set of sentences in the language L.

We assume, without loss of generality, that our sentences, sets of sentences and sequences of sets of sentences are such that no variable occurs in more than one set of variables immediatly after a quantifier, and that $L_{k,k}$ has neither individual constants nor functions.

Since k is a strong limit cardinal of denumerable cofinality, we may assume that $k = \bigcup \{k_n: n \in \omega\}$, where $2^{k_n} \leqslant k_{n+1}$.

These assumptions will hold throughout this paper.

We recall from [3] that we can also speak of ω-satisfiability of formulas in a language with individual constants provided that all the individual constants are interpreted within a fixed structure of the ω-chain of models that is supposed to ω-satisfy the formula.

Let C_n, $n \in \omega$, be sets of individual constants (not in $L_{k,k}$) such that $|C_n| = k_n$ and for all m, n in ω if $m \neq n$ then $C_m \cap C_n = \emptyset$.

For all $n \in \omega$ let L_n be the language obtained from $L_{k,k}$ by adding $\bigcup \{C_i: i \in n\}$ as individual constants.

DEFINITION. An ω-sequence $S = \langle s_n: n \in \omega\rangle$ of sets of sentences is called a *good ω-sequence* of sets of sentences if

a) $\left|\bigcup \{s_n: n \in \omega\}\right| \leqslant k$ and *b*) for all $n > 0$ all the sentences in s_n are of the form $-F(\bar{v}_F/f)$ where f is a 1-1 function, $f\colon \bar{v}_F \to C_n$, and the sentence $-\forall \bar{v}_F F \in \bigcup \{s_j\colon j < n\}$ and

c) there is a natural number n' such that for all $n > n'$ we have that $|s_n| \leqslant |s_{n'}| \leqslant k_{n'}$ and $s_0 \subset \mathrm{Stmt}(L_{n'})$, and

d) for all $n > 0$ $s_n \subset \mathrm{Stmt}(L_n)$.

REMARK. If all the sentences occurring in a good ω-sequence $S = \langle s_n\colon n \in \omega\rangle$ are in $\mathrm{Stmt}(L_0)$ then $s_n = \emptyset$ for $n > 0$.

DEFINITION. We say that an ω-chain of models $\bar{M}$ ω-satifies a good ω-sequence $S = \langle s_n\colon n \in \omega\rangle$ of sets of sentences, $\bar{M} \models^{\omega} S$, if for all $p \in \omega$ $\bar{M} \models^{\omega} \bigcup \{s_n\colon n \leqslant p\}$ (of course this means that $\bigcup \{C_j\colon j \leqslant p\}$ is interpreted within a fixed structure of $\bar{M}$).

A good ω-sequence S is said to be ω-satisfiable if there is an ω-chain of models $\bar{M}$ such that $\bar{M} \models^{\omega} S$.

If $S = \langle s_n\colon n \in \omega\rangle$ is a good ω-sequence, let $\bar{S} = \bigcup \{s_n\colon n \in \omega\}$ and $\bar{S}^p = \bigcup \{s_n\colon n \leqslant p\}$.

3. Seq-consistency property.

Let us now define the notion of seq-consistency property for $L_{k,k}$.

Σ is a seq-consistency property for $L_{k,k}$ with respect to $\{C_n\colon n \in \omega\}$ if Σ is a set of good ω-sequences $S = \langle s_n\colon n \in \omega\rangle$ of sets s_n of sentences such that all of the following conditions hold.

C0) If Z is an atomic sentence then either $Z \notin \bar{S}$ or $-Z \notin \bar{S}$ and if Z is of the form $-(t = t)$, t a constant, then $Z \notin \bar{S}$.

C1) Suppose $|I| < k$,

a) if $\{c_i = d_i\colon i \in I\} \subset s_0$ and $\langle s_n\colon n \in \omega\rangle = S \in \Sigma$ with c_i and d_i constants, then the good ω-sequence $S' = \langle s'_n\colon n \in \omega\rangle$ such that $s'_0 = s_0 \bigcup \{d_i = c_i\colon i \in I\}$ and $s'_n = s_n$ for $n > 0$, $n \in \omega$, belongs to Σ.

b) If $\{Z_i(c_i),\ c_i = d_i\colon i \in I\} \subset \bigcup \{s_n\colon n \leqslant m\}$ for some $m \in \omega$ and $\langle s_n\colon n \in \omega\rangle = S \in \Sigma$ and Z_i are atomic or negated atomic sentences and c_i and d_i are constants, then the good ω-sequence $S' = \langle s'_n\colon n \in \omega\rangle$ such that $s'_0 = s_0 \bigcup \{Z_i(d_i)\colon i \in I\}$ and $s'_n = s_n$ for $n > 0$, $n \in \omega$, belongs to Σ.

C2) If $\{--F_i: i \in I\} \subset \bar{S}^m$ for some $m \in \omega$ and $\langle s_n: n \in \omega\rangle = S \in \Sigma$ and $|I| < k$, then the good ω-sequence $S' = \langle s'_n: n \in \omega\rangle$ such that $s'_0 = s_0 \bigcup \{F_i: i \in I\}$, $s'_n = s_n$ for $n > 0$, $n \in \omega$, belongs to Σ.

C3) If $\{\& \bar{F}_i: i \in I\} \subset s_0$ and $|I| < k$ and there is $m' \in \omega$ such that for all $i \in I$, $0 < |\bar{F}_i| < k_{m'}$ and $\langle s_n: n \in \omega\rangle = S \in \Sigma$ then the good ω-sequence $S' = \langle s'_n: n \in \omega\rangle$ such that $s'_0 = s_0 \bigcup \{F: F \in \bar{F}_i, i \in I\}$, $s'_n = s_n$ for $n > 0$, $n \in \omega$, belongs to Σ.

C4) If $\{\forall \bar{v}_i F_i: i \in I\} \subset s_0$ and $|I| < k$ and there is $m' \in \omega$ such that for all $i \in I$, $0 < |\bar{v}_i| < k_{m'}$ and $\langle s_n: n \in \omega\rangle = S \in \Sigma$, and n' is the natural number mentioned in c) of the definition of good ω-sequence relative to S, then the good ω-sequence $S' = \langle s'_n: n \in \omega\rangle$ such that

$$s'_0 = s_0 \bigcup \{F_i(\bar{v}_i/f): f \in {}^{\bigcup\{\bar{v}_i: i \in I\}} \bigcup \{C_h: h \leqslant n' + 1\},\ i \in I\},\ s'_n = s_n$$

for $n > 0$, $n \in \omega$, belongs to Σ.

C5) If $\{-\& \bar{F}_i: i \in I\} \subset \bar{S}^m$ for some $m \in \omega$ and $|I| < k$ and there is m' in ω such that for all $i \in I$, $0 < |\bar{F}_i| < k_{m'}$ and $\langle s_n: n \in \omega\rangle = S \in \Sigma$, then there is $g \in \bigtimes \{\bar{F}_i: i \in I\}$ such that the good ω-sequence $S' = \langle s'_n: n \in \omega\rangle$ such that $s'_0 = s_0 \bigcup \{-g(i): i \in I\}$, $s'_n = s_n$ for $n > 0$, $n \in \omega$, belongs to Σ.

C6) If $\{- \forall \bar{v}_i F_i: i \in I\} \subset \bar{S}^m$ for some $m \in \omega$ and there is m' the least natural number such that $|I| < k_{m'}$ and for all $i \in I$, $0 < |\bar{v}_i| < k_{m'}$ and $m \leqslant m'$ and $\bar{S}^m \subset \mathrm{Stmt}(L_{m'})$, and $\langle s_n: n \in \omega\rangle = S \in \Sigma$, then there is an ω-partition $P = \langle I_p: p \in \omega\rangle$ of I such that for any set $\{f_p: p \in \omega\}$ of 1-1 functions $f_p \in {}^{\bigcup\{\bar{v}_i: i \in I_p\}}\big(C_{m'+p} - \{c: c$ is a constant occurring in $\bigcup\{s_n: n \in \omega\}\}\big)$ the good ω-sequence $S' = \langle s'_n: n \in \omega\rangle$ such that for $n \leqslant m'$, $s'_n = s_n$ and for all $p \in \omega$ $s'_{m'+p+1} = s_{m'+p+1} \bigcup \{- F_i(\bar{v}_i/f_p): i \in I_p\}$ belongs to Σ.

4. Model existence theorem.

Model existence theorem. If $S = \langle s_n: n \in \omega\rangle$ is a good ω-sequence of sets of sentences of $L_{k,k}$ and $S \in \Sigma$ a seq-consistency property with respect to $\{C_i: i \in \omega\}$, and $|\bar{S}| = k_0 < k$, then S is ω-satisfiable in an ω-chain of models $\bar{M}$. Moreover the n-th structure of $\bar{M}$ has cardinality at most k_n.

The following proof is somehow similar to the proof of the analogous theorem for ω-consistency properties given in [6] and again it is an Hintikka type argument.

PROOF. By a good split of a set s of at most k sentences we shall mean a partition $\langle s_m: m \in \omega \rangle$ of s such that $|s_m| \leqslant k_m$, every sentence of the form either $\& \overline{F}$ or $-\& \overline{F}$ in s_m has $|\overline{F}| \leqslant k_m$, every sentence of the form either $\forall \overline{v}_F F$ or $-\forall \overline{v}_F F$ in s_m has $|\overline{v}_F| \leqslant k_m$.

Let us define, by induction on n, good ω-sequences $S_n = \langle s_{n,m}: m \in \omega \rangle \in \Sigma$ such that $s_{n+1,m} \supset s_{n,m}$ and all the sentences occurring in $\overline{S}^n_n$ are in L_n, $|\overline{S}^n_n| \leqslant k_n$; good splits $\langle \overline{S}^n_{n,m'}: m' \in \omega \rangle$ of each $\overline{S}^n_n$ such that $\overline{S}^{n+1}_{n+1,m'} = \overline{S}^n_{n,m'}$ for $m' \leqslant n$ and $\overline{S}^{n+1}_{n+1,m'} \supset \overline{S}^n_{n,m'}$ for $m' > n$; and for all $p \in \omega$ sets S^p_n of existential sentences in $\overline{S}^{n-1}_{n-1}$ and 1-1 functions $f_{n,p}$ from

$$\bigcup \{\overline{v}_F: -\forall \overline{v}_F F \in S^p_n\}$$

into C_{n+p+1} such that for all i and j, $i \leqslant n$, $j \leqslant n$, if $i \neq j$ and $i + p = j + q$ then range$(f_{i,p}) \cap$ range$(f_{j,q}) = \emptyset$.

$S_0 = S$; $\langle \overline{S}^0_{0,m'}: m' \in \omega \rangle$ is any good split of $\overline{S}^0_0$; for all $p \in \omega$ $S^p_0 = \emptyset$ and $f_{0,p} = \emptyset$.

Suppose that S_h, $\langle \overline{S}^h_{h,m'}: m' \in \omega \rangle$, S^p_h, $f_{h,p}$ have been defined for all $p \in \omega$ and for all $h \leqslant n$ with the above mentioned properties.

Let

$$S'_n = \overline{S}^n_{n,n} \cup \bigcup \{\forall \overline{v}_F F: \forall \overline{v}_F F \in \bigcup \{\overline{S}^n_{n,i}: i < n\}\} \cup$$
$$\cup \bigcup \{c = d: c = d \in \bigcup \{\overline{S}^n_{n,i}: i < n\}\} \cup$$
$$\cup \bigcup \{Z: Z \text{ is an atomic or negated atomic sentence in } \bigcup \{\overline{S}^n_{n,i}: i < n\}\}.$$

Clearly $\overline{S}'_n \subset \bigcup \{s_{n,i}: i \leqslant n\}$, $|S'_n| \leqslant k_n$, all conjunction and quantification sets in S'_n have cardinality at most k_n.

Let

$$S^{(1)}_n = \langle s^{(1)}_{n,m}: m \in \omega \rangle$$

where $s^{(1)}_{n,m} = s_{n,m}$ if $m > 0$ and

$$s^{(1)}_{n,0} = s_{n,0} \cup \{d = c: c = d \in S'_n\} \cup$$
$$\cup \{Z(d): Z(c) \text{ is an atomic or negated atomic sentence in } S'_n \text{ and } c = d \in S'_n\};$$

$$S^{(2)}_n = \langle s^{(2)}_{n,m}: m \in \omega \rangle$$

where $s^{(2)}_{n,m} = s^{(1)}_{n,m}$ if $m > 0$ and

$$s^{(2)}_{n,0} = s^{(1)}_{n,0} \bigcup \{F\colon --F \in S'_n\};$$

$S^{(3)}_n = \langle s^{(3)}_{n,m}\colon m \in \omega\rangle$

where $s^{(3)}_{n,m} = s^{(2)}_{n,m}$ if $m > 0$ and

$$s^{(3)}_{n,0} = s^{(2)}_{n,0} \bigcup \{F\colon F \in \overline{F},\ \& \overline{F} \in S'_n\};$$

$S^{(4)}_u = \langle s^{(4)}_{n,m}\colon m \in \omega\rangle$

where $s^{(4)}_{n,m} = s^{(3)}_{n,m}$ if $m > 0$ and

$$s^{(4)}_{n,0} = s^{(3)}_{n,0} \bigcup \{F(\overline{v}_F/f)\colon f \in^{\bigcup\{\overline{v}_F\colon \forall\overline{v}_F F \in S'_n\}} \bigcup \{C_j\colon j < n,\ \forall\overline{v}_F F \in S'_n\}\};$$

$$S^{(5)}_n = \langle s^{(5)}_{n,m}\colon m \in \omega\rangle$$

where $s^{(5)}_{n,m} = s^{(4)}_{n,m}$ if $m > 0$ and

$$s^{(5)}_{n,0} = s^{(4)}_{n,0} \bigcup \{-g(\overline{F})\colon -\& \overline{F} \in S'_n\}$$

where g is a function, $g \in \bigtimes\{\overline{F}\colon -\& \overline{F} \in S'_n\}$, such that if $S^{(4)}_n \in \Sigma$ then also $S^{(5)}_n \in \Sigma$ (such a function exists due to C5));

Let $\langle s^p_{n+1}\colon p \in \omega\rangle$ be an ω-partition of $\{-\forall\overline{v}_F F\colon -\forall\overline{v}_F F \in S'_n\}$ such that for any set $\{f^*_{n+1,p}\colon p \in \omega\}$ of 1-1 functions

$$f^*_{n+1,p} \in^{\bigcup\{\overline{v}_F\colon -\forall\overline{v}_F F \in S^p_{n+1}\}} \Big(C_{n+p} - \{c\colon c \text{ is a constant occurring in } S_n\}\Big)$$

if $S^{(5)}_n \in \Sigma$ then the ω-sequence $S^*_n = \langle s^*_{n,m}\colon m \in \omega\rangle \in \Sigma$ where for $m \leqslant n$, $s^*_{n,m} = s^{(5)}_{n,m}$ and for all p

$$s^*_{n,n+p+1} = s^{(5)}_{n,n+p+1} \bigcup \{-F(\overline{V}_F/f^*_{n+1,p})\colon -\forall\overline{v}_F F \in s^p_{n+1}\}\,.$$

(Such partition and functions exist due to C6)).

Let s^p_{n+1} be the set just mentioned and $f_{n+1,p}$ a choice of functions as above.

Let $S^{(6)}_n = \langle s^{(6)}_{n,m}\colon m \in \omega\rangle$ where for $m \leqslant n$, $s^{(6)}_{n,m} = s^{(5)}_{n,m}$ and for all $p \in \omega$, $s^{(6)}_{n,n+p+1} = s^{(5)}_{n,n+p+1} \bigcup \{-F(\overline{v}_F/f_{n+1,p})\colon -\forall\overline{v}_F F \in s^p_{n+1}\}$.

Define $S_{n+1} = S_n^{(6)}$, $\langle \bar{S}^{n+1}_{n+1,m}: m \in \omega \rangle$ and good split of $\bar{S}^{n+1}_{n+1}$ such that $\bar{S}^{n+1}_{n+1,m} = \bar{S}^n_{n,m}$ if $m \leqslant n$ and $\bar{S}^{n+1}_{n+1,m} \supset \bar{S}^n_{n,m}$ if $m > n$.

To complete the definition by induction we have only to remark that all the conditions on S_n, $\langle \bar{S}^n_{n,m}: m \in \omega \rangle$, S^p_n, $f_{n,p}$ are preserved.

Indeed $S_{n+1} \in \Sigma$ if S_n does too thanks to conditions C1), C2), C3), C4), C5), C6,)and also the other conditions are satisfied due to the type of construction that we used.

Remark that for all n and m in ω $s_{n,m} \subset s_{n+1,m}$ and $\bar{S}_n \subset \bar{S}_{n+1}$.

Now let $s_\omega = \bigcup \{\bar{S}_n: n \in \omega\}$.

This set s_ω has the same properties as the analogous s_ω in [4] and it can be used to build an ω-chain of models whose universes have the prescribed cardinalities and which ω-satisfies the good ω-sequences that we have considered in the seq-consistency property and in particular S.

5. The inverse of the model existence theorem.

Observe that if Σ is a seq-consistency property then $\{S: \langle s_n: n \in \omega \rangle = S$ is a good ω-sequence and there is $S' = \langle s'_n: n \in \omega \rangle \in \Sigma$ such that for all $n \in \omega$, $s_n \subset s'_n\}$ is a seq-consistency property.

THEOREM. Let $S = \langle s_n: n \in \omega \rangle$ be a good ω-sequence of sets of sentences of $L_{k,k}$ that is ω-satisfiable. Let $|\bar{S}| \leqslant k$. Let C_j be sets such that for all $i \in \omega$, $|C_i| = k_i$ and for all i, $j \in \omega$ if $i \neq j$ then $C_i \cap C_j = \emptyset$.

Then there is a seq-consistency property Σ with respect to $\{C_i: i \in \omega\}$ such that $S \in \Sigma$.

PROOF. Partition each C_i in exactly $i+1$ parts, $\langle C_{i,j}: j \leqslant i \rangle$, such that $|C_{i,j}| = k_j$. Let L'_m be the language obtained from $L_{k,k}$ by adding $\bigcup \{C_{i,m'}: m' < m,\ i \in \omega\}$ as a set of constants, $L'_0 = L_{k,k}$. Remark that

$$L'_m = L_m \bigcup \Big(\bigcup \{C_{i,j}: i \geqslant m,\ j \leqslant m\}\Big); \qquad L'_m \bigcup \Big(\bigcup \{C_{i,m}: i \geqslant m\}\Big) = L'_{m+1}$$

Let $\langle s^p_n: p \in \omega \rangle$ be an ω-partition of s_n such that $|s^p_0| \leqslant k_p$ and for $n > 0$ if $-F(\bar{v}_F/f) \in s^p_n$ then $-\forall \bar{v}_F F \in \bigcup \{s^p_{n'}: n' < n\}$.

Let $\bar{M}$ be an ω-chain of models that ω-satisfies S.

Let us define by induction on m:

a) good ω-sequences $S_m = \langle s_{m,n} \colon n \in \omega \rangle$ in L'_m such that

$$|\bar{S}_m - \bar{S}_{m-1}| \leqslant k_m \quad \text{and} \quad \bar{S}_m \supset \bar{S}_{m-1};$$

b) ω-partitions $\langle s^p_{m,n} \colon p \in \omega \rangle$ of $s_{n,m}$ such that $|s^p_{m,n}| \leqslant k_p$ and $s^p_{m,n} \supset s^p_{m-1,n}$ and if $-F(\bar{v}_F/f) \in s^p_{m,n}$ then $-\forall \bar{v}_F F \in \bigcup \{s^p_{m,n'} \colon n' < n\}$;

c) 1-1 functions $f_{m,q}$ with $q \in \omega$; and

d) ω-chains of models $\bar{M}_m$ which are expansions of $\bar{M}_{m-1}$ to the language L'_m such that $\bar{M}_m \models^\omega S_m$, as follows.

$$S_0 = S, \ \bar{M}_0 = \bar{M}, \ f_{0,q} = \emptyset, \ s^p_{0,n} = s^p_n.$$

Suppose that S_h, $\bar{M}_h$, $f_{h,q}$, $s^p_{h,n}$ have already been defined for all $h \leqslant m$.

Let $\bar{s}'_m = \bigcup \{s^p_{m,0} \colon p \leqslant m\}$, $\bar{S}'_m = \bigcup \{s^p_{m,n} \colon n \leqslant m, \ p \leqslant m\}$.

Let $s'_m = \{c = d \colon c = d \in \bar{s}'_m\} \bigcup \{Z \colon Z \in \bar{S}'_m$ and Z is an atomic or negated atomic sentence$\} \bigcup \{--F \colon ---F \in (\bar{S}'_m - \bar{S}'_{m-1})\} \bigcup \{\& \bar{F} \colon$ either $\& \bar{F} \in (\bar{s}'_m - \bar{s}'_{m-1})$ and $|\bar{F}| \leqslant k_m$ or $\& \bar{F} \in \bar{s}'_{m-1}$ and $|\bar{F}| = k_m\} \bigcup \{\forall \bar{v}_F F \colon \forall \bar{v}_F F \in \bar{s}'_m$ and $|\bar{v}| \leqslant k_m\} \bigcup \{- \& \bar{F} \colon$ either $- \& \bar{F} \in (\bar{S}'_m - \bar{S}'_{m-1})$ and $|\bar{F}| \leqslant k_m$ or $- \& \bar{F} \in \bar{S}'_{m-1}$ and $|\bar{F}| = k_m\}$.

Let $s'_{m,q} = \{- \forall \bar{v}_F F \colon q$ is the least natural number such that there is a bounded assignment $\bar{a}_F$ within the $(m+q)$-th structure of $\bar{M}_m$ such that $\bar{M}_m, \bar{a}_F \models^\omega F$ and either $-\forall \bar{v}_F F \in (\bar{S}'_m - \bar{S}'_{m-1})$ and $|\bar{v}_F| \leqslant k_m$ or $-\forall \bar{v}_F F \in \bar{S}'_{m-1}$ and $|\bar{v}_F| = k_m\}$.

Let $f_{m+1,q}$ be a 1-1 function from $\bigcup \{\bar{v}_F \colon -\forall \bar{v}_F F \in s'_{m,q}\}$ in $C_{m+q,m}$. Such functions exist since $|\mathrm{dom}(f_{m+1,q})| \leqslant k_m$.

Let $\bar{a}_{m,q} = \bigcup \{\bar{a}_F \colon -\forall \bar{v}_F F \in s'_{m,q}\}$.

Define $\bar{M}_{m+1}$ as the expansion of $\bar{M}_m$ to the language L'_{m+1} obtained by interpreting each constant c belonging to

$$\bigcup \{f_{m+1,q}(\bar{v}_F) \colon -\forall \bar{v}_F F \in s'_{m,q}\}$$

for all q in ω in $\bar{a}_{m,q}(f^{-1}_{m+1,q}(c))$ and each constant of

$$C_{m+q,m} - \bigcup \{f_{m+1,q}(\bar{v}_F) \colon -\forall \bar{v}_F F \in s'_{m,q}\}$$

for all $q \in \omega$ in a fixed element of the universe of the first structure in $\bar{M}$.

Let $g_m \in \bigtimes\{\bar{F}: -\&\bar{F} \in s'_m\}$ such that $-g(\bar{F})$ is ω-satisfied in $\bar{M}_m$.

Let $S_{m+1} = \langle s_{m+1,n}: n \in \omega\rangle$ be the good ω-sequence defined as follows.

$s_{m+1,0} = s_{m,0} \bigcup \{d = c: c = d \in s'_m\} \bigcup \{Z(d): Z(c) \in s'_m,\ c = d \in s'_m$ and Z is an atomic or negated atomic sentence$\} \bigcup \{F: --F \in s'_m\} \bigcup \{F: F \in \bar{F}$ and $\&\bar{F} \in s'_m\} \bigcup \{F(\bar{v}_F/f_F): \forall\bar{v}_F F \in s'_m$ and $f_F \in^{\bar{v}_m} \bigcup\{C_i: i \leqslant m\}\} \bigcup \bigcup\{-g(\bar{F}): -\&\bar{F} \in s'_m\}$.

If $n > m$ let

$$s_{m+1,n} = s_{m,n} \bigcup \left(\bigcup \{-F(\bar{v}_F/f_{m+1,n-m-1}): -\forall\bar{v}_F F \in s'_{m,n-m-1}\}\right).$$

If $0 < n \leqslant m$ let $s_{m+1,n} = s_{m,n}$.

Let $s^p_{m+1,n} = s^p_{m,n}$ for $p \leqslant m$.

Let $\langle I_{n,m,r}: r - 1 \in \omega\rangle$ be an ω-partition of $s_{m+1,n} - s_{m,n}$ such that $|I_{n,m,r}| \leqslant k_{m+r}$.

If $p > m$ let $s^p_{m+1,n} = s^p_{m,n} \bigcup I_{n,m,p-m}$.

So we have completed the definition by induction on m once we have remarked that it is easy to see that the defined entities have the properties that they should have.

Now let $\Gamma = \{S': S' = \langle 's_n: n \in \omega\rangle$ is a good ω-sequence and there is $m \in \omega$ such that for all $n \in \omega$, $'s_n \subset s_{m,n}$ where $\langle s_{m,n}: n \in \omega\rangle = S_m\}$.

It is not difficult to prove that this Γ is a seq-consistency property with respect to $\{C_i: i \in \omega\}$ to which S belongs.

6. Interpolation theorem.

By a *bipartition* of a good ω-sequence $S = \langle s_n: n \in \omega\rangle$ of sets of sentences we mean an ordered pair (S_1, S_2) where $S_1 = \langle s^1_n: n \in \omega\rangle$ and $S_2 = \langle s^2_n: n \in \omega\rangle$ are good ω-sequence of sets of sentences satisfying the following conditions:

1) for all n (s^1_n, s^2_n) is a partition of s_n;
2) for $n > 0$ if $-F(\bar{v}_F/f) \in s_n$ and $-\forall\bar{v}_F F \in s^1_m$ (s^2_m) for some $m < n$ then $-F(\bar{v}_F/f) \in s^1_n$ (s^2_n).

Let $S = \langle s_n: n \in \omega\rangle$ be a good ω-sequence of sets of sentences and let (S_1, S_2) be a bipartition of S. We say that a sentence α is an ω-interpolant for S with respect to (S_1, S_2) if the extralogical symbols in α occur both in $\bigcup\{s^1_n: n \in \omega\}$ and in $\bigcup\{s^2_n: n \in \omega\}$ and if the the good ω-sequences $S_1^{-\alpha} = \langle s^{1\alpha}_n: n \in \omega\rangle$ and $S_2^{\alpha} = \langle s^{2\alpha}_n: n \in \omega\rangle$ defined as $s_0^{1\alpha} = s_0^1 \bigcup \{-\alpha\}$, $s_0^{2\alpha} = s_0^2 \bigcup \{\alpha\}$, $s_n^{1\alpha} = s_n^1$, $s_n^{2\alpha} = s_n^2$ for $n > 0$, are not ω-satisfiable.

We now prove the following.

LEMMA. Let Γ be the set of good ω-sequences S of sets of sentences such that there is a bipartition (S_1, S_2) of S without ω-interpolant for S with respect to (S_1, S_2). Then Γ is a seq-consistency property.

PROOF. We have to check all the defining conditions of a seq-consistency property. Indeed it is not difficult to prove that C0), C1), C2), C3), C4), C5) are satisfied using standard methods. Therefore, here we treat only the remaining condition.

C6) Suppose that there is $m \in \omega$ such that $\{-\forall \bar{v}_i F_i : i \in I\} \subset \subset \{s_n : n \leqslant m\}$ and let m' be the least natural number such that $|I| < k_{m'}$, for all $i \in I$ we have $0 < |\bar{v}_i| < k_{m'}$ and $\{s_n : n \leqslant m\} \in \mathrm{Stmt}(L_{m'})$.

Let (S_1, S_2) be a bipartition of $S = \langle s_n : n \in \omega \rangle$ without ω-interpolant for S with respect to (S_1, S_2). Let $S_1 = \langle s_n^1 : n \in \omega \rangle$, $S_2 = = \langle s_n^2 : n \in \omega \rangle$.

Suppose that for all ω-partitions $P = \langle I_p^P : p \in \omega \rangle$ of I there is a set of 1-1 functions $\{f_p^P : p \in \omega\}$,

$$f_p^P \in {}^{\bigcup\{\bar{v}_i : i \in I\}}\Big(C_{m'+p} - \{c\colon c \text{ is a constant occurring in } \bigcup \{s_n : n \in \omega\}\}\Big),$$

such that for all bipartitions (S_1^P, S_2^P) of $S^P = \langle s_n^P : n \in \omega \rangle$ where $s_n^P = s_n$ if $n \leqslant m'$ and $s_{m'+p+1}^P = s_{m'+p+1} \bigcup \{-F_i(\bar{v}_i / f_p^P) : i \in I_p^P\}$, there is an ω-interpolant, say α^P, for S^P with respect to the bipartition (S_1^P, S_2^P), with $S_1^P = \langle s_n^{1P} : n \in \omega \rangle$ and $S_2^P = \langle s_n^{2P} : n \in \omega \rangle$.

Let $I_1 = \{i : -\forall \bar{v}_i F_i \in \bigcup \{s_n^1 : n \in \omega\}\}$, $I_2 = I - I_1$.

Any ω-partition P of I gives rise to a pair of ω-partitions P_1 and P_2 of I_1 and I_2 respectively according to the following definitions:

$$P_1 = \langle I_p^P \cap I_1 : p \in \omega \rangle\,, \quad P_2 = \langle I_p^P \cap I_2 : p \in \omega \rangle\,.$$

On the other hand any pair of ω-partitions P_1 and P_2 of I_1 and I_2 respectively determines an ω-partition P of I according to the following definition $P = \langle I_p^{P_1} \cup I_p^{P_2} : p \in \omega \rangle$.

Remark that the two operations described above are inverse one of the other.

Each ω-interpolant α^P may now be denoted as α^{P_1, P_2}.

Remark also that for all ω-partitions P of I the extralogical symbols in common between $\bigcup\{s_n^{1P}: p\in\omega\}$ and $\bigcup\{s_n^{2P}: n\in\omega\}$ are the same as those in common between $\bigcup\{s_n^1: n\in\omega\}$ and $\bigcup\{s_n^2: n\in\omega\}$.

Let $\alpha = \&\{-\&\{-\alpha^{P_1,P_2}: P_1$ is an ω-partition of $I_1\}: P_2$ is an ω-partition of $I_2\}$.

Claim: α is an ω-interpolant for S with respect to (S_1, S_2).

Indeed α satisfies the condition on the extralogical symbols.

Furthermore $S_1^{-\alpha}$ is not ω-satisfiable. If not, there would be an ω-chain of models $\overline{M}$ such that $\overline{M}\models^{\omega} S_1^{-\alpha}$.

Let $\overline{P}_1 = \langle I_p^{\overline{P}_1}: p\in\omega\rangle$ be the ω-partition of I_1 such that $I_p^{\overline{P}_1} = \{i: p$ is the least natural number such that there is a bounded assignment $\overline{b}_i$ to the variables in $\overline{v}_i$ within the p-th structure of $\overline{M}$ such that $\overline{M}, \overline{b}_i\models^{\omega} - F_i(\overline{v}_i)\}$.

Let $\overline{M}'$ be the expansion of $\overline{M}$ obtained by interpreting each constant c in $f_p^{\overline{P}_1}(\overline{v}_i)$ in $\overline{b}_i((f_p^{\overline{P}_1})^{-1}(c))$ for all $i\in I_p^{\overline{P}_1}$ and $p\in\omega$ where $f_p^{\overline{P}_1}$ are the restrictions to $\bigcup\{\overline{v}_i: i\in I_p^{\overline{P}_1}\}$ of any f_p^P with P that coincides with $\overline{P}_1$ on $\bigcup\{\overline{v}_i: i\in I_1\}$ for all $p\in\omega$.

Due to the previous results $\overline{M}'\models^{\omega} S_1^{\overline{P}_1-\alpha}$.

Therefore there is an ω-partition of I_2, say $\overline{P}_2$, such that $\overline{M}'\models^{\omega} S_1^{\overline{P}_1\alpha^{\overline{P}_2}}$ where $\alpha^{\overline{P}_2}$ is $\&\{-\alpha^{P_1,\overline{P}_2}: P_1$ is an ω-partition of $I_1\}$ which is one of the disjuncts of $-\alpha$.

In particular $\overline{M}'\models^{\omega} S_1^{\overline{P}_1-\alpha^{\overline{P}_1,\overline{P}_2}}$, but this contradicts the fact that $\alpha^{\overline{P}_1,\overline{P}_2}$ is an ω-interpolant for $S^{\overline{P}}$ with respect to $(S_1^{\overline{P}}, S_2^{\overline{P}})$ where

$$\overline{P} = \overline{P}_1 \cap P_2 .$$

This contradiction shows that $S_1^{-\alpha}$ is not ω-satisfiable.

A similar argument shows that also S_2^{α} is not ω-satisfiable, and this proves the claim.

Therefore there is an ω-partition P of I such that for any set $\{f_p: p\in\omega\}$ of 1-1 functions,

$$f_p \in^{\bigcup\{\overline{v}_i:\, i\in I_p\}}\left(C_{m'+p} - \{c: c \text{ is a constant occurring in } S\}\right)$$

we have that $S^P\in\Gamma$ and we have checked C6).

Having proved the lemma, we can now proceed to show

Cunningham's interpolation theorem. Suppose that $F_1\to F_2$ is an ω-valid sentence of $L_{k,k}$. Then there is a sentence α, called an interpolant, whose extralogical symbols occur both in F_1 and in F_2 such that $\models^{\omega} F_1\to\alpha$ and $\models^{\omega}\alpha\to F_2$.

PROOF. If not, for all sentences α whose extralogical symbols are both in F_1 and in F_2 we would have that either $-(F_1 \to \alpha)$ is ω-satisfiable or $-(\alpha \to F_2)$ is ω-satisfiable. Then for the ω sequence of sets of sentences $S = \langle s_n : n \in \omega \rangle$ with $s_0 = \{F_1, -F_2\}$ and $s_n = \emptyset$ for $n > 0$, there is a bipartition (S_1, S_2) with $S_1 = \langle s_n^1 : n \in \omega \rangle$ and $S_2 = \langle s_n^2 : n \in \omega \rangle$ where $s_0^1 = \{F_1\}$, $s_0^2 = -\{F_2\}$, $s_n^1 = s_n^2 = \emptyset$ for $n > 0$ such that S has no ω-interpolant with respect to (S_1, S_2).

Therefore S belongs to the seq-consistency property of the previous lemma and hence it is ω-satisfiable. But this contradicts the ω-validity of $F_1 \to F_2$ and therefore $F_1 \to F_2$ must have an interpolant.

7. Conclusion.

In [8] Maehara and Takeuti proposed a stronger type of interpolation theorem for $L^{2+}_{\omega,\omega}$ which was extended in [4] to $L^{2+}_{k,k}$ with respecto the notion of ω-satisfiability but assuming validity instead of ω-validity in the definition of an interpolant as in Karp's interpolat tion theorem.

While the techniques used by Cunningham in [2] do not seem to have an immediate extension to $L^{2-}_{k,k}$ languages, those used in this paper are easily extendable to the language $L^{2-}_{k,k}$ by:

1) adding to $L^{2-}_{k,k}$ not only sets of new individual constants C_n, $n \in \omega$, but also sets of new p-placed precidate constants, $C_{n,p}$, such that $|C_{n,p}| = k_n$, $n \in \omega$, $p \in \omega$.
2) modifying the notion of good ω-sequence of sets of sentences $\langle s_n : n \in \omega \rangle$ as to allow in the n-th set, for $n > 0$, sentences of the form $-F(\overline{V}_F/f)$ where f is a 1-1 function from $\overline{V}_F$ into $\bigcup \{C_{n,p} : p \in \omega\}$, $\overline{V}_F$ is a set of second order variables and the sentence $-\forall \overline{V}_F F \in \bigcup \{s_j : j < n\}$.
3) modifying the notion of seq-consistency property for $L^{2-}_{k,k}$ with respect to $\{C_n : n \in \omega\}$ and $\{C_{n,p} : n \in , p \in \omega\}$ by adding to the clauses already stated a new one obtained from C6) by replacing the individual variables with predicate variables and adequatly adjusting the range of the functions f_p.

The above mentioned extension of the technique to $L^{2-}_{k,k}$ permits to improve the interpolation theorem in [4] to read ω-validity also in the definition of an interpolant as it is done by Baldo in [1].

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