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Projectivity of Pure Ideals.

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0. Introduction.

A very interesting theorem of Bkouche ([Bk_2], Th. 6) characterizes projectivity of pure ideals in terms of the topology of Max (A), when A is a « soft » ring.

The original purpose of this paper was to exploit Bkouche's result for rings of real valued continuous functions, in order to see if some classes of spaces can be characterized in terms of projectivity of ideals in their ring of continuous functions. The investigation disclosed the fact that Bkouche's result, when suitably formulated, gives a characterization of projectivity for pure ideals in terms of the spectral topology, for every commutative ring.

This characterization yields proofs of results already obtained by Vasconcelos [V]; which are here obtained at once, in a compact way. In the case of rings of continuous functions, several classes of spaces X are characterized in terms of projective ideals of $C(X)$.

1. Algebraic results.

1.1. Ring means commutative ring with an identity 1.

If A is a ring, Spec (A) denotes the set of all (proper) prime ideals of A with the Zariski (or hull-kernel) topology, Max (A) denotes the

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subspace of maximal ideals. If I is an ideal of A, then $V(I) = \{p \in \operatorname{Spec}(A): P \supseteq I\}$ denotes the associated closed set, $D(I) = \operatorname{Spec}(A) \setminus V(I)$ the associated open set; subscript M denotes relativization to $\operatorname{Max}(A)$, e.g. $D_M(I) = D(I) \cap \operatorname{Max}(A)$, etc.. For principal ideals a A, $V(aA)$ and $D(aA)$ are shortened, as usual, to $V(a)$, $D(a)$. If X is a subset of $\operatorname{Spec}(A)$, its specialization (resp: generalization) $\mathcal{S}(X)$ (resp: $\mathcal{G}(X)$) is the set of all primes which contain (resp: are contained in) some prime belonging to X; X is said to be $\mathcal{S}$-stable (resp: $\mathcal{G}$-stable) iff $\mathcal{S}(X) = X$ (resp: $\mathcal{G}(X) = X$).

Notice that closed sets of $\operatorname{Spec}(A)$ are $\mathcal{S}$-stable, and that open sets are $\mathcal{G}$-stable. Since the closure of a singleton $\{P_0\} \subseteq \operatorname{Spec}(A)$ is $\{P \in \operatorname{Spec}(A): P \supseteq P_0\} = \mathcal{S}(\{P_0\})$, it easily follows that a set is $\mathcal{S}$-stable iff it is a union of closed subsets of $\operatorname{Spec}(A)$. Trivially, the complement in $\operatorname{Spec}(A)$ of an $\mathcal{S}$-stable subset is a $\mathcal{G}$-stable subset, and conversely.

Following Lazard [L_1], call $\mathfrak{D}$-topology the topology on the set of prime ideals of A whose open sets are the open $\mathcal{S}$-stable subsets of $\operatorname{Spec}(A)$ (it is a topology coarser than the spectral topology, strictly so, in general); denote by $\mathfrak{D}_M$ its relativization to the subspace of maximal ideals.

The $\mathfrak{D}$ topology and the $\mathfrak{D}_M$ topology are in one-to-one correspondence:

PROPOSITION. *The mapping* $U \to U_M = U \cap \operatorname{Max}(A)$ *is a bijection of the set of* $\mathfrak{D}$-open *sets onto the set of* $\mathfrak{D}_M$-open *sets; and* $G \to G_M = G \cap \operatorname{Max}(A)$ *is a bijection of the* $\mathfrak{D}$-closed *sets onto the* $\mathfrak{D}_M$-closed sets.

PROOF. It is immediate to see that if U is open and $\mathcal{S}$-stable then $U = \mathcal{G}(U \cap \operatorname{Max}(A))$; and if G is closed and $\mathcal{G}$-stable then, again, $G = \mathcal{G}(G \cap \operatorname{Max}(A))$.

1.2. A pm-ring A is defined to be a ring in which every prime ideal is contained in a unique maximal ideal. If A is a pm-ring, then the mapping $\mu: \operatorname{Spec}(A) \to \operatorname{Max}(A)$ which sends every prime ideal of A into the unique maximal ideal containing it is a continuous closed map, and $\operatorname{Max}(A)$ is compact T_2 (see [DO] or [Bk_1]; the «soft» rings in [Bk_1] are the pm-rings with zero Jacobson radical). It follows that in a pm-ring A the open $\mathcal{S}$-stable subsets of $\operatorname{Spec}(A)$ are of the form $\mu^{-1}(V)$, with V spectrally open in $\operatorname{Max}(A)$. It is easy to get the following:

PROPOSITION. *A is a pm-ring if and only if the spectral topology and the $\mathfrak{D}$-topology coincide on* Max (A).

1.3. A typical pm-ring is the ring $C(X)$ of all continuous real valued functions on a topological space X. If X is completely regular Hausdorff, and βX is the Stone-Čech compatification of X, then the mapping $\iota\colon \beta X \to \operatorname{Max}(C(X))$ defined by $\iota(p) = M^p = \{f \in C(X)\colon p \in \operatorname{cl}_{\beta X}(Z(f))\}$ is a homeomorphism of $\beta(X)$ onto $\operatorname{Max}(C(X))$. We shall freely identify βX and $\operatorname{Max}(C(X))$, via this map ι.

1.4. Given a ring A, we define the *support* of $a \in A$ (in $\operatorname{Spec}(A)$) as $V(\operatorname{Ann}(a))$, where $\operatorname{Ann}(a)$ is the annihilator ideal of a in A. (We always have $\operatorname{Supp}(a) \supseteq \operatorname{cl}_{\operatorname{Spec}(A)}(D(a))$, with equality if A is reduced, i.e., if A has no nilpotents). When I is an ideal of A, we define $\operatorname{Supp}(I) = \bigcup_{a\in I} \operatorname{Supp}(a)$ (the same set is obtained if a ranges over any generating system of I). These definitions are equivalent to the usual ones for modules, given, e.g., in [B₂].

PROPOSITION. (i) *If I is an ideal of A, then* $D(I) \subseteq \operatorname{Supp}(I)$.

(ii) *Let J be an ideal of A, and $a \in A$. Then* $\operatorname{Supp}(a) \subseteq D(J)$ *holds iff $a \in aJ$.*

PROOF. (i) If P is prime, and $P \not\supseteq I$, then $P \supseteq \operatorname{Ann}(a)$ for every $a \in I \setminus P$.

(ii) $\operatorname{Supp}(a) \subseteq D(J) \Leftrightarrow V(\operatorname{Ann}(a)) \cap V(J) = \emptyset \Leftrightarrow \operatorname{Ann}(a) + J = A$. Take $x \in \operatorname{Ann}(a)$ and $y \in J$ such that $x + y = 1$; then $a = ay \in aJ$; conversely, from $a = ay$, $y \in J$, it follows that $1 - y \in \operatorname{Ann}(a)$. Then $\operatorname{Ann}(a) + J = A$, and the proof is concluded.

If $A = C(X)$, and $f \in C(X)$, then f has a *zero-set* (in X) $Z(f)$ $(= Z_X(f)) = f^{\leftarrow}(\{0\})$ and a cozero set (in X), $Cz(f) = X \setminus Z(f)$. The *support* of f *in* X is usually defined as $\operatorname{Supp}_X(f) = \operatorname{cl}_X(Cz(f))$; it is easy to see that $\operatorname{Supp}_X(f) = X \cap \iota^{\leftarrow}(\operatorname{Supp}(f) \cap \operatorname{Max}(C(X)))$, ι being the map defined in 1.3.

1.5. Pure ideals. There are various definitions for the concept of pure submodule of a given A-module (see, e.g. [F]). However, for ideals of commutative rings, they are all equivalent.

We say that an ideal I of a ring A is pure if $J \cap I = JI$ for every ideal J of A.

Proposition. *Let I be an ideal of A. The following are equivalent:*

(i) *I is pure.*

(ii) *For every $a \in A$, $(aA) \cap I = aI$.*

(iii) *For every $a \in I$ there exists $b \in I$ such that $a = ab$.*

(iv) *For every finite subset $\{a_1, \ldots, a_m\}$ of I, there esists $b \in I$ such that $a_i b = a_i$, for all $i = 1, \ldots, m$.*

(v) $D(I) = \operatorname{Supp}(I)$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are immediate, from the definition of pure ideal; (iv) $\Rightarrow$ (iiii) is trivial. To prove that (iii) $\Rightarrow$ (iv): for every $i = 1, \ldots, m$ pick $b_i \in I$ such that $a_i b_i = a_i$. Since $1 + I$ is multiplicatively closed, there exists $b \in I$ such that $1 - b = \prod_{i=1}^{m} (1 - b_i)$; it is immediate to see that b is the required element. That (iii) and (v) are equivalent follows from proposition 1.4.

Remark. As is stated in [Bk_2], one can prove also that I is pure if and only if A/I is a flat A-module. Another interesting characterization of pure ideals is the following: I is pure if and only if it is the kernel of the canonical homomorphism of A into the fraction ring $A[(1 + I)^{-1}]$ (see [B_1]); this is a trivial consequence of (iii) above. This and the machinery of fraction rings could be used to introduce some minor improvements in the proofs to follow. But we are not interested in a full discussion of the concept of purity: the above Proposition is all we will need.

1.6. For every ideal I, $\operatorname{Supp}(I)$ is a union of closed sets, hence it is $\mathcal{S}$-stable. Hence 1.5, Proposition, (v), shows that if I is a pure ideal, then $D(I) = \operatorname{Supp}(I)$ is an open $\mathcal{S}$-stable subset of $\operatorname{Spec}(A)$. Conversely, for every $\mathcal{S}$-stable open subset U od $\operatorname{Spec}(A)$ there exists one and only one pure ideal I such that $U = D(I)$. This result is stated in [Bk_2], without proof. We give a proof of it later.

Here we note the following simple consequence of $\mathcal{G}$-stability of $V(I)$ for I pure:

Proposition. *The radical of a pure ideal I is the intersection of all minimal prime ideals of A containing I.*

1.7. For an ideal J of A, denote by $\gamma(J)$ the smallest cardinality of the generating systems of J.

PROPOSITION. *The mapping* $I \to D(I)$ (resp: $I \to D_M(I)$) *is an inclusion preserving bijection of the set of all pure ideals of* A *onto the set of all* $\mathfrak{D}$-open (resp: all $\mathfrak{D}_M$-open) *sets. Moreover for every pure ideal* I, $\gamma(I)$ *equals the smallest cardinality of families of* (spectrally) *closed sets of* Max (A) *whose union is* $D_M(I)$.

PROOF. We prove only the part concerning Spec (A); the part concerning Max (A) then follows easily from Prop. 1.1. If I is pure, then $D(I)$ is $\mathcal{S}$-stable, as remarked in 1.6.

We have to show that if U is open and $\mathcal{S}$-stable then there exists one and only one pure ideal I such that $U = D(I)$. Put $G = \text{Spec}(A) \setminus U$, $J = \bigcap_{p \in G} P$; then $V(J) = G$. Let $I = \{a \in A : a \in aJ\}$; clearly I is an ideal, $I \subseteq J$; by 1.4, $I = \{a \in A : \text{Supp}(a) \subseteq D(J) = U\}$. We claim that I is pure, and that $\text{Supp}(I) = U$: for this evidently it suffices to show that $V(I) \subseteq V(J)$. Given $P \in V(I)$, observe that the multiplicative subset $S = (A \setminus P)(1 + J)$ does not contain 0 (the annihilator of $1 + J$ is I, 1.4); thus there exists a prime ideal Q disjoint from it.

Since $Q \cap (1 + J) = \emptyset$, $Q + J$ is a proper ideal; every maximal ideal containing it belongs to $V(J)$; but hence Q belongs to $V(J)$, which is $\mathcal{G}$-stable; and since $Q \subseteq P$, P belongs to $V(J)$, which is closed.

Uniqueness of I is clear from its description in terms of U, $I = \{a \in A : \text{Supp}(a) \subseteq U\}$. For the part concerning $\gamma(I)$ first observe that $D(I) = \text{Supp}(I) = \bigcup_{\lambda \in \Lambda} \text{Supp}(f_\lambda)$, where $(f_\lambda)_{\lambda \in \Lambda}$ is any generating system for I. And if I is pure, and $(F_\lambda)_{\lambda \in \Lambda}$ is any family of closed subsets of Spec (A) such that $\bigcup_{\lambda \in \Lambda} F_\lambda = D(I)$, one can see that there exists a family $(a_\lambda)_{\lambda \in \Lambda}$ of elements of I such that $F_\lambda \subseteq D(a_\lambda) \subseteq \text{Supp}(a_\lambda) \subseteq D(I)$; it is easy to prove that $(a_\lambda)_{\lambda \in \Lambda}$ generates the ideal I. To prove existence of such family $(a_\lambda)_{\lambda \in \Lambda}$ notice that for every $\lambda \in \Lambda$ the ideals I and $J_\lambda = \cap \{M : M \in F_\lambda\}$ are co-prime, i.e. there exist $a_\lambda \in I$, $b_\lambda \in J_\lambda$ such that $a_\lambda + b_\lambda = 1$.

1.8. We collect here some elementary facts on pure ideals.

PROPOSITION. (*a*) *If* $(I_\lambda)_{\lambda \in \Lambda}$ *is a family of pure ideals, then* $\sum_{\lambda \in \Lambda} I_\lambda$ *is a pure ideal.*

(*b*) *If* $(I_1, \ldots, I_m)$ *is a finite family of pure ideals, then* $I = \bigcap_{k=1}^{m} I_k = \prod_{k=1}^{m} I_k$ *is a pure ideal.*

(*c*) *If I is pure ideal and J is any ideal, then $I+J/J$ is a pure ideal of A/J.*

(*d*) *A pure ideal contained in the Jacobson radical of A is zero.*

(*e*) *Let I be a pure ideal. Then: for every ideal J of A, $I \subseteq J$ holds if and only if $D_M(I) \subseteq D_M(J)$.*

(*f*) *A finitely generated pure ideal is generated by an idempotent.*

(*g*) *In a reduced ring, pure ideals coincide with their radical.*

PROOF. Easy computations: use 1.5, 1.6, 1.7.

1.9. The decompositions of a pure ideal I into direct sums of (necessarily pure) ideals are in one-to-one correspondence with open partitions of $D(I)$ (notice that if an open $\mathcal{S}$-stable set is partitioned into spectrally open sets, then each of these sets is also $\mathcal{S}$-stable; so there is no need to distinguish between spectrally open partitions and $\mathcal{S}$-open partitions).

PROPOSITION. *Let I be a pure ideal.*

(*a*) *If $(U_\lambda)_{\lambda\in\Lambda}$ is an open partition of $D(I)$, then $I=\bigoplus_{\lambda\in\Lambda} I$, where I_λ is the pure ideal such that $D(I_\lambda)=U_\lambda$, for every $\lambda\in\Lambda$.*

(*b*) *If $I=\sum_{\lambda\in\Lambda} J$, and $J_\lambda\cdot J_\mu=0$ whenever $\lambda\neq\mu$ then each J_λ is a pure ideal, and $I=\bigoplus_{\lambda\in\Lambda} J_\lambda$ (and, of course $D(I)$ is the disjoint union of $\{D(J_\lambda)\colon \lambda\in\Lambda\}$).*

PROOF. Easy.

1.10. An A-module M is projective if and only if for every system of generators $(f_\lambda)_{\lambda\in\Lambda}$ of M there exists a family $(\varphi_\lambda)_{\lambda\in\Lambda}$, where each $\varphi_\lambda\in\operatorname{Hom}_A(M,A)$ $(=M^*)$ such that for every $x\in M$ the set $\Lambda(x)=\{\lambda\in\Lambda\colon \varphi_\lambda(x)\neq 0\}$ is finite, and $x=\sum_{\lambda\in\Lambda(x)}\varphi_\lambda(x)f_\lambda$ (see, e.g. [B₁]). When this holds, $(\varphi_\lambda, f_\lambda)_{\lambda\in\Lambda}$ is called a projective basis for M.

Mutuating the terminology from general topology, we say that a subset E of a ring A is star-finite (resp: star-countable) if for every $a\in E$ the set $\Delta(a)=\{b\in E\colon ab=0\}$ is finite (resp. countable).

This implies that $\{D(a)\colon a\in E\}$ is a star-finite (resp. star-countable) family of open subsets of Spec (A); and if A has no nonzero nilpotents, these facts are equivalent.

1.11. The following interesting result is a particular case of a theorem of [L_2]; a direct proof of it is however much simpler.

PROPOSITION. *A countably generated pure ideal is projective; moreover, I has a generating system $(c_n)_{n\in\mathbf{N}}$ such that for every $x \in I$, the set $\Delta(x) = \{n \in \mathbf{N}\colon xc_n = 0\}$ is finite, and $x = \sum_{n\in\Delta(x)} xc_n$.*

The proof is in the following number, which contains another important result (also found in [L_2]).

1.12. PROPOSITION. *Let J be a pure ideal. Every countably generated ideal I contained in J is contained in a pure countably generated ideal K contained in J.*

Before the proof, notice:

COROLLARY. *The open F_σ-subsets of* Spec (A) *are a base for the $\mathfrak{D}$-topology.*

PROOF OF PROP. 1.12. Let $(a_n)_{n\in\mathbf{N}}$ be a generating system for I. Define $b_n \in J$ inductively as follows: $b_0 \in J$ is such that $a_0 b_0 = a_0$; given b_n, $b_{n+1} \in J$ is such that $a_{n+1}b_{n+1} = a_{n+1}b_n = b_n$ (1.3 (iv)). It is easy to see that the ideal K generated by $(b_n)_{n\in\mathbf{N}}$ has the required properties; notice also that $b_i b_n = b_i$ for $i < n$.

PROOF OF PROP. 1.11. The above proof, with $J = I$, yields that I has a generating system $(b_n)_{n\in\mathbf{N}}$ such that $b_i b_n = b_i$ for $i < n$; thus $x \in I$ holds iff $xb_m = x$ for all m larger than some $n(x) \in \mathbf{N}$. Put $c_0 = b_0$, $c_{n+1} = b_{n+1} - b_n$; for $m > n(x)$ we have $xc_m = xb_{m+1} = x - x = 0$; moreover $\sum_{i=0}^{n(x)} xc_i = x\big(b_0 + (b_1 - b_0) + \ldots + (b_{n(x)+1} - b_{n(x)})\big) = xb_{n(x)+1} = x$. Hence $(c_n)_{n\in\mathbf{N}}$ is the required generating system. To see projectivity: for each $n \in \mathbf{N}$ take $d_n \in I$ such that $c_n d_n = c_n$; it is simple to check that $(c_n, d_n)_{n\in\mathbf{N}}$ is a projective basis for I (cfr. (1.7); here c_n has to be interpreted as « multiplication by c_n », to make it an element of $\mathrm{Hom}_A(I, A)$).

1.13. Here we give a characterization of pure and projective ideals. Whenever G is a $\mathfrak{G}$-closed subset of Spec (A), or even a $\mathfrak{D}_M$-closed subset of Max (A), we denote by O^G the corresponding pure ideal, i.e. the unique pure ideal I such that $V(I) = G$ (or $V_M(I) = G$).

THEOREM. *Let I be an ideal of A. The following are equivalent:*

(i) *I is pure and projective.*

(ii) *I has a generating system $\{a_\lambda\colon \lambda \in \Lambda\}$ such that for every $x \in I$ the set $\Lambda(x) = \{\lambda \in \Lambda\colon xa_\lambda \neq 0\}$ is finite, and $x = \sum_{\lambda \in \Lambda(x)} xa_\lambda$.*

(iii) *I is pure, and has a star-finite generating system.*

(iv) *I is pure, and has a star-countable generating system.*

(v) *$I = \bigoplus_{\gamma \in \Gamma} I_\gamma$, where each I_γ is pure and countably generated*

(vi) *$I = \bigoplus_{\gamma \in \Gamma} O^{Z_\gamma}$, where each Z_γ is a closed G_δ in* Spec (A).

(vii) *I is pure, and $D(I)$ is a disjoint union of open F_σ subsets of* Spec (A).

PROOF. (i) implies (ii). Let $(\varphi_\lambda, f_\lambda)_{\lambda \in \Lambda}$ be a projective basis for I. For every $\lambda \in \Lambda$ take by $b_\lambda \in I$ such that $f_\lambda b_\lambda = f_\lambda$. Then $\varphi_\lambda(f_\lambda b_\lambda) = \varphi_\lambda(f_\lambda) b_\lambda = \varphi_\lambda(f_\lambda)$, i.e. $\varphi_\lambda(f_\lambda) \in I$. Put $a_\lambda = \varphi_\lambda(f_\lambda)$: this is the required generating system: in fact, for every $x \in I$ we have $\varphi_\lambda = (x) \neq 0$ only for a finite set $\Lambda(x)$ of $\lambda \in \Lambda$, and $x = \sum_{\lambda \in \Lambda(x)} \varphi_\lambda(x) f_\lambda = \sum_{\lambda \in \Lambda(x)} \varphi_\lambda(f_\lambda) x = \sum_{\lambda \in \Lambda(x)} a_\lambda x$ (notice also that, for $\lambda \notin \Lambda(x)$, $\varphi_\lambda(x) = 0 \Rightarrow \varphi_\lambda(x) f_\lambda = 0 \Leftrightarrow \varphi_\lambda(f) x = 0 \Leftrightarrow a_\lambda x = 0$). (iii) $\Rightarrow$ (iv) Trivial. (iv) implies (v). Let E be a star countable generating system for I. Introduce an equivalence relation on E by saying that $a \sim b$ if there exist $a_0, \ldots, a_n \in E$ with $a_0 = a$, $a_n = b$ and $a_{i+1} \in \Delta(a_i)$ for $i = 0, \ldots, n-1$ (cfr. 1.7). Call Γ the set of equivalence classes so obtained. Each $\gamma \in \Gamma$ is countable since the equivalence class of $a \in E$ may described as $\bigcup_{n \in \mathbf{N}} \Delta^n(a)$, where $\Delta^0(a) = \{a\}$ and $\Delta^{n+1}(a) = \bigcup \{\Delta(c)\colon c \in \Delta^n(a)\}$. Denote by I_γ the ideal generated by γ; if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$, we have $I_{\gamma_1} \cdot I_{\gamma_2} = \{0\}$; moreover $I = \sum_{\gamma \in \Gamma} I_\gamma$, and I is pure. The conclusion follows from 1.9 (b). By 1.11, (v) implies (i).

Equivalence of (vi), (vii) is trivial; that (vi) is equivalent to (v) follows from 1.7.

1.14. The trace $\tau\, I$ of and ideal I is the ideal $\sum_{\varphi \in I^*} \varphi(I)$, where $I^* = \mathrm{Hom}_A(I, A)$; $\tau\, I$ is the image of the *trace homomorphism* $\tau\colon I^* \otimes_A I \to A$ defined by $\tau(\varphi \otimes x) = \varphi(x)$.

PROPOSITION. *Let I be a projective ideal of A. Then:*

(i) *τI is pure and projective; τI is the smallest pure ideal containing I; and $D(\tau I) = \mathrm{Supp}\,(I)$.*

(ii) *The decomposition of I and τI into direct sums of ideal are in one-to-one correspondece via τ: that is if $I = \bigoplus_{\lambda \in \Lambda} I_\lambda$ then $\tau I = \bigoplus_{\lambda \in \Lambda} \tau I_\lambda$; and if $\tau I = \bigoplus_{\gamma \in \Gamma} J_\gamma$, then $I = \bigoplus_{\gamma \in \Gamma} (IJ_\gamma)$, and $\tau(IJ_\gamma) = J_\gamma$.*

(iii) *I is finitely generated if and only if τI is finitely generated (and in this case τI is generated by an idempotent, see* 1.8 *(f)).*

(iv) *If I is not finitely generated, then $\gamma I = \gamma(\tau I)$.*

PROOF. Let $(\varphi_\lambda, f_\lambda)_{\lambda \in \Lambda}$ be a projective basis for I and put $a_\lambda = \varphi_\lambda(f_\lambda)$.

(i) Easy calculations prove that $(a_\lambda)_{\lambda \in \Lambda}$ is a generating system for τI, satisfying to 1.13 (ii); hence τI is pure and projective. Moreover $\mathrm{Ann}\,(f_\lambda) \subseteq \mathrm{Ann}\,(a_\lambda)$, hence $\mathrm{Supp}\,(I) = \mathrm{Supp}\,(\tau I) \subseteq D(\tau I)$. The minimality of τI follows from 1.7 and $D(\tau I) = \mathrm{Supp}\,(I)$.

(ii) Easy. To conclude, observe that plainly $\gamma(\tau I) \leqslant \gamma(I)$; using the generating system $(a_\lambda)_{\lambda \in \Lambda}$ as above, from 1.13 (v) and (ii) it follows that we are reduced to Λ contable, i.e. $\gamma(I) \leqslant \aleph_0$; it remains then only to prove that if τI is generated by an idempotent e, then I is finitely generated; this is an easy computation $\left(e = \sum_{\lambda \in \Lambda(e)} e a_\lambda \Rightarrow x = xe = \right.$ $\left.= \sum_{\lambda \in \Lambda(e)} e\varphi_\lambda(x) f_\lambda, \text{ for every } x \in I\right)$.

As a corollary, we obtain a particular case of Kaplansky's theorem [K]:

COROLLARY 1. *A projective ideal is a direct sum of countably generated ideals.*

COROLLARY 2. *A projective ideal has a star finite generating system.*

PROOF. It is easy to see that such a system if $\{a_\lambda f_\mu : (\lambda, \mu) \in \Lambda \times \Lambda\}$, the meaning of a_λ, f_μ being as in the preceding proof. Observe that a pure ideal which contains a non zero divisor is necessarily A. Then (iii) above implies the following known fact ([B$_1$], p. 84).

COROLLARY 3. *A projective ideal which contains a non zero divisor is finitely generated.*

2. Applications.

2.1. If A is a pm-ring, then Max (A) is compact Hausdorff, and the spectral topology on Max (A) is the $\mathfrak{D}_M$-topology (1.2).

The zero-sets of Max (A) are the closed G_δ sets, hence the countably generated pure ideals are, in a pm-ring, exactly the ideals O^Z, where Z is a zero-set of Max (A).

THEOREM. *Let A be a pm-ring, and let I be a pure ideal of A. The following are equivalent*:

(i) *I is projective.*

(ii) $I = \bigoplus_{\gamma\in\Gamma} O^{Z_\gamma}$, *where each Z_γ is a zero set of* Max (A).

(iii) *$D_M(I)$ is paracompact* (Bkouche, [Bk_2]).

PROOF. Immediate consequence of 1.13. Observe also that $D_M(I)$ is a locally compact space, and recall that a locally compact space is paracompact if and only if it is a topological sum of σ-compact spaces.

2.2. In $C(X)$, pure and projective ideals are related to star-finite partitions of unity (cf. [Br_1], [D_2]).

PROPOSITION. *An ideal I of $C(X)$ is pure and projective if and only if it is generated by a family $(u_\lambda)_{\lambda\in\Lambda}$ of continuous functions such that $(u_\lambda|Cz(I))$ is a star-finite partition of unity on* $Cz(I) = \bigcup_{f\in I} Cz(f)$.

PROOF. This is essentially 1.13 (ii): we only have to prove that the a_λ's described there can be assumed positive; and this is easily done by replacing them with the functions u_λ defined by $u_\lambda(x) = |a_\lambda(x)|/\sum_{\mu\in\Lambda} |a_\mu(x)|$ for $x \in Cz(I)$, $u_\lambda(x) = 0$ otherwise.

2.3. PROPOSITION. *Let A be a ring. The following are equivalent*:

(i) *Every projective ideal of A is finitely generated.* (i.e., A is an F-ring, [V]).

(ii) *Every pure ideal of A is generated by an idempotent.*

(iii) *Every open* $\mathcal{S}$-stable *subset of* Spec (A) *is closed in* Spec (A).

(iv) *In the $\mathfrak{D}$-topology,* Spec (A) *is a finite sum of indiscrete spaces.*

PROOF. Use 1.13, 1.14. For (iv) (cf. [L_2]) observe that a compact space in which every open set is also closed is necessarily a finite sum of indiscrete subspaces. Equivalence of (i) and (ii) has been proved in [V]; (iii) and (iv) are found in [L_1]. Notice tat:

COROLLARY. *A pm-ring A is an F-ring* iff Max (A) *is finite*; *in particular*, $C(X)$ *is an F-ring iff X is finite* (we assume X Tychonoff).

2.4. *If a countably generated ideal I of the ring A is generated by idempotents, then* $I = \bigoplus_{n\in\mathbb{N}} e_n A$, *where each* e_n *is an idempotent.*

PROOF. Any ideal generated by idempotents is clearly pure. By the hypothesis, $D(I)$ is a union of clopen subsets of Spec (A). Since $D(I)$ is an F_σ, an obvious compactness argument shows that $D(I) = \bigcup_{n\in\mathbb{N}} D(b_n)$, where each b_n is an idempotent and $D(b_n) \subseteq D(b_{n+1})$. Letting $e_n = b_{n+1} - b_n$, each e_n is an idempotent, and $I = \bigoplus_{n\in\mathbb{N}} e_n A$.

PROPOSITION. *Let A be a ring. The following are equivalent*:

(i) *Every pure ideal is generated by idempotents* (i.e., *A is an f-ring*, [V]).

(ii) *The* $\mathfrak{D}$-topology *has the clopen subsets of* Spec (A) *as a basis.*

(iii) *Every projective ideal of A is a direct sum of finitely generated ideals*; *see also* 1.7, 1.9.

PROOF. 1.13, 1.14. Equivalence of (i) and (iii) is proved in [V] (there it is also remarked that S. Jøndrup has obtained a purely spectral characterization of f-rings).

Recall that a compact Hausdorff space is totally disconnected if and only if it has a clopen basis; if X is Tychonoff, then βX is totally disconnected if and only if X is (*strongly*) *zero-dimensional* [GJ, Ch. 14].

COROLLARY. *A pm-ring A is an f-ring iff* Max (A) *is totally disconnected*; $C(X)$ *is an f-ring iff X is strongly zero-dimensional.*

This is an algebraic characterization of strongly zero-dimensional spaces, as those spaces such that every projective ideal of $C(X)$ is a direct sum of finitely generated ideals.

2.5. We deduce here some more results on pm-rings.

PROPOSITION. *Let A be a* pm-ring.

(a) Max (A) *is hereditarily paracompact if and only if every pure ideal of A is projective.*

(*b*) Max (A) *is perfectly normal if and only if every pure ideal of* A *is countably generated.*

(*c*) *The following are equivalent*:

(i) Max (A) *has countable cellularity.*

(ii) *Every projective ideal of* A *is countably generated.*

(Recall that the cellularity of a space is the supremum of cardinalities of disjoint families of open susets of the space; recall also that a topological space is said to be hereditarily paracompact when every subspace of it is paracompact: this is equivalent to assume that every subspace is paracompact).

PROOF. (*a*), (*b*) 2.1. (*c*) Easy (by complete regularity of Max (A), for every disjoint family of open subsets of Max (A) there exists a disjoint family of cozero-sets with same cardinality).

3. Applications to $C(X)$.

For unexplained terminology in this section the reader is referred to [GJ].

3.1. If I is an ideal such that $I = I$, then $\tau I = I$. Calling *semiprime* an ideal which coincides with its radical, we observe that for every semiprime ideal I of $C(X)$ equality $I^2 = I$ holds (every $f \in C(X)$ has a cubic root in $C(X)$); thus $\tau I = I$ holds for every semiprime ideal of $C(X)$. Call z-free an ideal I of $C(X)$ such that $Cz(I) = X$ (thus z-free ideal = free ideal of [GJ]); since ([FGL]) every $\varphi \in \mathrm{Hom}_{C(X)}(I, C(X))$ is a multiplication by some $b \in C(X)$, we have $\tau I = I$ for every z-free ideal of $C(X)$. Thus (1.14):

PROPOSITION. *If a projective ideal* I *of* $C(X)$ *is either semiprime, or* z*-free, then it is a pure ideal.*

REMARK. In [Br_1] it is proved that projective ideals which are z-free, or z-ideals, are pure. The projective z-ideal associated in [Br_1] to a given projective ideal I is, of course, the trace ideal τI.

3.2. We now address the question of projectivity of some pure ideals of $C(X)$. First we need a simple Lemma.

LEMMA. *Let T be an open non-compact, paracompact subspace of the compact Hausdorff space Y. Let S be a subset of T such that* $\mathrm{cl}_Y(S) \not\subseteq T$. *Then S contains an infinite discrete subset Δ which is C*-embedded (see [GJ]) *in T; and* $\mathrm{cl}_Y(\Delta) \subseteq Y \setminus T$.

PROOF. White T as a disjoint union of open σ-compact subsets U_γ, $\gamma \in \Gamma$. If $S \cap U_\gamma$ is non-empty for infinitely many $\gamma \in \Gamma$, construct Δ by picking a point form each non empty $S \cap U_\gamma$; otherwise, there exists $\bar{\gamma} \in \Gamma$ such that $\mathrm{cl}_Y(S \cap U_{\bar{\gamma}}) \not\subseteq T$. Take $g \in C(Y)$ such $Cz_Y(g) = U_{\bar{\gamma}}$; the range $\{x_n: n \in \mathbf{N}\}$ of any sequence $x_n \in S \cap U_{\bar{\gamma}}$ such that $\lim_n g(x_n) = 0$ gives the required set Δ.

COROLLARY. *X is pseudocompact if and only if no z*-free *ideal of $C(X)$ is projective.*

PROOF. It is well known [GJ] that X is pseudocompact if it contains no C-embedded copy of an infinite discrete subspace, and also if and only if no non-empty zero set of βX is contained in $\beta X \setminus X$; the conclusion follows then from the above Lemma, with X in place of S, $\mathrm{Supp}_{\beta X}(I)$ in place of T, I being an hypothetical z-free projective X hence pure (3.1) ideal) and 2.1.

3.3. PROPOSITION. (*a*) *A z*-free *projective ideal of $C(X)$ is contained in at least $2^{\mathfrak{c}}$ maximal ideals. Thus, if $p \in \beta X \setminus X$, the ideal O^p is not projective.*

(*b*) *If p is non isolated in X, and O^p is projective, then every subsets S of $X \setminus \{p\}$ which contains p in its closure contains a sequence which converges to p. In particular, some sequence of $X \setminus \{p\}$ converges to p in X.*

(*c*) *A prime ideal of $C(X)$ is projective if and only if it is generated by an idempotent.*

PROOF. (*a*) Apply lemma 3.2 with $T = \mathrm{Supp}_{\beta X}(I)$, $S = X$; since $T \supseteq X$, Δ is C-embedded in X, hence C^*-embedded in X; then $|\mathrm{cl}_{\beta X}(\Delta) \setminus \Delta| = |\beta\Delta \setminus \Delta| = 2^{2^{|\Delta|}} \geqslant 2^{\mathfrak{c}}$.

(*b*) Apply lemma 3.2 with $T = \beta X \setminus \{p\}$; observe that βX is the one-point compactification of T.

(*c*) Since P is prime, its trace τP coincides with P(3.1); if P is projective, then $\tau P = P$ is pure and projective; then we have $P = O^p$, where O^p is the pure ideal corresponding to the maximal ideal M^p which contains P. By (*b*), there exists a sequence $\{x_n: n \in \mathbf{N}\}$

of distinct points of $X\setminus\{p\}$ which converges to p in X. Put $K = \{p\} \cup \cup \{x_n : n \in \mathbf{N}\}$ and define $h \in C(K)$ by means of $h(p) = 0$, $h(x_n) = = (-1)^n 2^{-n}$. Since K is compact, there exists $g \in C(\beta X)$ such that $g|K = h$. Put $f = g|X$. We have $f^+ f^- = 0$, but $f^+, f^- \notin O^p$. Then O^p cannot be prime.

REMARK 1. (c) is proved also in [Br_2], with a more direct argument.

REMARK 2. (c) holds for any uniformly closed φ-algebra [HJ], with almost exactly the same proof. It may fail for non uniformly closed φ-algebras: in the sub-φ-algebra A of $\mathbb{R}^{\mathbf{N}}$ consisting of eventually constant functions, the ideal of functions with finite support is a projective pure maximal ideal, countably but not finitely generated.

3.4. COROLLARIES. (a) *$C(X)$ is hereditary if and only if X is finite.*

(b) *The following are equivalent*:

(i) *X is compact and hereditarily paracompact* (see 2.5).

(ii) *Every pure ideal of $C(X)$ is projective.*

(c) *The following are equivalent*:

(i) *X is compact and perfectly normal.*

(ii) *Every pure ideal of $C(X)$ is countably generated.*

PROOF. (a) By 3.3 (c), M^p is projective iff it is generated by an idempotent. This forces X to be discrete, hence finite; but then $\beta X = X$; (b), (c). From 3.3 (a), projectivity of the pure ideals O^p for every p implies $X =$; the remaining statements now follow easily from 2.5.

REMARK. (a) is proved also in [Br1]; I have reproduced here Brooks-hear's proof, more direct than my original one.

3.5. In a ring A, a principal aA is projective if and only if Ann (a) is generated by an idempotent, i.e. if Supp (a) is open. It follows that $fC(X)$ is projective iff $\mathrm{Supp}_X (f)$ is open in X. Hence: every principal ideal of $C(X)$ is projective if and only if X is basically disconnected [GJ]. Since basically disconnected spaces are F-spaces, i.e. spaces such that every finitely generated ideal of $C(X)$ is principal, we have

PROPOSITION. *The following are equivalent*:

(i) $C(X)$ *is semihereditary* (i.e., every finitely generated ideal of $C(X)$ is projective).

(ii) $C(X)$ *is principally projective* (i.e., every principal ideal of $C(X)$ is projective).

(iii) X *is basically disconnected.*

REMARK. This proposition is proved also in [Br_2].

3.6. An ideal generated by idempotents is pure. And pure, finitely generated ideals are generated by idempotents.

Rings in which every ideal is generated by idempotents are exactly the *absolutely flat rings*, (called also Von Neumann regular rings; see, e.g. [GJ, 4K]). We have proved:

LEMMA. *Let* A *be a ring.*

The following are equivalent:

(i) *Every ideal of* A *is pure.*

(ii) A *is absolutely flat.*

In $C(X)$, the radical of a principal ideal $fC(X)$ is countably generated, having $\{|f|^{1/n}: n = 2, 3, 4, \ldots\}$ as a set of generators. If such an ideal is projective, then it is pure (3.1) and this implies $Z(f)$ open in X, as is easy to see. Recall that the P-spaces are exactly those spaces X for which $C(X)$ is a regular ring, and those in which every zero-set is open [GJ, 4K]; using also 1.11 it is easy to get the following:

PROPOSITION. *The following are equivalent*:

(i) X *is a* P*- space.*

(ii) *Every ideal of* $C(X)$ *is pure.*

(iii) *Every countably generated ideal of* $C(X)$ *is projective.*

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