RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 69 (1983), p. 277-288
<http://www.numdam.org/item?id=RSMUP_1983__69__277_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 69 (1983)

On a Property of the One-Dimensional Torus.

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RIASSUNTO - Il toro di dimensione uno è caratterizzato come l'unico gruppo abeliano compatto che contiene sottogruppi non minimali, ognuno dei quali non ammette alcuna topologia minimale meno fine della topologia indotta.

Introduction.

Compact groups can be characterized by some properties of their subgroups. A Hausdorff topological group (G, τ) (and its topology τ) is said to be *minimal* if G admits no Hausdorff group topologies strictly coarser than τ. Prodanov [P2] characterized the p-adic numbers $\mathbb{Z}_p$ as the only up to isomorphism infinite compact Abelian groups which induce minimal topology on every subgroup. Another characterization of the groups $\mathbb{Z}_p$ is given in [D1] as the only infinite compact groups in which every non-zero closed subgroup has finite index. It is well known that closed subgroups of minimal Abelian groups are minimal ([P2], [P5]) and all compact Hausdorff groups are minimal. Stojanov and the first author [DS] described all minimal groups in which every subgroup is minimal. They are isomorphic either to subgroups of $\mathbb{Z}_p$ for some p, or to rank-one subgroups of $\mathbb{Z}_p \times F_p$ for some p and some finite Abelian p-group F_p, or to direct sums $\bigoplus F_p$, where

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for every prime p F_p is a finite Abelian p-group and the sum is provided with the product topology.

In the present paper we give a similar characterization of the one-dimensional torus $\mathbb{T}^1$. In the first section we introduce the notion of minimalizable topology and give examples of minimalizable groups. In the second section we show that all minimalizable subgroups of $\mathbb{T}^1$ are minimal and every compact Abelian group which contains non-minimal subgroups and possesses this property is isomorphic to $\mathbb{T}^1$.

Notations.

We denote by $\mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_p, \mathbb{Q}, \mathbb{R}$ respectively the sets of prime, natural, integer, interger p-adic, rational and real numbers. $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional torus, $\mathbb{Z}(p^\infty)$ is the p-torsion part of $\mathbb{T}^1$.

Throughout the paper all groups are Abelian. Let G be a group, then $S(G)$ is the socle of G, $r(G)$ is the free rank of G and $r_p(G)$ is the p-rank of G. For $x \in G$ $\langle x \rangle$ is the subgroup of G generated by x. Let G be a topological group then $\hat{G}$ is the completion of G, G^* is the group of continuous characters of G and bG is the Bohr compactification of G. We denote by $\mathcal{T}_G$ the finest precompact topology on G. Finally we denote by τ_p the p-adic topology on $\mathbb{Z}$. In general definitions and notations follow those from [HR].

1. Minimalizable groups.

First of all we recall some facts about minimal groups. The following minimality criterion is given by Banaschewski [B] and Stephenson [St]. A subgroup G' of a topological group G is called *essential* if every nonzero closed subgroup H of G intersects non-trivially G'.

(*MC*) A dense subgroup G' of the Hausdorff topological group G is minimal iff G is minimal and G' is essential in G.

It was shown that in many cases the minimal groups are precompact ([P3], [P4], [P5] and [S3]). For example all complete minimal reduced torsion-free groups are compact and every minimal topology is precompact on groups G with $r(G/D) < c$, where D is the maximal divisible subgroup of G ([S3]). All known minimal Abelian groups at present are precompact.

The existence of minimal topology on a given group furnish information about the algebraic structure of the group. For instance it follows immediately from (MC) that a torsion free-group G admits precompact topologies iff G can be embedded as an essential dense subgroup in $\mathbf{Q}^{*\tau} \times \prod_p \mathbf{Z}_p^{\tau_p}$ for some cardinals τ and τ_p ($p \in \mathbf{P}$). In particular a non-reduced torsion-free group of finite rank does not admit minimal topologies (for rank one see [P2]).

Now we remind briefly what is known about the algebraic structure of groups admitting minimal topologies. It follows from (MC) that minimal groups contain the socle of its completion and the p-primary component of this socle are compact. Therefore for a minimal group G

(1) for every $p \in \mathbf{P}$, $r_p(G)$ is either finite or $r_p(G) = 2^{\varrho_p}$ with $\varrho_p \geqq \aleph_0$.

According to [D2] a divisible group G admits minimal topologies in the following two cases:

a) $r(G) = 2^{\sigma}$ with $\sigma \geqq \aleph_0$ and (1) holds with $\varrho_p \leqq \sigma$.

b) $r(G) < c$ and there exist $n \in \mathbf{N}$ and a subset π of $\mathbf{P}$ such that card $\pi \leqq r(G)$ and $r_p(G) = n$ for $p \notin \pi$ and $r_p(G) = n - 1$ for $p \in \pi$.

Let now G be a periodic group with maximal divisible subgroup D, denote by R_p the p-primary component of G/D. By the results from [DP] and [S3] G admits minimal topologies iff each R_p is bounded and there exists a non-negative integer n such that for every p $r_p(G) \geqq n \geqq r_p(D)$ and (1) holds. These results are extended in [D3] to groups G of rank less than c.

In [S2] the cardinalities of the minimal precompact groups are characterized and the free groups admitting minimal precompact topologies are described.

Till now we discussed the existence of minimal topologies on a group G which a priori has no topology.

DEFINITION 1.1. A topological group (G, τ) and the topology τ are said to be *minimalizable* if G admits a minimal topology σ coarser than τ.

Non-Hausdorff topologies are obviously non-minimalizable. The above discussion provides a lot of examples of non-minimalizable groups, since any group which does not admit minimal topologies is always non-minimalizable. On the other hand various groups are minimalizable or non-minimalizable depending on the topology considered on them.

EXAMPLE 1.2. Every infinite topological group G, which possesses a smallest closed non-zero subgroup H, is non-minimalizable. Indeed, assume there is a minimal topology τ on G coarser than the given one. Clearly H is essential in its τ-closure, hence H is a minimal group with no proper closed subgroups. By a theorem of Mutylin [M] $H \cong \mathbb{Z}(p) = \mathbb{Z}/p\mathbb{Z}$ for some $p \in \mathbb{P}$. Now the periodic part of G is an essential extension of H, so it can be embedded in $\mathbb{Z}(p^\infty)$ which does not admit minimal topologies. Hence there exists a non-periodic element $x \in G$. Denote by G' the τ-closure of the subgroup generated by x and H in G. Then G' is a minimal precompact group ([P5]) and every non-zero closed subgroup of G' contains H. By (MC) every closed non-zero subgroup of the compact group $\hat{G}'$ contains H. This implies that $\hat{G}'$ is a finite cyclic p-group which contradicts the choice of x.

EXAMPLE 1.3 ([P1]). The additive group of every linear topological space over $\mathbb{R}$ is non-minimalizable.

EXAMPLE 1.4. *Maximal* topology on a group G is a maximal Hausdorff non-discrete topology on G. The infimum of all maximal topologies on G is called *submaximal topology* of G and is denoted by $\mathcal{M}_G$ ([P4]). Denote by $\mathcal{N}_G$ the topology on G with fundamental system of neighbourhoods of 0 $\{nG + S(G)\}_{n=1}^{\infty}$. It is shown in [P4] that $\mathcal{M}_G = \sup\{\mathcal{T}_G, \mathcal{N}_G\}$ and every minimal topology on G is coarser than $\mathcal{M}_G$. Therefore a topology τ on G is minimalizable iff $\inf\{\tau, \mathcal{M}_G\}$ is minimalizable, so studying minimalizable topologies on G one can consider only topologies coarser then $\mathcal{M}_G$.

For a topological group (G, τ) the Bohr compactification bG is the completion of $(G, \bar{\tau})$, where $\bar{\tau} = \inf\{\tau, \mathcal{T}_G\}$ ([HR]). Let $b_G\colon G \to bG$ be the canonical homomorphism, then b_G is injective iff $\bar{\tau}$ is Hausdorff, i.e. the continuous characters of (G, τ) separate the points. The Bohr compactification is characterized by the following universal property: for every continuous homomorphism $f\colon G \to H$, where H is a compact group and the image of f is dense in H, there exists a continuous homomorphism $g\colon bG \to H$ such that $f = g \circ b_G$.

EXAMPLE 1.5. Let G be a group on which every minimal topology is precompact, then every topology τ on G, for which $b_G\colon (G, \tau) \to bG$ is not injective, is non-minimalizable. In [P6] an elementary example of a Hausdorff topology on $\mathbb{Z}^{(N)}$ is given for which the Bohr compactification is trivial, so this topology is non-minimalizable.

In the next proposition the non-minimalizable topologies on $\mathbb{Z}$ are characterized by means of the Bohr compactification.

PROPOSITION 1.6. *Let τ be a topology on $\mathbb{Z}$, G be the Bohr compactification of $(\mathbb{Z}, \tau)$ and C be the connected component of zero in G. Then τ is non-minimalizable iff*

$$G/C \cong \prod_{p\in P} \mathbb{Z}(p^{n_p}) . \tag{2}$$

PROOF. By virtue of example 1.5 τ is non-minimalizable if $\bar{\tau} = \inf\{\tau, \mathfrak{T}_{\mathbb{Z}}\}$ is not Hausdorff. In this case G is finite, so (2) holds obviously. From now on we suppose that $\bar{\tau}$ is Hausdorff, i.e. the canonical homomorphism $b_{\mathbb{Z}}$: $(\mathbb{Z}, \tau) \to G$ is injective. It is known that the minimal topologies on $\mathbb{Z}$ are exactly the p-adic ones ([P1] and [P5]). By the universal property of G and the total disconnectedness of $\mathbb{Z}_p$, it follows that τ is minimalizable iff there exists a continuous epimorphism from G/C onto $\mathbb{Z}_p$, for some p. Now G/C is a monothetic totally disconnected group, hence epimorphisms of the above kind do not exist iff (2) holds. Q.E.D.

The above proposition shows that there does not exist a finest non-minimalizable topology on $\mathbb{Z}$. Is this true for any group admitting minimal topologies? Another curious property of the precompact topologies on $\mathbb{Z}$ is the following: every precompact topology on $\mathbb{Z}$ is the supremum of a non-minimalizable topology and some minimal topologies. For the proof consider a Hausdorff precompact topology τ on $\mathbb{Z}$ and the completion G of $(\mathbb{Z}, \tau)$. Then $\tau \geqslant \tau_p$ iff $G = G' \times G_p$, where $G_p \cong \mathbb{Z}_p$ and G' does not contain subgroups isomorphic to $\mathbb{Z}_p$. Denote by Π the set of all primes p such that $\tau \geqslant \tau_p$. Then $G = G'' \times \prod_{p\in\Pi} G_p$ with $G_p = \mathbb{Z}_p$ for $p \in \Pi$ and G'' contains no copies of $\mathbb{Z}_p$ $(p \in \mathbf{P})$, *i.e.* G'' satisfies (2). Now $\tau = \sup\{\tau', \tau_0\}$, where $\tau_0 = \sup\{\tau_p\colon p \in \Pi\}$ and τ' is the topology induced on $\mathbb{Z}$ by the embedding $\mathbb{Z} \to G/\prod_{p\in\Pi} G_p \cong G''$.

Now we give a criterion analogous to (MC) for minimalizable groups.

PROPOSITION 1.7. *Let G be a Hausdorff precompact group. Then G is minimalizable iff there exists a closed subgroup H of $\hat{G}$ satisfying:*

$$G \cap H = 0 \tag{3}$$

and maximal with this property.

PROOF. By the precompactness of G, $\hat{G} = bG$. Suppose more generally that (G, τ) is a topological group for which $b_G\colon (G, \tau) \to bG$ is injective. Then to every closed subgroup H of bG satisfying (3), corresponds a precompact Hausdorff topology σ on G coarser than τ: the induced topology by the embedding $G \to \hat{G}/H$. On the other hand for every precompact topology σ on G with $\sigma \leqslant \tau$, consider the compact completion $\tilde{G}$ of (G, σ). By the universal property of bG there exists a continuous homomorphism $g\colon bG \to \tilde{G}$. Now $H = \ker g$ is a closed subgroup of bG satisfying (3) and clearly σ is the topology corresponding to H by the above correspondence. This correspondence is one-to-one and order-reversing, therefore closed subgroups H of bG which are maximal with (3) correspond to minimal precompact topologies on G coarser than τ. Q.E.D.

By virtue of this proposition a Hausdorff precompact group G is non-minimalizable iff, for every closed subgroup H of $\hat{G}$ satisfying (3), there exists a closed subgroup of $\hat{G}$ containing properly H and satisfying (3). The next corollary follows directly from the above proposition.

Topological groups in which every ascending (descending) chain of closed subgroups stabilizes are called A.C.C. (D.C.C.) groups.

COROLLARY 1.8. *Every subgroup of a compact* A.C.C. *group is minimalizable.*

In fact, if G' is a compact group and G a subgroup of G', then G is minimalizable if there exists a closed subgroup H of G' satisfying (3) and such that G'/H is a compact A.C.C. group.

If G is a topological group, then $G^* \cong (bG)^*$. By the above corollary, G is minimalizable if G^* is a D.C.C. group and separates the points of G.

The completion of any minimal finitely generated group is a compact A.C.C. group [P2]. Therefore, by virtue of the above corollary, a finitely generated topological group G is minimalizable iff G can be mapped by a continuous monomorphism in a compact A.C.C. group. In particular all Hausdorff quotients of a minimalizable finitely generated group are minimalizable. In the next section we give examples of minimalizable groups which have not this property. On the other hand, closed subgroups of minimalizable groups may be non-minimalizable. Take for example the subgroup $\mathbb{Z}(p^\infty)$ of $\mathbf{T}^1$ provided with the discrete topology.

It is well known that the product of minimal groups is often non-minimal. Doitchinov [Do] showed that, for every $p \in \mathbb{P}$, $(\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_p)$ is not minimal. Later Stojanov [S3] established that, whith respect to products, $(\mathbb{Z}, \tau_p)$ are the «worst» behaving minimal groups, i.e. if for some minimal group H all products $H \times (\mathbb{Z}, \tau_p)$ are minimal, then for any minimal group G the product $H \times G$ is minimal. On the other hand $(\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_p)$ is minimalizable, according to corollary 1.8. Moreover the following lemma shows that from the point of view of minimalizable topologies, the groups $(\mathbb{Z}, \tau_p)$ are «well» behaving. When we consider short exact sequence of topological groups, we always assume that all homomorphisms involved are continuous and open on their image.

LEMMA 1.9. *Let* (4) $0 \to G \to A \to H \to 0$ *be an exact short sequence of Hausdorff topological groups. If G is minimal and precompact and H is minimal and finitely generated, then A is minimalizable.*

PROOF. From the information given about H we use only that $\hat{H}$ is a compact A.C.C. group and H is essential in $\hat{H}$, according to (MC). By the exactness of (4) A is precompact. Consider the compact completion $\hat{A}$ of A. The closure of G in $\hat{A}$ is compact, so it is the completion $\hat{G}$ of G. Now $H = A/G$ can be identified with its dense image in $\hat{A}/\hat{G}$. This is why we set $\hat{H} = \hat{A}/\hat{G}$ and, by abuse of laguage, we consider H as subgroup of $\hat{H}$. In this way we get the exact short sequence:

$$0 \to \hat{G} \to \hat{A} \xrightarrow{\varphi} \hat{H} \to 0 . \tag{5}$$

Suppose N is a closed subgroup of $\hat{A}$ satisfying:

$$N \cap A = 0 . \tag{6}$$

Then $N \cap G = 0$ and by virtue of the minimality of G, (MC) yields $N \cap \hat{G} = 0$. Thus $\varphi|_N \colon N \to \varphi(N) \subset \hat{H}$ is an isomorphism. Moreover, if $N' \supsetneq N$ are closed subgroups of $\hat{A}$ satisfying (6), then $\varphi(N') \supsetneq \varphi(N)$. Indeed, take $x \in N' \setminus N$ and assume $\varphi(x) \in \varphi(N)$. Then by (5), $x = y + g$, where $y \in N$ and $g \in \hat{G}$. Now $g = x - y \in N' \cap \hat{G} = 0$, which gives $x = y \in N$, in contradiction with the choice of x. The family $\{\varphi(N)\colon N$ is a closed subgroup of $\hat{A}$ satisfying (6)$\}$ of closed subgroups of $\hat{H}$ has a maximal element since $\hat{H}$ satisfies A.C.C. Therefore there exists a closed subgroup N of $\hat{A}$ which is maximal with the property (6). By virtue of proposition 1.7 A is minimalizable. Q.E.D.

COROLLARY 1.10. *Let G and H be minimalizable groups. If H is finitely generated and G is precompact, then $G \times H$ is minimalizable.*

In particular, if G and H are finitely generated, then $G \times H$ is minimalizable iff G and H are minimalizable. The following example shows that in corollary 1.10 one cannot replace H-finitely generated with $r(H)$-finite.

EXAMPLE 1.11. Fix two distinct prime numbers q and r and consider a closed subgroup L of Q^* isomorphic to $\prod_{p \neq r} \mathbb{Z}_p$. Now set $\hat{G} = \mathbb{Q}^*/L$, then the periodic part of $\hat{G}$ is $\bigoplus_{p \neq r} \mathbb{Z}(p^\infty)$ and $\hat{G}$ contains a closed subgroup K isomorphic to $\mathbb{Z}_r$. Choose an arbitrary non zero element x of K and denote by G the subgroup of $\hat{G}$ generated by x and $\left(\bigoplus_{p \neq r,q} \mathbb{Z}(p)\right) \oplus \mathbb{Z}(q^\infty)$. Then G is a dense minimal subgroup of $\hat{G}$ and G^2 is non minimalizable.

The group $G = \left(\bigoplus_{p \neq r,q} \mathbb{Z}(p)\right) \oplus \mathbb{Z}(q^\infty) \oplus \mathbb{Z}$ considered in the above example is the minimal possible in the following sense. If G_1 is a minimal subgroup of G and G_1^2 is non-minimalizable, then $G_1 = \left(\bigoplus_{p \neq r,q} \mathbb{Z}(p)\right) \oplus \mathbb{Z}(q^\infty) \oplus k\mathbb{Z}$ for some $k \in \mathbb{N}$. On the other hand if we take $\mathbb{Z}(p^\infty)$ instead of $\mathbb{Z}(p)$ for every $p \neq q,r$ and the subgroup of $\mathbb{Q}^*$ generated by $\{1/p^n \cdot x\}_{n=1}^\infty$ instead of $\langle x \rangle \simeq \mathbb{Z}$, we get a totally minimal group G such that G^2 is non-minimalizable.

2. Topological characterization of T^1.

In this section will be established that every minimalizable subgroup of T^1 is minimal and this property characterizes T^1, up to isomorpism, in the class of all compact groups which admit non-minimal subgroups. Remark that it is easy to characterize all compact groups in which every minimal subgroup is compact—these groups are isomorphic to products $\mathbb{Z}(p_1)^{\alpha_1} \times \ldots \times \mathbb{Z}(p_n)^{\alpha_n} \times F$, where F is a finite group.

By (MC) a Hausdorff group containing a dense minimal subgroup is minimal too. The next example shows that this is not true for minimalizable groups.

EXAMPLE 2.1. Let $\mathbb{P}'$ be an infinite subset of $\mathbb{P}$ and $\mathbb{P}_1 \subset \mathbb{P}_2 \subset \ldots \subset \mathbb{P}_n \subset \ldots$ be an ascending chain of subsets of $\mathbb{P}'$ such that $\mathbb{P}' = \bigcup_{n=1}^{\infty} \mathbb{P}_n$

and for every $n \in \mathbb{N}$ $\mathbb{P}_{n+1} \setminus \mathbb{P}_n$ is infinite. For $n \in \mathbb{N}$ consider the subgroups of $\hat{G} = \prod_{p \in \mathbb{P}'} \mathbb{Z}(p)$ $K_n = \bigoplus_{p \in \mathbb{P}_n} \mathbb{Z}(p)$ and $N_n = \prod_{p \notin \mathbb{P}_n} \mathbb{Z}(p)$. Clearly

$$K_n \cap N_n = 0 \tag{7}$$

for every $n \in \mathbb{N}$ and N_n is the greatest closed subgroup of $\hat{G}$ with this property (to see this it is enough to to mention that every closed subgroup N of $\hat{G}$ has the form $\prod_{p \in \mathbb{P}'} N_p$, where each N_p is a subgroup of $\mathbb{Z}(p)$). Now for every $p \in \mathbb{P}'$ fix a generator c_p of $\mathbb{Z}(p)$. Consider for every $n \in \mathbb{N}$ the element x_n of $\hat{G}$ such that $x_n(p) = 0$ for $p \in \mathbb{P}_n$ and $x_n(p) = c_p$ for $p \notin \mathbb{P}_n$, consider also the element x_0 of $\hat{G}$ with $x_0(p) = c_p$ for every $p \in \mathbb{P}'$. Finally set $G_0 = \langle x_0 \rangle$ and for $n \in \mathbb{N}$ $G_{2n-1} = \langle x_0, x_1, \ldots, x_{n-1} \rangle + K_n$, $G_{2n} = \langle x_0, x_1, \ldots, x_n \rangle + K_n$. Obviously $x_n \in N_n \setminus N_{n+1}$ and $x_0, x_1, \ldots, x_n, \ldots$ are linearly independent, so $G \cong \mathbb{Z}^n \oplus K_n$ and $G_{2n} \cong \mathbb{Z}^{n+1} \oplus K_n$ algebraically. This yields by virtue of (7) that $G_{2n-1} \cap N_n \subset \subset \langle x_0, \ldots, x_{n-1} \rangle \cap N_n = 0$ hence N_n is the greatest closed subgroup of $\hat{G}$ satisfying $G_{2n-1} \cap N_n = 0$. By proposition 1.2 G_{2n-1} is minimalizable for every n. Remark that the topology of G_{2n-1} majorizes the minimal topology on G_{2n-1} induced by the embedding $G_{2n-1} \to \hat{G}/N_n \cong \cong \prod_{p \in \mathbb{P}_n} \mathbb{Z}(p)$. Next we prove that for every n the subgroup G_{2n} is non-minimalizable. Remark first that for a closed subgroup $N = \prod_{p \in S} \mathbb{Z}(p)$ of $\hat{G}$ (here $S \subset \mathbb{P}'$) $N \cap G_{2n} = 0$ iff $S \cap \mathbb{P}_n = \emptyset$ and $\mathbb{P}' \setminus (S \cup \mathbb{P}_n)$ is infinite. This yields that there do not exist closed subgroups N of $\hat{G}$ satisfying $N \cap G_{2n} = 0$ and maximal with this property.

The group G_1 in the above example is minimalizable and K_1 is a closed subgroup of G_1 such that G_1/K_1 is non-minimalizable.

THEOREM 2.2. *A compact group G contains non-minimal subgroups and has the property*

(*P*) *every minimalizable subgroup of G is minimal*,

iff G is isomorphic to $\mathbb{T}^1$.

PROOF. By (*MC*) an infinite subgroup of $\mathbb{T}^1$ is minimal iff it contains the socle of $\mathbb{T}^1$, thus $\mathbb{T}^1$ contains a lot of non-minimal subgroups. To verify (*P*) we have to show that all they are non-minimalizable.

In fact, assume H is an infinite subgroup of $\mathbb{T}^1$ with $S(\mathbb{T}^1) = = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p) \not\subset H$, hence $\mathbb{Z}(p) \cap H = 0$ for some $p \in \mathbb{P}$. Let now N be a closed subgroup of $\mathbf{T}^1$ with $N \cap H = 0$. Then $N \neq \mathbf{T}^1$, so $\mathbb{Z}(p^\infty) \cap N = \mathbb{Z}(p^{n-1})$ for some $n \in \mathbf{N}$. Now set $N' = N + \mathbb{Z}(p^n)$, obviously $pN' \subset N \underset{\neq}{\subset} N'$. Thus $p(N' \cap H) \subset N \cap H = 0$, so $N' \cap H \subset \mathbb{Z}(p) \cap \cap H = 0$. This shows that H is non-minimalizable according to proposition 1.7.

Next we show that every compact group satisfying (P) is isomorphic either to zome $\mathbb{Z}_p$ or to $\mathbf{T}^1$. Since every subgroup of $\mathbb{Z}_p$ is minimal this proves the theorem. During the first steps we suppose only that G is a minimal precompact group satisfying (P) and this will give a lot of information about the completion $\hat{G}$. Only in the final part of the proof we assume that $G = \hat{G}$ is compact.

Suppose $\hat{G}$ contains a copy of $\mathbb{Z}_p^2$ for some $p \in \mathbf{P}$. By the minimality of G each copy of $\mathbb{Z}_p$ contains a copy of $(\mathbb{Z}, \tau_p)$ in G, so G contains $(\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_p)$ which is minimalizable by corollary 1.8 and non-minimal. This contradicts (P), therefore for every p $\hat{G}$ contains no copy of $\mathbb{Z}_p^2$.

Assume now that $r_p(\hat{G})$ is infinite for some p, then G contains $\mathbb{Z}(p)^{\mathbf{N}}$ because of the minimality of G and $S(G) = S(\hat{G})$. Take a dense hyperplane L of $\mathbb{Z}(p)^N$ considered as a linear space over $GF(p)$. This is possible since $\mathbb{Z}(p)^N$ contains at least 2^c hyperplanes and only c closed ones, according to the duality of Pontrjagin. Then L is not minimal since does not contain the socle of $\hat{L} = \mathbb{Z}(p)^N$. On the other hand $\hat{L}/L$ is finite, hence L is minimalizable. This contradicts (P), therefore all $r_p(\hat{G})$ are finite. The structure of compact groups $\hat{G}$ such that for every p $r_p(\hat{G})$ is finite and $\hat{G}$ does not contain copies of $\mathbb{Z}_p^2$ is studied in [P3]. In particular there exist $n \in \mathbf{N}$, $\pi \subset \mathbf{P}$ and an exact short sequence

$$0 \to \prod_{p\in\pi} \mathbb{Z}_p \times \prod_{p\in\mathbf{P}} F_p \to \hat{G} \xrightarrow{\varphi} T^n \to 0\,, \tag{8}$$

where each F_p is a finite p-group.

Next we show that card $\pi \leqq 1$. In fact, assume there are at least two distinct primes p and q in π, i.e. $\mathbb{Z}_p \times \mathbb{Z}_q$ is contained in $\hat{G}$. Then by the minimality of G $\mathbb{Z}_p \cap G$ contains a copy of $(\mathbb{Z}, \tau_p)$ and $\mathbb{Z}_q \cap G$ contains a copy of $(\mathbb{Z}, \tau_p)$. Hence $K = (\mathbb{Z}, \tau_p) \times (\mathbb{Z}, \tau_q)$ is contained in G. Denote by D the diagonal subgroup of K, then D is minimalizable by corollary 1.8 and non-minimal, since the induced by K topology on D is sup $\{\tau_p, \tau_q\}$.

Consider now the case $\pi = \{p\}$ and take a prime $q \neq p$. Assume $F_q \neq 0$, then $\mathbb{Z}(q)$ is contained in $\hat{G}$ and so $\mathbb{Z}_p \times \mathbb{Z}(q)$ is contained in $\hat{G}$

with its product topology. We remark here that if A and B are subgroups of a Hausdorff topological group H, such that $A \cap B = 0$ and the product topology on $A + B$ is minimal, then the induced by H topology on $A + B$ coincides with the product topology. Returning to our case we see that the minimality of G yields $S(\hat{G}) \subset G$, hence $\mathbf{Z}(q) \subset S(F_q) \subset G$. Choose a non-zero element x in $G \cap \mathbf{Z}_p$, then $\langle x \rangle \cong$ $\cong (\mathbf{Z}, \tau_p)$ and $G' = (\mathbf{Z}, \tau_p) \times \mathbf{Z}(q)$ is contained in G provided with the product topology (the product of a minimal group by a compact group is always minimal [Do]). Choose a non-zero element z in $\mathbf{Z}(q)$, then $\mathbf{Z} \cong \langle x + z \rangle$ is minimalizable by corollary 1.8 and non-minimal. This contradiction shows that $F_q = 0$ for all $q \neq p$ in (8). This implies $n = 0$ in (8), i.e. $\hat{G} = \mathbf{Z}_p \times F_p$. By corollary 1.8 every subgroup of $\hat{G}$ is minimalizable, on the other hand if $r(G) > 1$ and $F_p \neq 0$, then G contains non-minimal subgroups according to [DS]. Therefore in the case $\pi = \{p\}$ (P) implies that either G is a rank-one subgroup of $\mathbf{Z}_p \times F_p$, where F_p is a non-trivial finite p-group, or G is a subgroup of $\mathbf{Z}_p$. In both cases all subgroups of G are minimal according to [DS]. If $G = \hat{G}$ is compact only the second case is possible.

From now on we assume $G = \hat{G}$ and satisfies (P). We showed above that $\pi = \{p\}$ implies $G \cong \mathbf{Z}_p$. Consider now the case $\pi = \emptyset$ and set $S = \{p \in \mathbf{P}: F_p \neq 0\}$. According to example 2.1 S is finite, otherwise G contains non-minimal minimalizable subgroups. By the finiteness of S (8) splits, hence $G = \mathbf{T}^n \times F$, where F is a finite group.

Next we show that $n = 1$ and $F = 0$. Since (P) is hereditary we can assume that $F = \mathbf{Z}(p)$ for some p, so it is enough to show that $G' = \mathbf{T}^1 \times \mathbf{Z}(p)$ does not satisfy (P). Denote by D the diagonal subgroup of $\mathbf{Z}(p)^2$ in G', then $H = \left(\bigoplus_{q \neq p} \mathbf{Z}(q)\right) \oplus D$ is dense in G' and non-minimal since $H \not\supset S(G')$. This contradicts (P) since H is minimalizable by the maximality of F as a closed subgroup of G' with $F \cap H = 0$. In this way $F = 0$ was proved. In the same way can be shown that $n = 1$, hence $G \cong \mathbf{T}^1$. Q.E.D.

We have proved in fact that if G is a minimal precompact group satisfying (P), then either all subgroups of G are minimal or $\hat{G}$ is a compact group with a short exact sequence (8) where $\pi = \emptyset$ (compact groups with this property are studied in [DP]). In the case $\hat{G}$ is connected and one-dimensional it can be shown that G satisfies (P) iff each non-periodic element of G generates a dense subgroup of G, i.e. for every non-periodic element x of G $\langle \varphi(x) \rangle$ is dense in $\mathbf{T}^1$.

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Manoscritto pervenuto in redazione il 22 ottobre 1982.