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# On Weighted Estimated for Some Systems of Partial Differential Operators.

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## Introduction.

Let $\Omega$ be an open set in $\mathbb{R}^n$ and let $A(x, D): \mathcal{E}^p(\Omega) \to \mathcal{E}^q(\Omega)$ be a linear partial differential operator with smooth coefficients in $\Omega$.

We want to solve the equation

$$(1) \qquad u \in \mathcal{E}^p(\Omega), \quad A(x, D)u = f \quad \text{on } \Omega$$

when the right hand side $f \in \mathcal{E}^q(\Omega)$ satisfies suitable integrability conditions, that we assume to be of the form

$$(2) \qquad B(x, D)f = 0$$

for a differential operator

$$B(x, D): \mathcal{E}^q(\Omega) \to \mathcal{E}^r(\Omega) \quad \text{with } B(x, D) \circ A(x, D) = 0 .$$

This problem generalizes that of the integrability of closed exterior differential forms on a differentiable manifold or of closed antiholomorphic forms on a complex manifold.

This last problem in particular (Dolbeault complex), related to the solution of E. E. Levi problem, motivated many researches on

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overdetermined systems. In 1952 Garabedian and Spencer [6] introduced the $\bar{\partial}$-Neumann problem, a non-elliptic boundary value problem that by a regularity theorem of Kohn and Nirenberg [10] yielded solvability of (1), (2) for $\bar{\partial}$ in strictly pseudoconvex domains. This kind of approach was pursued in full generality, in the context of the theory of pseudodifferential operators, by Hörmander in [9].

In this paper I want to outline the extension to general complexes of an alternative method, also developed for the study of $\bar{\partial}$, but not implying solving the $\bar{\partial}$-Neumann problem. It consists in the use of a priori estimates involving weight functions, that are related to a method developed by Carleman [5] to prove uniqueness for solutions of the Cauchy problem. The idea of using this method was suggested to Andreotti and Vesentini [3], [4] by the observation that problem (1), (2) is easily dealt with in the case of compact manifolds without boundary and then a next reasonable step was to investigate manifolds endowed with a complete metric (the weight function played an essential role for the completeness of the metric). For the use of weight functions for $\bar{\partial}$, cf. also Hörmander [7] and [8].

While the two methods are giving equivalent results for $\bar{\partial}$, it turns out that the first, having stronger implications (regularity up to the boundary) requires a priori estimates more difficult to establish, while it cannot be applied directly on domains either unbounded or with non smooth boundaries.

## 1. Sobolev spaces with weights and regularity theorems.

*a.* Let $\Omega$ be an open set in $\mathbb{R}^n$ and let $\psi\colon \Omega \to \mathbb{R}$ be a $C^\infty$-function. We set

$$\langle\psi\rangle = (1 + |\operatorname{grad}\psi|^2)^{1/2}.$$

If $m$ is a nonnegative integer, we denote by $W^m(\Omega, \psi)$ the space of functions $u$ in $W^m_{loc}(\Omega)$ ($=$ space of functions that are locally square summable with all weak derivatives up to order $m$) for which is finite the norm:

$$\|u\|_{m,\psi} = \Big(\sum_{|\alpha|\leqslant m} \int_\Omega \langle\psi\rangle^{2(m-|\alpha|)} |D^\alpha u|^2 e^{-\psi}\, dx\Big)$$

This is the norm associated to the scalar product

$$(u, v)_{m,\psi} = \sum_{|\alpha|\leqslant m} \int_{\Omega} \langle\psi\rangle^{2(m-|\alpha|)} D^\alpha u\, \overline{D^\alpha v}\, e^{-\psi}\, dx\,,$$

that gives to $W^m(\Omega, \psi)$ a structure of Hilbert space.

We also set $W^\infty(\Omega, \psi) = \varprojlim_m W^m(\Omega, \psi)$ with the Fréchet topology of inverse limit of a sequence of Hilbert spaces.

We will restrict our consideration to $\sigma$-smooth open subsets of $\mathbf{R}^n$, i.e. such that there exists a $C^\infty$ function $\psi\colon \Omega \to \mathbf{R}$ with the properties:

(3) $\forall c \in \mathbf{R}$ the set $\Omega_c = \{x \in \Omega \,|\, \psi(x) < c\}$ is relatively compact in $\Omega$;

(4) the set $\{x \in \Omega \,|\, d\psi(x) = 0\}$ is a compact subset of $\Omega$;

and to the class $\Psi(\Omega)$ of weight functions $\psi$ that satisfy (3), (4) and moreover

(5) $\forall$ integer $m \geqslant 0$ and real $\varepsilon > 0$ we can find a constant $c(m, \varepsilon)$ such that

$$\sum_{|\alpha|\leqslant m} |D^\alpha \psi(x)| \leqslant c(m, \varepsilon) \langle\psi\rangle^{1+\varepsilon} \qquad \text{on } \Omega\,.$$

The following lemma is fundamental for the use of weight functions:

LEMMA 1. *Assume that $\Omega$ is $\sigma$-smooth and let $\varphi \in C^\infty(\Omega, \mathbf{R})$ satisfy* (3) *and* (4).

*Then for every upper semicontinuous function $\lambda\colon \Omega \to \mathbf{R}$ we can find a $C^\infty$ function $h\colon R \to R$ such that*

$$\psi = h(\varphi) \in \Psi(\Omega) \qquad \textit{and} \qquad \psi \geqslant \lambda \qquad \textit{on } \Omega\,.$$

Let $m$ be either a nonnegative integer or $+\infty$. From the previous lemma we obtain the following:

PROPOSITION 1. *If $\Omega$ is $\sigma$-smooth and $\varphi \in C^\infty(\Omega, \mathbf{R})$ satisfies* (3) *and* (4), *then for any sequence $\{f_n\}$ in $W^m_{\mathrm{loc}}(\Omega)$ we can find $h \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $\psi = h(\varphi) \in \Psi(\Omega)$ and $f_n \in W^m(\Omega, \psi)$, $\forall n$. If moreover $f_n \to g$ in $W^m_{\mathrm{loc}}(\Omega)$, then we can choose $h$ in such a way that $f_n \to g$ in $W^m(\Omega, \psi)$.*

This proposition implies in particular that $W^m_{\mathrm{loc}}(\Omega)$ is the direct limit of the spaces $W^m(\Omega, \psi)$ for $\psi$ in $\Psi(\Omega)$.

Having fixed $\psi$ in $\Psi(\Omega)$, we will also consider for non negative integers $m$ and real $\delta$, the spaces $W^{m,\delta}(\Omega, \psi) = W^m(\Omega, \psi + \delta \ln \langle\psi\rangle)$.

By linear interpolation we consider also the spaces $W^{s,\delta}(\Omega, \psi)$ for $s$ real $\geqslant 0$. After identifying the dual of $W^0(\Omega, \psi)$ with itself by Riesz isomorphism, we define the space $W^{s,\delta}(\Omega, \psi)$ for $s < 0$ as the dual of $W^{-s,-\delta}(\Omega, \psi)$; as the Riesz isomorphism yields natural inclusions $W^{s,\delta}(\Omega, \psi) \hookrightarrow \mathfrak{D}'(\Omega)$, we identify all these spaces to spaces of distributions. We denote by

$$\|u\|_{s,\psi,\delta}$$

a continuous norm in $W^{s,\delta}(\Omega, \psi)$, $(s, \delta \in R)$.

The spaces we have introduced have the following properties:

PROPOSITION 2. *For every $s, \delta \in \mathbb{R}$ and $\psi \in \Psi(\Omega)$, the space $\mathfrak{D}(\Omega)$ of $C^\infty$ functions with compact support in $\Omega$ is dense in $W^{s,\delta}(\Omega, \psi)$.*

*If $s, s', \delta, \delta' \in R$ and $s \leqslant s'$, $\delta \leqslant \delta' + s' - s$, then we have a continuous inclusion*

$$W^{s',\delta'}(\Omega, \psi) \to W^{s,\delta}(\Omega, \psi).$$

*If $s < s'$ and $\delta < \delta' + s' - s$, then the inclusion is compact.*

Let $P(x, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha$ be a linear differential operator of order $\leqslant m$.

We say that $P(x, D)$ has type $(m, \delta)$ with respect to $\psi \in \Psi(\Omega)$ if for every multiindex $\beta$ and real $\varepsilon > 0$ we can find a constant $c(\beta, \varepsilon) > 0$ such that

$$|D^\beta a_\alpha| \leqslant c(\beta, \varepsilon) \langle\psi\rangle^{m-|\alpha|+\delta+\varepsilon|\beta|} \qquad \forall |\alpha| \leqslant m.$$

We denote by $P_\psi^*(x, D)$ the formal adjoint of $P(x, D)$ for the scalar product of $W^0(\Omega, \psi)$, characterized by:

$$\big(P(x, D)u, v\big)_{0,\psi} = \big(u, P_\psi^*(x, D)v\big)_{0,\psi} \qquad \forall u, v \in \mathfrak{D}(\Omega).$$

If $Q(x, D)$ is another differential operator with smooth coefficients on $\Omega$, we denote by $[P, Q] = P \circ Q - Q \circ P$ the commutator of $P$ and $Q$.

Then we have:

PROPOSITION 3. *a) If $P(x, D)$ is of type $(m, \delta)$ with respect to $\psi \in \Psi(\Omega)$, then for every $s, \sigma \in \mathbf{R}$ it defines a continuous linear map*

$$P(x, D)\colon W^{s,\sigma}(\Omega, \psi) \to W^{s-m,\sigma-\delta}(\Omega, \psi)\,.$$

*b) The operator $P_\psi^*(x, D)$ is also of type $(m, \delta)$.*

*c) If $Q(x, D)$ is of type $(k, \sigma)$, then the commutator $[P, Q]$ is of type $(m + k - 1, \lambda)$ for every $\lambda > \delta + \sigma$.*

If $s = (s_1, \ldots, s_p) \in \mathbf{R}^p$ and $\delta \in \mathbf{R}$, we will write $W^{s,\delta}(\Omega, \psi)$ for $W^{s_1,\delta}(\Omega, \psi) \times \ldots \times W^{s_p,\delta}(\Omega, \psi)$. We will also use the notations

$$(u, v)_{s,\psi,\delta} = (u^1, v^1)_{s_1,\psi,\delta} + \ldots + (u^p, v^p)_{s_p,\psi,\delta}$$

for the scalar product on $W^{s,\delta}(\Omega, \psi)$ if $u = (u^1, \ldots, u^p)$, $v = (v^1, \ldots, v^p)$ and for each $j = 1, \ldots, p$, we denoted by $(\cdot,\cdot)_{s_j,\psi,\delta}$ a continuous scalar product in $W^{s_j,\delta}(\Omega, \psi)$; we set also

$$\|u\|_{s,\psi,\delta} = (u, u)^{\frac{1}{2}}_{s,\psi,\delta}\,.$$

For $s \in \mathbf{R}^p$ and $t \in \mathbf{R}$, we set also $\boldsymbol{t} = (t, \ldots, t) \in \mathbf{R}^p$ and $s + \boldsymbol{t} = (s_1 + t, \ldots, s_p + t)$.

An operator $A(x, D) = \big(A_{ij}(x, D)\big)_{i=1,\ldots,q;\, j=1,\ldots,p}$ is said to be of type $(m, k, \delta)$ for a $p$-uple of integers $m = (m_1, \ldots, m_p)$, a $q$-uple of integers $k = (k_1, \ldots, k_q)$ and a real $\delta$ with respect to $\psi \in \Psi(\Omega)$ if for every pair of indices $i, j$ the operator $A_{ij}(x, D)$ is of type $(m_j - k_i, \delta)$. Such an operator defines a linear and continuous map

$$A(x, D)\colon W^{m+\boldsymbol{t},\sigma}(\Omega, \psi) \to W^{k+\boldsymbol{t},\sigma-\delta}(\Omega, \psi)$$

for all real $t$, $\sigma$.

*b.* Let $m = (m_1, \ldots, m_p)$ be a $p$-uple of nonnegative integers and let $\psi \in \Psi(\Omega)$. A differential operator with smooth coefficients

$$E(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^N(\Omega)$$

will be said to be $W^m(\Omega, \psi)$-elliptic if it is of type $(m, \mathbf{0}; 0)$ and there is a constant $c > 0$ such that

$$\|E(x, D)u\|^2_{0,\psi} \geq c\|u\|^2_{m,\psi} \qquad \forall u \in \mathcal{D}^p(\Omega)\,.$$

We have the following:

Proposition 4. *If $E(x, D)$ is $W^m(\Omega, \psi)$-elliptic, then for every $s, \delta \in \mathbf{R}$*
$L(x, D) = E^*_\psi(x, D) \circ E(x, D)\colon W^{m+s,\delta}(\Omega, \psi) \to W^{s-m,\delta}(\Omega, \psi)$

*is an isomorphism.*

As an example of such an operator $L(x, D)$, we can consider the operator $\mathbf{\Delta}_{m,\psi}\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^p(\Omega)$ characterized by the identity:

$$(\mathbf{\Delta}_{m,\psi} u, v)_{0,\psi} = (u, v)_{m,\psi} \qquad \forall u, v \in \mathcal{D}^p(\Omega)\,.$$

Let now $0 < \delta \leqslant 1$ be fixed. We say that $E(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^N(\Omega)$ is $W^{m-1,\delta}(\Omega, \psi)$-coercive if $E(x, D)$ is of type $(m, \mathbf{0}; 0)$ and there are constants $c > 0$ and $\lambda \geqslant 0$ such that

$$c\|u\|^2_{m-1,\psi,\delta} \leqslant \|Eu\|^2_{0,\psi} + \lambda\|u\|^2_{0,\psi} \qquad \forall u \in \mathcal{D}^p(\Omega)\,. \tag{6}$$

Note that, while $W^m(\Omega, \psi)$-ellipticity implies that $E^*_\psi(x, D) \circ E(x, D)$ is in $\Omega$ an elliptic operator in the sense of Douglis and Nirenberg (cf. [12]), neither ellipticity nor sub-ellipticity are implied by $W^{m-1,\delta}(\Omega, \psi)$-coerciveness. Thus we shall need also the following assumption:

(7) *$E(x, D)$ is sub-elliptic, i.e. there is a real number $\sigma$, with $0 \leqslant \sigma < 1$, such that every distribution $u \in \mathcal{D}'(\Omega^p)$ for which $E(x, D)u \in (L^2_{\text{loc}}(\Omega))^N$ belongs to $W^{m-\sigma}_{\text{loc}}(\Omega)$.*

(For $\sigma = 1/2$ necessary and sufficient conditions for subellipticity have been studied by Hörmander in [9]).

We have the following:

Proposition 5 (*Regularity Theorem*). *Let us assume that* (6) *and* (7) *hold. Then, if $f \in W^{s+1-m,\sigma}(\Omega, \psi)$ with $s \geqslant 0$ and $s + \sigma + \delta \geqslant 0$ and $u \in W^{m-1,\delta}(\Omega, \psi)$ with $E(x, D)u \in W^0(\Omega, \psi)$ solves*

$$\big(E(x, D)u, E(x, D)v\big)_{0,\psi} = f(e^{-\psi}\bar{v}) \qquad \forall v \in \mathcal{D}^p(\Omega)\,; \tag{8}$$

*we have*

$$u \in W^{m+s-1,2\delta+\sigma}(\Omega, \psi) \qquad \text{and} \qquad E(x, D)u \in W^{s,\delta+\sigma}(\Omega, \psi)\,.$$

This is the key result for the application of estimates involving weight functions, and plays here a role analogous of the regularization method of Kohn and Nirenberg for the $\bar{\partial}$-Neumann problem. The proof is done by elliptic regularization.

## 2. Application to complexes of partial differential operators.

Let us consider a complex

$$\mathcal{E}^p(\Omega) \xrightarrow{A(x,D)} \mathcal{E}^q(\Omega) \xrightarrow{B(x,D)} \mathcal{E}^r(\Omega) \tag{9}$$

of differential operators with smooth coefficients on $\Omega$ $\big(B(x, D)\circ \circ A(x, D) = 0\big)$.

We assume that for $s \in \mathbb{Z}^p$, $m \in \mathbb{Z}^q$, $t \in \mathbb{Z}^r$ and $\psi \in \Psi(\Omega)$ the operator $A(x, D)$ is of type $(s, m; 0)$ and the operator $B(x, D)$ is of type $(m, t; 0)$.

Let us choose $\lambda \leqslant \inf m_i$ and an operator $F(x, D)\colon \mathcal{E}^q(\Omega) \to \mathcal{E}^N(\Omega)$ $W^{m-\lambda}(\Omega, \psi)$-elliptic. Then we choose an integer $l$ in such a way that $\boldsymbol{l} + s$ and $\boldsymbol{l} + 2\boldsymbol{\lambda} - t$ have all components $\geqslant 0$ and we define $E_\psi(x, D)$ by

$$\big(E_\psi(x, D)u, E_\psi(x, D)v\big)_{0,\psi} = \big(A^*_\psi(x, D)u, A^*_\psi(x, D)v\big)_{s+\boldsymbol{l},\psi} + \\ + \big(B(x, D)F^*_\psi Fu, B(x, D)F^*_\psi\circ Fv\big)_{\boldsymbol{l}+2\boldsymbol{\lambda}-t},$$

for every $u, v \in \mathcal{D}^q(\Omega)$.

Then $E_\psi(x, D)$ is of type $(m + \boldsymbol{l}, \boldsymbol{0}; 0)$ with respect to $\psi$.

We have the following:

PROPOSITION 6. *The properties of $E_\psi(x, D)$ of being either subelliptic or $W^{m+\boldsymbol{l}-1,\delta}(\Omega, \psi)$-coercive for some $0 < \delta \leqslant 1$ are independent of the choice of $\lambda$, $l$ and $F$.*

From the regularity theorem (Proposition 5) we obtain:

PROPOSITION 7. *Assume that the operator $E_\psi$ defined above satisfies* (6) *and* (7). *If $\delta_1 > -\delta$ and $f \in W^{-m+\boldsymbol{\lambda},\delta_1}(\Omega, \psi)$ satisfies $B(x, D)f = 0$, then for any solution $u \in W^{m+\boldsymbol{l}-1,\delta}(\Omega, \psi)$ with $E_\psi(x, D)u \in W^{\boldsymbol{0}}(\Omega, \psi)$ of*

$$\big(E_\psi(x, D)u, E_\psi(x, D)v\big)_{0,\psi} = f(e^{-\psi}\bar{v}) \qquad \forall v \in \mathcal{D}^q(\Omega)$$

*we have*

$$w = \boldsymbol{\Delta}_{s+\boldsymbol{l}}\circ A^*_\psi(x, D)u \in W^{\boldsymbol{\lambda}-s-\boldsymbol{1},\delta+\delta_1}(\Omega, \psi) \qquad \textit{and} \qquad A(x, D)w = f \textit{ on } \Omega\,.$$

*Moreover,*

$$B_\psi^*(x,D)\circ\Delta_{l+2\lambda-t}B(x,D)\circ F_\psi^*(x,D)\circ F(x,D)u=0 \qquad on\ \Omega\,.$$

Let us denote by $A(x,D)\colon W^{s+h,\sigma}(\Omega,\psi)\ldots\to W^{h+m+1,\sigma-\delta}(\Omega,\psi)$ the closed densely defined linear operator obtained by considering the differential operator $A(x,D)$ on the domain

$$D(A)=\{u\in W^{h+s,\sigma}(\Omega,\psi)\,|A(x,D)u\in W^{h+m+1,\sigma-\delta}(\Omega,\psi)\}$$

and analogously for $B(x,D)$, $A_\psi^*(x,D)$ and $B_\psi^*(x,D)$. Then we set

$$H(h,\sigma;\,\Omega,\psi)=$$
$$=\frac{\operatorname{Ker}\big(B(x,D)\colon W^{h+m+1,\sigma-\delta}(\Omega,\psi)\ldots\to W^{h+t+2,\sigma-2\delta}(\Omega,\psi)\big)}{\operatorname{Image}\big(A(x,D)\colon W^{h+s,\sigma}(\Omega,\psi)\ldots\to W^{h+m+1,\sigma-\delta}(\Omega,\psi)\big)}\,.$$

$$N(\Omega,\psi)=\Big\{u\in\big(W^\infty(\Omega,\psi)\big)^q|A_\psi^*(x,D)u=0,$$
$$B(x,D)\circ F_\psi^*(x,D)\circ F(x,D)u=0\Big\}\,.$$

Then from the regularity theorem we obtain the following:

PROPOSITION 8. *Under the same assumptions of Proposition* 7:

($a$) $\dim_{\mathbf{C}}N(\Omega,\psi)=d<\infty$

($b$) $\forall h,\sigma\in\mathbb{R}$, $\dim_{\mathbf{C}}H(h,\sigma;\,\Omega,\psi)=d.$

If $A(x,D)$ and $B(x,D)$ are differential operators with coefficients bounded with all derivatives in $\Omega$, then all operators $E_\psi(x,D)$ obtained as explained above from different weight functions $\psi\in\Psi(\Omega)$ are of type $(m+l,0;0)$. Then we obtain the following:

PROPOSITION 9. *Assume that for every upper semicontinuous function* $\varphi\colon\Omega\to\mathbb{R}$ *there is* $\psi\in\Psi(\Omega)$ *such that* $\psi\geqslant\varphi$ *and the operator* $E_\psi(x,D)$ *satisfies* (6) *and* (7), *then the space*

$$H(\Omega)=\frac{\operatorname{Ker}\big(B(x,D)\colon\mathcal{E}^q(\Omega)\to\mathcal{E}^r(\Omega)\big)}{\operatorname{Image}\big(A(x,D)\colon\mathcal{E}^p(\Omega)\to\mathcal{E}^q(\Omega)\big)}$$

*is finite dimensional.*

## 3. Localization of the estimates.

Let us consider now a stronger coerciveness estimate: namely we assume that for the operator $E(x, D)\colon \mathcal{E}^p(\Omega) \to \mathcal{E}^N(\Omega)$ of type $(m, \mathbf{0}; 0)$ we have

$$(10) \qquad \|u\|^2_{m-1/2,\psi} \leqslant c\{\|E(x, D)u\|^2_{0,\psi} + \|u\|^2_{0,\psi}\} \qquad \forall u \in \mathcal{D}^p(\Omega)\,.$$

For

$$E(x, D) = \big(E_{ij}(x, D)\big) \qquad \text{with} \qquad E_{ij}(x, D) = \sum_{|\alpha| \leqslant m_j} E^\alpha_{ij}(x) D^\alpha\,,$$

we set

$$\hat{E}_{ij}(x, \xi) = \sum_{|\alpha| = m_j} E^\alpha_{ij}(x)\xi^\alpha \qquad \text{and} \qquad \hat{E}(x, \xi) = \big(\hat{E}_{ij}(x, \xi)\big)\,.$$

Then the following theorem holds:

Proposition 10. *A necessary and sufficient condition in order that estimate* (10) *holds, is that there exist a constant* $C > 0$ *such that*

$$\begin{aligned}(11) \quad & \sum_{j=1}^{p} (\langle\psi(x)\rangle + |\xi|)^{2m_j - 1}\int |v_j(y)|^2 dy \\ & \leqslant C\Big\{\int |\hat{E}(x, i\xi)v(y) + \langle\psi(x)\rangle^{-1/2} \sum_h \partial\hat{E}(x, i\xi)/\partial x_h \cdot y_h \cdot v(y) \\ & + \langle\psi(x)\rangle^{1/2} \sum_h \partial\hat{E}(x, i\xi)/\partial\xi_h \cdot \partial v/\partial y_h|^2 dy \\ & + \sum_{j=1}^{p} \sum_{|\alpha| \leqslant m_j} (\langle\psi(x)\rangle + |\xi|)^{2m-2} \int |D^\alpha v_j(y)|^2 dy\Big\} \\ & \qquad\qquad \forall v \in \mathcal{D}^p\big(B(0, 1)\big),\ \forall x \in \Omega,\ \forall \xi \in \mathbf{R}^n\,.\end{aligned}$$

The proof of this statement is similar to that of the analogous statement in Hörmander [9].

We also note that, if $\Omega$ is relatively compact and $E(x, D)$ is subelliptic with $\sigma = 1/2$ on a neighborhood of the closure of $\Omega$, then (10) is a consequence of (6) with $\delta = 1/2$, while (6) cannot be easily localized.

## 4. An application to the case of complexes differential operators with constant coefficients.

Let $\mathcal{T}$ denote the ring of polynomials in $n$ indeterminates $\xi_1, \ldots, \xi_n$, filtered by the degree. Given a $\mathcal{T}$-module $M$ of finite type, we choose a filtration

$$0 = M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \ldots \quad \text{of } M$$

compatible with that of $\mathcal{T}$ and we denote by $M^0$ the associated graded ring:

$$M^0 = \oplus M_j/M_{j-1}\,.$$

To any Hilbert resolution of $M^0$ by homogeneous matrices of polynomials

$$0 \leftarrow M^0 \leftarrow \mathcal{T}^{p_0} \xleftarrow{{}^t\hat{A}_0} \mathcal{T}^{p_1} \xleftarrow{{}^t\hat{A}_1} \mathcal{T}^{p_2} \leftarrow \ldots \leftarrow \mathcal{T}^{p_d} \leftarrow 0$$

corresponds a resolution of $M$

$$0 \leftarrow M \leftarrow \mathcal{T}^{p_0} \xleftarrow{{}^tA_0} \mathcal{T}^{p_1} \xleftarrow{{}^tA_1} \mathcal{T}^{p_2} \leftarrow \ldots \leftarrow \mathcal{T}^{p_d} \leftarrow 0$$

where, for a suitable choice of multigraduations, the $\hat{A}_j$'s can be considered as the homogeneous parts of higher degree of the $A_j$'s. (Cf. [2]).

The modules $\operatorname{Ext}^j(M, \mathcal{E}(\Omega))$ (where $\mathcal{E}(\Omega)$ is considered as a left-$\mathcal{T}$-module by $p(\xi)\cdot f = p(D)f$) are isomorphic to the cohomology groups of the complex of differential operators with constant coefficients:

$$\mathcal{E}^{p_0}(\Omega) \xrightarrow{A_0(D)} \mathcal{E}^{p_1}(\Omega) \xrightarrow{A_1(D)} \mathcal{E}^{p_2}(\Omega) \rightarrow \ldots \rightarrow \mathcal{E}^{p_d}(\Omega) \rightarrow 0\,.$$

For $\xi^0 \in \mathbb{C}^n$, we denote by $L_{\xi^0}$ the localization at $\xi^0$ of $\mathcal{T}$, i.e. the ring of fractions $p/q$ for $p, q \in \mathcal{T}$ and $q(\xi^0) \neq 0$.

We say that $M^0$ is simple of principal type if the characteristic variety $V(M^0) = \{\xi \in \mathbb{C}^n | M^0 \otimes_{\mathcal{T}} \mathcal{T}/\mathfrak{m}_\xi \neq 0\}$ (where $\mathfrak{m}_\xi$ is the ideal of polynomials vanishing at $\xi$) is smooth outside 0 and $\forall \xi^0 \in V(M^0) - \{0\}$, having chosen $p_1, \ldots, p_k$ such that $V(M^0)$ is defined by $p_1 = \ldots = p_k = 0$

near $\xi^0$, with $dp_1 \wedge \ldots \wedge dp_k \neq 0$ at $\xi^0$, we have

$$M^0 \otimes_{\mathcal{T}} L_{\xi_0} \cong L_{\xi_0}/(p_1, \ldots, p_k)$$

where $(p_1, \ldots, p_k)$ is the ideal of $L_{\xi^0}$ generated by $p_1, \ldots, p_k$.

The following proposition, that is a consequence of the results of the preceding sections, is a generalization of the vanishing theorems for $\bar{\partial}$ on strictly pseudoconvex domains of $\mathbf{C}^n$ for $M^0$ simple of principal type:

PROPOSITION 11. *Let us assume that* $V(M^0) \cap \mathbf{R}^n \subset \{0\}$, *and let* $\Omega$ *be a* $\sigma$*-smooth open set of* $\mathbf{R}^n$ *with a* $C^\infty$ *function* $\varphi: \Omega \to \mathbf{R}$ *satisfying* (3) *and* (4) *and the following convexity assumption*:

*There is a compact set* $K \subset \Omega$ *such that*

$$\forall x \in \Omega - K\,, \qquad \forall \xi \in \mathbf{R}^n \qquad \textit{such that} \qquad \zeta = i\xi + \operatorname{grad}\varphi(x) \in V(M^0)\,,$$

*the quadratic form*

$$\sum \partial^2\varphi(x)/\partial x_k \partial x_k \cdot v^h \cdot \bar{v}^k$$

*restricted to the complex linear space* $H$ *of vectors*

$$v = \operatorname{grad} p(\zeta) \quad \textit{where } p \in \mathcal{T} \textit{ vanishes on } V(M^0) \textit{ and}$$

$$\langle \operatorname{grad}\varphi(x), \operatorname{grad} p(\zeta)\rangle = 0$$

*has either at least* $j$ *negative or at least* $\dim_{\mathbf{C}} H - j + 1$ *positive eigenvalues. Then* $\operatorname{Ext}^j(M, \mathcal{E}(\Omega))$ *is finite dimensional over* $\mathbf{C}$.

*If moreover* $K$ *is contained in a convex open subset of* $\Omega$,

$$\operatorname{Ext}^j(M, \mathcal{E}(\Omega)) = 0\,.$$

## 5. Concluding remarks.

The results of the preceding paragraphs apply also to complexes of linear partial differential operators with variable coefficients; for instance we can study the Cauchy-Riemann complex induced on a generic real submanifold of $\mathbf{C}^n$. However we will not discuss these applications here. We hope also to develop by means of the result of § 4 a « function theory » for some complexes of p.d.e. with constant coefficients that could be of help in the study of analytic hypoellipticity and propagation of analytic singularities (cf. Schapira [14]).

We also want to note that the results of sections 1, 2, 3 can be extended to the case of linear differential operators between vector bundles over a complete, $\sigma$-smooth Riemannian manifold, endowed with affine connections. (cf. [2]).

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