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EUGENE LEIMANIS

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On Integration of the Differential Equation of Central Motion, I.

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SUMMARY - Assuming that the force acting on a particle is of the form $f(r)g(\theta)$, the theory of infinitesimal transformations is applied to determine the forms of $f(r)$ and $g(\theta)$ for which the differential equation of central motion is integrable by quadratures or reducible to a first order differential equation.

1. It is well known that the problem of motion under central forces is always solvable by quadratures when the force $f(r)$ is a function of the distance r only [13]. If we ask the further question, in which cases the integration can be effected in terms of known functions, then Newton [10] showed that if the central force varies as some positive or negative integral power of the distance, say the n-th, the problem is solvable by circular functions in the cases $n = 1, -2$ and -3. The case $n = -3$ was studied in detail by R. Cotes [3]. Next Legendre [5] showed that the integration can be effected by means of elliptic functions for $n = 0, 3, 5, -4, -5$ and -7. Afterwards Stader [12] investigated in great detail the cases $n = -3$, $-4, -5, -6, -7$, Kärger [4] the case $n = -4$ and Macmillan [9] the case $n = -5$. Finally Nobile [11] discussed the integration in the cases when n is a negative integer, and in particular the cases $n = -2$, $-3, -4, -5$, and when n is a rational fraction, in particular the cases $n = -3/2$ and $-5/2$.

(*) Indirizzo dell'A.: 3839 Selkirk Street, Vancouver, B.C. V6H 2Z2, Canada.

In what follows we shall assume that the force depends on both variables r and θ, and that it can be written in the form of a product $f(r)g(\theta)$. Here r, θ are the polar coordinates of the particle P with respect to the center of force O. Our aim is to determine the forms of $f(r)$ and $g(\theta)$ for which the equation (1) is integrable by quadratures or for which the integration can be reduced to some particular type of differential equations. The method used is that of infinitesimal transformations which I applied [6, 7, 8] some 35 years ago to the particle problem of exterior ballistics. It is a well known fact, that the knowledge of an infinitesimal transformation under which the given system of differential equations remains invariant, can be used to lower the order of the system. In our problem the integration of the second order differential equation of the central motion will be reduced to the integration of a first order differential equation, satisfied by the invariants of the infinitesimal transformation used, and an additional quadrature for the time.

2. Consider the motion of a particle P of mass one acted on by a central force of the form $Q(r, \theta) = f(r)g(\theta)$. Then the differential equation of the trajectory can be written in the form [13]

$$\frac{d^2u}{d\theta^2} + u = c^2 F(u) g(\theta)\,, \tag{1}$$

where $u = 1/r$, $c^2 = 1/h^2$ (h is a constant which represents the angular momentum of P about O) and $F(u) = (1/u^2)f(1/u)$.

Let

$$\mathfrak{C} = \Theta \frac{\partial}{\partial \theta} + U \frac{\partial}{\partial u} + V \frac{\partial}{\partial v} \tag{2}$$

be an infinitesimal transformation which leaves the system

$$\begin{aligned} \frac{du}{d\theta} &= v \\ \frac{dv}{d\theta} &= c^2 F(u) g(\theta) - u \end{aligned} \tag{3}$$

which is equivalent to equation (1), invariant. Let

$$\mathfrak{C}_0 = \frac{\partial}{\partial\theta} + v\frac{\partial}{\partial u} + [c^2F(u)\,g(\theta) - u]\frac{\partial}{\partial v}$$

be the trivial transformation. Then $\mathfrak{C} + \Lambda\mathfrak{C}_0$, where Λ is an arbitrary function of u, v, θ, is also an infinitesimal transformation which leaves system (3) invariant. If we put $\Lambda = -\Theta$, then to any transformation (2) there corresponds a transformation

$$\bar{\mathfrak{C}} = \mathfrak{C} - \Theta\mathfrak{C}_0 = \bar{U}\frac{\partial}{\partial u} + \bar{V}\frac{\partial}{\partial v}, \tag{4}$$

where

$$\bar{U} = U - \Theta v, \qquad \bar{V} = V - [c^2F(u)g(\theta) - u]\Theta. \tag{5}$$

If u, v and u_1, v_1 are the coordinates of two neighboring points in the u, v-plane, then the equations

$$u_1 = u + \bar{U}\delta\theta, \qquad v_1 = v + \bar{V}\delta\theta$$

represent explicitly the infinitesimal transformation (4), and the conditions to be satisfied in order that (4) leaves system (3) invariant are

$$\frac{d\bar{U}}{d\theta} = \bar{V}, \qquad \frac{d\bar{V}}{d\theta} = [c^2F'(u)g(\theta) - 1]\bar{U}. \tag{6}$$

In order to integrate system (6), let us rewrite the first equation in the form

$$\bar{V} = \frac{d\bar{U}}{d\theta} = \bar{U}_u v + \bar{U}_v[c^2F(u)g(\theta) - u] + \bar{U}_\theta$$

or

$$\bar{V} = Mc^2 + N, \tag{7}$$

where

$$M(u, v, \theta) = F(u)g(\theta)\bar{U}_v,$$

$$N(u, v, \theta) = \bar{U}_u v - \bar{U}_v u + \bar{U}_\theta.$$

Here the subscripts denote partial derivatives of $\overline{U}$ with respect to the variables represented by the indices. If by means of (7) we calculate $d\overline{V}/d\theta$, substitute its expression into the second equation of system (6) and arrange that equation according to the powers of c^2, then we obtain the equation

$$\begin{aligned}(8)\quad & c^4F^2(u)g^2(\theta)\overline{U}_{vv}+c^2\{-2uF(u)g(\theta)\overline{U}_{vv}+2vF(u)g(\theta)\overline{U}_{uv}+\\ &+2F(u)g(\theta)\overline{U}_{v\theta}+F(u)g(\theta)\overline{U}_u+[vF'(u)g(\theta)+F(u)g'(\theta)]\overline{U}_v-\\ &-F'(u)g(\theta)\overline{U}\}+\{v^2\overline{U}_{uu}+u^2\overline{U}_{vv}+\overline{U}_{\theta\theta}-2uv\overline{U}_{uv}+2v\overline{U}_{u\theta}-\\ &\qquad\qquad -2u\overline{U}_{v\theta}-u\overline{U}_u-v\overline{U}_v+\overline{U}\}=0\,.\end{aligned}$$

In order that the infinitesimal transformations (2) and (4) be independent of c^2, the functions U, V, Θ and $\overline{U}$ must be independent of c^2. This implies that the quadratic equation (8) in c^2 must vanish identically. In other words, the coefficients of c^4, c^2 and c^0 must vanish separately. Hence we must have

1) $\overline{U}_{vv}=0$, Consequently $\overline{U}_v=\varphi(u,\theta)$ and

$$(9)\qquad \overline{U}=v\varphi(u,\theta)+\psi(u,\theta)\,,$$

where φ and ψ are two functions to be determined later.

2) The coefficient of c^2, put equal to zero, gives the equation

$$\begin{aligned}&2vF(u)g(\theta)\overline{U}_{uv}+2F(u)g(\theta)\overline{U}_{v\theta}+F(u)g(\theta)\overline{U}_u+\\ &\qquad+[vF'(u)g(\theta)+F(u)g'(\theta)]\overline{U}_v-F'(u)g(\theta)\overline{U}=0\,,\end{aligned}$$

or, interms of φ and ψ and their derivatives, the equation

$$\begin{aligned}&v\left[3F(u)g(\theta)\frac{\partial\varphi}{\partial u}\right]+2F(u)g(\theta)\frac{\partial\varphi}{\partial\theta}+F(u)g(\theta)\frac{\partial\psi}{\partial u}+\\ &\qquad+F(u)g'(\theta)\varphi-F'(u)g(\theta)\psi=0\,.\end{aligned}$$

Since φ and ψ are independent of v, then we must have

$$\frac{\partial\varphi}{\partial u}=0\,,\qquad \text{and consequently } \varphi=\varphi(\theta)\,,$$

and

$$(10)\qquad F(u)g(\theta)\frac{\partial\psi}{\partial u}+F(u)g'(\theta)\varphi+2F(u)g(\theta)\varphi'-F'(u)g(\theta)\psi=0\,.$$

3) The coefficient of c^0, equated to zero, leads to the equation

$$(11)\qquad v^2\frac{\partial^2\psi}{\partial u^2}+v\left(2\frac{\partial^2\psi}{\partial u\,\partial\theta}+\varphi''\right)+\frac{\partial^2\psi}{\partial\theta^2}-u\frac{\partial\psi}{\partial u}-2u\varphi'+\psi=0\,.$$

Since φ and ψ are independent of v, then in the last equation the coefficients of the quadratic equation in v must vanish separately. Hence first we must have

$$\frac{\partial^2\psi}{\partial u^2}=0\,;\qquad \text{consequently}\qquad \frac{\partial\psi}{\partial u}=\psi_1(\theta)\,,$$

and

$$(12)\qquad \psi(u,\theta)=u\psi_1(\theta)+\psi_2(\theta)\,.$$

Second, we must have

$$(13)\qquad 2\psi_1'(\theta)+\varphi''(\theta)=0\,,$$

and hence

$$(14)\qquad 2\psi_1(\theta)+\varphi'(\theta)=k\qquad (k \text{ is a constant of integration})\,.$$

Third, the equation

$$u[-2\varphi'(\theta)+\psi_1''(\theta)]+\psi_2''(\theta)+\psi_2(\theta)=0$$

must be satisfied. This implies that

$$(15)\qquad \psi_1''(\theta)-2\varphi'(\theta)=0\,,$$

$$(16)\qquad \psi_2''(\theta)+\psi_2(\theta)=0\,.$$

If we eliminate $\varphi'(\theta)$ from equations (14) and (15), then we obtain for $\psi_1(\theta)$ the differential equation

$$\psi_1''(\theta) + 4\psi_1(\theta) - 2k = 0, \tag{17}$$

the general solution of which is

$$\psi_1(\theta) = (1/2)[(4C + k^2)^{1/2} \sin 2(\theta + a) + k], \tag{18}$$

where C and a are two constants of integration. Since

$$\varphi'(\theta) = k - 2\psi_1(\theta) = -(4C + k^2)^{1/2} \sin 2(\theta + a)$$

and $\varphi(\theta)$ satisfies the same differential equation as $\psi_1(\theta)$, then

$$\varphi(\theta) = (1/2)[(4C + k^2)^{1/2} \cos 2(\theta + a) + k]. \tag{19}$$

Finally equation (10) can be rewritten in the form

$$F(u)[g'(\theta)\varphi(\theta) + g(\theta)\psi_1(\theta) + 2g(\theta)\varphi'(\theta)] - \\ - F'(u)g(\theta)[u\psi_1(\theta) + \psi_2(\theta)] = 0. \tag{20}$$

In order that the variables u and θ be separable in (20), $\psi_1(\theta)$ or $\psi_2(\theta)$ must vanish identically. If $\psi_1(\theta) = 0$, then also $\varphi'(\theta) = \varphi(\theta) = 0$ (since $C = k = 0$) and equation (20) reduces to $F'(u)g(\theta) = 0$. Hence 1) $F'(u) = 0$, or 2) $g(\theta) = 0$. The second case is of no interest. In the first case $F(u) = \text{const.}$, say equal to one, $f(r) = 1/r^2$ and $g(\theta)$ remains arbitrary. This is the case discovered by Armellini [2]. If $\psi_2(\theta) = 0$ which represents the trivial solution of (16), then

$$\frac{uF'(u)}{F(u)} = \frac{g'(\theta)\varphi(\theta) + g(\theta)\psi_1(\theta) + 2g(\theta)\varphi'(\theta)}{g(\theta)\psi_1(\theta)} = m, \tag{21}$$

where m is an arbitrary constant. Hence

$$F(u) = Du^m \tag{22}$$

(D is a constant which without any loss of generality can be assumed equal to one),

$$f(r) = 1/r^{m+2} \tag{23}$$

and

$$\frac{g'(\theta)}{g(\theta)} = \frac{(m-1)\psi_1(\theta) - 2\varphi'(\theta)}{\varphi} = \frac{(m+3)\psi_1(\theta) - 2k}{\varphi}$$

or

$$\frac{g'(\theta)}{g(\theta)} = \frac{(m+3)(4C+k^2)^{1/2}\sin 2(\theta+a) + (m-1)k}{(4C+k^2)^{1/2}\cos 2(\theta+a) + k}. \tag{24}$$

Accordingly $\overline{U}$ assumes the expression

$$\begin{aligned} \overline{U} &= v\varphi(\theta) + u\psi_1(\theta) \\ &= (v/2)[(4C+k^2)^{1/2}\cos 2(\theta+a) + k] + \\ &\qquad + (u/2)[(4C+k^2)^{1/2}\sin 2(\theta+a) + k] \end{aligned} \tag{25}$$

and

$$M = F(u)g(\theta)\overline{U}_v = F(u)g(\theta)\varphi(\theta),$$

$$N = v\overline{U}_u - u\overline{U}_v + \overline{U}_\theta = v\psi_1(\theta) - u\varphi(\theta) + v\varphi'(\theta) + u\psi_1'(\theta).$$

Then from formulas (5) and (7) it follows that

$$\begin{aligned} \overline{V} &= Mc^2 + N = [c^2F(u)g(\theta) - u]\varphi(\theta) + u\psi_1'(\theta) + v[\psi_1(\theta) + \varphi'(\theta)] \\ &= V - [c^2F(u)g(\theta) - u]\Theta. \end{aligned}$$

Hence, by comparison of the coefficients we obtain that

$$\Theta = -\varphi(\theta) = -(1/2)[(4C+k^2)^{1/2}\cos 2(\theta+a) + k] \tag{26}$$

and

$$\begin{aligned} V &= u\psi_1'(\theta) + v[\psi_1(\theta) + \varphi'(\theta)] \\ &= u(4C+k^2)^{1/2}\cos 2(\theta+a) - (v/2)[(4C+k^2)^{1/2}\sin 2(\theta+a) - k]. \end{aligned} \tag{27}$$

Since by (5) $\overline{U} = U - v\Theta$, then

$$U = \overline{U} + v\Theta = u\psi_1(\theta) = (u/2)[(4C + k^2)^{1/2} \sin 2(\theta + a) + k]. \tag{28}$$

Hence we have determined the three coefficient functions U, V and Θ of the infinitesimal transformation (2). They contain three arbitrary constants C, a and k. If we put $C = a = 0$, $k = 1$ or $k = a = 0$, $C = 1$, then we obtain two infinitesimal transformations which leave system (3) invariant. A fourth arbitrary constant m enters formulas (23) and (24) which give the functions $f(r)$ and $g(\theta)$.

Let us consider now the two special cases just mentioned.

3. *Case I*: $k = a = 0$, $C = 1$.

In this case we have

$$\Theta = -\cos 2\theta, \qquad U = u \sin 2\theta, \qquad V = 2u \cos 2\theta - v \sin 2\theta$$

and the invariants X and Y of the infinitesimal transformation, determined by the equations

$$\frac{d\theta}{-\cos 2\theta} = \frac{du}{u \sin 2\theta} = \frac{dv}{2u \cos 2\theta - v \sin 2\theta},$$

are

$$X = u \cos^{-1/2} 2\theta, \qquad Y = uv + u^2 \tan 2\theta. \tag{29}$$

$F(u)$ is given by (22) and from (24) it follows that

$$g(\theta) = 1/\cos^{(m+3)/2} 2\theta. \tag{30}$$

The derivatives of the invariants with respect to θ, taking into account formulas (3), (22) and (30), satisfy the equations

$$\begin{aligned} \cos 2\theta X^2 \frac{dY}{d\theta} &= c^2 X^{m+3} + X^4 + Y^2, \\ \cos 2\theta X \frac{dX}{d\theta} &= Y. \end{aligned} \tag{31}$$

Hence by division we obtain the equation

$$XY\frac{dY}{dX} = c^2X^{m+3} + X^4 + Y^2$$

or, if we put $Y^2 = Z$, the equation

(32) $$\frac{dZ}{dX} - \frac{2}{X}Z = 2c^2X^{m+2} + 2X^3 .$$

The general solution of (32) is

$$Z = [2c^2/(m+1)]X^{m+3} + X^4 + C_1X^2$$

(C_1 is a constant of integration) and hence

(33) $$Y = X\sqrt{[2c^2/(m+1)]X^{m+1} + X^2 + C_1} = X\sqrt{G(X)} \quad (m+1 \neq 0) .$$

From the first equation of (31) it then follows that

(34) $$I = \int\frac{dX}{\sqrt{G(X)}} = \int\frac{d\theta}{\cos 2\theta} = \frac{1}{2}\int \sec 2\theta \, d2\theta = \frac{1}{2}\ln(\sec 2\theta + \tan 2\theta)$$

and after integration $X = H(\theta)$. Then by (33) also

$$Y = H(\theta)\sqrt{G[H(\theta)]} = K(\theta)$$

and the first formula of (29) gives us

$$1/r = u = X\cos^{1/2}2\theta = H(\theta)\cos^{1/2}2\theta .$$

In order that the integral I on the left of (34) could be evaluated in terms of circular functions, $G(X)$ must be a polynomial of at most the second degree. This condition is fulfilled in the following three cases:

1) $m = 0$, i.e. $F(u) = 1$, $f(r) = 1/r^2$, $g(\theta) = 1/\cos^{3/2}2\theta$.

2) $m = 1$, i.e. $F(u) = u$, $f(r) = 1/r^3$, $g(\theta) = \cos^2 2\theta$.

3) $m = -3$, i.e. $F(u) = 1/u^3$, $f(r) = r$, $g(\theta) = 1$.

In fact, in the first two cases $G(X)$ is a polynomial of degree two in X and in the third case $G_1(X_1)$ is quadratic in $X_1 = X^2$ and $I = (1/2)\int dX_1/\sqrt{G_1(X_1)}$. Hence the integrability in terms of circular functions, first considered by Newton for $n = 1, -2, -3$, can be extended to the above first two more general cases.

Next, let us list the cases in which the integration on the left of (34) can be effected by means of elliptic integrals. For this it is necessary that $G(X)$ be of the third or fourth degree in X. This condition is fulfilled in the following six cases:

1) $m = -7$: $F(u) = 1/u^7$, $f(r) = r^5$, i.e. $n = 5$, $g(\theta) = \cos^2 2\theta$.

2) $m = -5$: $F(u) = 1/u^5$, $f(r) = r^3$, i.e. $n = 3$, $g(\theta) = \cos 2\theta$.

3) $m = -2$: $F(u) = 1/u^2$, $f(r) = 1$, i.e. $n = 0$, $g(\theta) = 1/\cos^{1/2} 2\theta$.

4) $m = 2$: $F(u) = u^2$, $f(r) = 1/r^4$, i.e. $n = -4$, $g(\theta) = 1/\cos^{5/2} 2\theta$.

5) $m = 3$: $F(u) = u^3$, $f(r) = 1/r^5$, i.e. $n = -5$, $g(\theta) = 1/\cos^3 2\theta$.

6) $m = 5$: $F(u) = u^5$, $f(r) = 1/r^7$, i.e. $n = -7$, $g(\theta) = 1/\cos^4 2\theta$.

In fact, it is easy to verify that in the above six cases the integral I is of the form

$$I_1 = \int \frac{dX}{\sqrt{G(X)}},$$

where $G(X)$ is a polynomial of degree three or four, or of the form

$$I_2 = \int \frac{X\,dX}{\sqrt{G(X)}},$$

where $G(X)$ is a polynomial of degree four in X, or it can be reduced to one of the integrals I_1 or I_2 in terms of $X_1 = X^2$.

Hence the integrability in terms of elliptic functions, first considered by Legendre for $n = 5, 3, 0, -4, -5, -7$, can also be extended to the above six more general cases.

4. *Case II*: $C = a = 0$, $k = 1$.

As before $F(u)$ is given by (22) and $g(\theta)$ is defined by (24). By integration we obtain that

$$g(\theta) = [1/\cos^{m+3}\theta] \exp\,[[(m-1)/2] \tan\theta]\,. \tag{35}$$

Further we have

$$\Theta = -(1/2)(\cos 2\theta + 1), \qquad U = (u/2)(\sin 2\theta + 1),$$
$$V = u \cos 2\theta - (v/2)(\sin 2\theta - 1),$$

and the invariants of the infinitesimal transformation, determined by the equations

$$\frac{d\theta}{-(\cos 2\theta + 1)} = \frac{du}{u(\sin 2\theta + 1)} = \frac{dv}{2u \cos 2\theta - v(\sin 2\theta - 1)},$$

are

$$(36) \qquad \begin{aligned} X &= [u \exp[(1/2)\tan\theta]]/\cos\theta, \\ Y &= (u \sin\theta + v\cos\theta)\exp[(1/2)\tan\theta]. \end{aligned}$$

Their derivatives with respect to θ satisfy the equations

$$(37) \qquad \begin{aligned} 2\cos^2\theta \frac{dX}{d\theta} &= X + 2Y, \\ 2\cos^2\theta \frac{dY}{d\theta} &= 2c^2X^m + Y \end{aligned}$$

and hence, by division we obtain the equation

$$(38) \qquad \left(Y + \frac{X}{2}\right)\frac{dY}{dX} = c^2X^m + \frac{1}{2}Y.$$

If we put $Y = -X/2 - Z$, then equation (38) becomes

$$(39) \qquad Z\,dZ + [P(X) + Z]\,dX = 0,$$

where

$$(40) \qquad P(X) = \frac{X}{4} - c^2X^m.$$

If $m = 0$, then

$$F(u) = 1, \qquad f(r) = 1/r^2, \qquad g(\theta) = 1/[\cos^3\theta \exp[(1/2)\tan\theta]]$$

and equations (37) can be easily integrated. Their integrals are

$$X = 4c^2 + A \tan\theta \exp[(1/2)\tan\theta] + B \exp[(1/2)\tan\theta],$$

$$Y = -2c^2 + A \exp[(1/2)\tan\theta],$$

where A and B are two constants of integration. From the first formula of (36) it then follows that

$$(41)\qquad u = 1/r = A \sin\theta + B \cos\theta + 4c^2 \cos\theta \exp[-(1/2)\tan\theta].$$

This is the general solution of equation (1) for $F(u) = 1$. Since in this particular case equation (1) is linear, its solution (41) can also be obtained by direct integration.

For $m = 1$, we have $F(u) = u$, $f(r) = 1/r^3$, $g(\theta) = 1/\cos^4\theta$ and the general solution of (38) is

$$(Y - cX)^{2c+1}(Y + cX)^{2c-1} = \text{const}.$$

On the other hand for $m = 1$ we have by (40) for $P(X)$ the expression

$$P(X) = (1/4 - c^2)X = (1/4)(1 - 4c^2)X.$$

It is interesting to note that for this particular case Abel [1] has given an integrating factor for equation (39). In general, however, this equation cannot be solved by quadratures.

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