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JORGE MARTINEZ

Almost finite-valued l-groups

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Almost Finite-Valued l-Groups.

Jorge Martinez (*)

1. Introduction.

Recall that in an l-group G a convex l-subgroup M is called a *value* if it is maximal with respect to missing some element $g \in G$. We also say that M is a value of g. This basic facts from the theory of l-groups that we shall require in this article are to be found in [1]; we mention the essential concepts here for completeness. By Zorn's Lemma each non-zero element of an l-group G has at least one value. If an element g has but a finite number of values we say g is *finite-valued*. If all the elements of an l-group are finite-valued we say the l-group is *finite-valued*. An element s is *special* if it has only one value; its single value is also said to be *special*. In these terms the structure of finite-valued l-groups is well-understood. Here is the main theorem.

Theorem ([2], Theorem 3.9). In an l-group G the following conditions are equivalent.

Following conditions are equivalent.

(*a*) G is finite-valued.

(*b*) Each value of G is special.

(*c*) Each $0 < g \in G$ can be written as a sum of pairwise disjoint-special elements.

(*) Indirizzo dell'A.: Department of Mathematics, University of Florida, Gainesville, Fl. 32611 U.S.A.

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This theorem has a « local » version, which can also be found in [2], but we shall omit it.

With these preliminaries we are able to define the class of l-groups we want: G is said to be *almost* finite-valued if for each $0 \neq g \in G$ every value of g except for finitely many, is special. (Locally, we speak of an *almost finite-valued* element if it has the stated property.) Clearly this class includes all the finite-valued l-groups.

For the fundamental concepts in l-groups we refer the reader to [1]. Our notation in l-groups is additive.

2. The main theorem.

We say that an element $g \neq 0$ in an l-group G is 1-*special* if all but one of its values are special. Note that if g is 1-special then it is not finite-valued, and in particular, it has infinitely many special values. If g is a 1-special element we call its single non-special value 1-*special.*

It is well-known that if M is a value then $M^* = \cap \{N | N > M$ properly$\}$ contains M and, indeed, covers M. If M is normal in M^* for each value M of G we say that G is *normal-valued.* If M is a special value then M is normal in M^*, (see [1]) and so a finite-valued l-group is necessarily normal-valued.

We start with the analogue of this for almost finite-valued l-groups.

1 Lemma. If Q is a 1-special value then Q is normal in Q^*.

Proof. Let $x > 0$ be an element for which Q is the only non-special value. In $G(x)$, the convex l-subgroup generated by x, $Q \cap G(x)$ is the only non-special maximal convex l-subgroup. It follows that $Q \cap G(x)$ is normal in $G(x)$, and hence that Q is normal in $Q^* = Q \vee G(x)$.

From a technical point of view the central result in this article is this local lemma, the analogue of Conrad's Local Structure Theorem [2].

2 Lemma. For an element $0 < g \in G$ the following are equivalent.

(A) Each value of g is either special or 1-special.

(B) g has finitely many non-special values.

(C) $g = g_1 + g_2 + \dots + g_n$, where $g_i \wedge g_j = 0$ for $i \neq j$, and each g_i is a 1-special element.

Before going on to prove Lemma 2 note that it has the following Corollary.

COROLLARY. If G is almost finite-valued then G is normal-valued.

PROOF OF LEMMA 2. It is immediate that (C) implies (A) because the values of g consist of the disjoint union of the sets of values of the g_i.

(A) implies (B). Let $\{Q_i | i \in I\}$ denote the set of distinct 1-special values of g, $\{V_\lambda | \lambda \in \Lambda\}$ the set of its distinct special values; we wish to show I is finite. For each $i \in I$ let $0 < g_i \in G$ be an element whose only non-special value is Q_i by replacing g_i by $g \wedge g_i$ we may suppose $g_i \leqslant g$ for all $i \in I$. In the same manner select for each $\lambda \in \Lambda$ a special element $0 < x_\lambda \in G$ having V_λ as its only value; as before, we may suppose $x_\lambda \leqslant g$ for each $\lambda \in \Lambda$. By replacing G by $G(g)$ we may suppose that generates G as a convex l-subgroup, and that the Q_i and V_λ are maximal in G. Now suppose $H = \left[\bigvee_{i \in I} G(g_i)\right] \bigvee_{\lambda \in \Lambda} [\boxplus G(x_\lambda)]$; that is, H is the convex l-subgroup generated by the g_i and the x_λ. If $H < G$ then $g \notin H$ is therefore contained in a value of g. This value must either be one of the Q_i or else one of the V_λ; but each g_i and each x_λ lies in H, which gives a contradiction. Consequently $G = H$.

Since G is compact in its own lattice of convex l-subgroups it takes only a finite number of the g_i and x_λ to generate G. However, no g_i can be omitted, and hence I must be finite.

(B) implies (C). As in the previous part of the proof suppose that $\{V_\lambda | \lambda \in \Lambda\}$ stands for the special values of g, and that the set $\{x_\lambda | \lambda \in \Lambda\}$ has been selected as we did there. Furthermore suppose $Q_1, \ldots, Q_m$ are picked as before. We may in addition assume (since they are only finitely many g_i to worry about), that they are pairwise disjoint. The $\{x_\lambda\}$ are necessarily pairwise disjoint.

Once again assume that $G = G(g)$, and form $H = \left[\boxplus_{i=1}^{m} G(g_i)\right] \vee \vee \left[\boxplus_{\lambda \in \Lambda} G(x_\lambda)\right]$. By a similar argument it turns out that $G = H$, and that only finitely many $\lambda_1, \ldots, \lambda_n$ are required among the $\lambda < \Lambda$. Thus we may express $G = \left[\boxplus_{i=1}^{m} G(g_i)\right] \vee \left[\boxplus_{j=1}^{n} G(x_{\lambda_j})\right]$; as before, each g_i must be used.) We must take care of the difficulty that some g_i may not be disjoint to the x_{λ_j}.

To that end define $h_i = g_i - \left(g_i \wedge (x_{\lambda_1} + \ldots + x_{\lambda_n})\right)$. The reader should verify that each h_i is disjoint to each x_{λ_j}, and that $G = \left[\boxplus_{i=1}^{m} G(h_i) \boxplus\right] \boxplus \left[\boxplus_{j=1}^{n} G(x_{\lambda}')\right]$. Note that each h_i is 1-special; indeed Q_i is its only non-special value. First, it is clear that Q_i is a value of h_i. Now if Q is a non-special value of h_i then Q must be a value of g, and hence coincide with some Q_{i_1}. Yet this makes Q_{i_1} a value of both h_i and h_{i_1}, which is absurd since they are disjoint.

The only thing left is to express

$$g = a_1 + \dots + a_m + z_1 + \dots + z_n ,$$

where $a_i \in G(h_i)$ $(i = 1, \dots, m)$ and $z_j \in G(x_{\lambda_j})$; the a_i and z_j together form a pairwise disjoint set. It is an easy matter to verify that each z_j is special (with value V_{λ_j}) while each a_i is 1-special (with Q_i as its only non-special value). This is the desired representation of (C).

The proof of Lemma is therefore complete.

Before leaving the above argument let us make an observation. Suppose $\lambda \in \Lambda$ is not one of the λ_j selected in the representation $G = \left[\boxplus_{i=1}^{m} G(g_i)\right] \bigvee \left[\boxplus_{j=1}^{n} G(x_{\lambda_j})\right]$. Since $x_\lambda \wedge x_{\lambda_j} = 0$ for all $j = 1, \dots, n$, V_λ must lie beneath a value of some g_i, and therefore coincide with it. Hence each «non-selected» V_λ is a value for some g_i.

Suppose now that G is an arbitrary l-group, and define $\mathscr{F}v(G)$ to be the intersection of all non-special values of G. This is nothing but the torsion-radical of G relative to the class $\mathscr{F}v$ of finite-valued l-groups; (see [4]). $\mathscr{F}v(G)$ then is the largest convex l-subgroup of G lying in $\mathscr{F}v$; $0 < g \in \mathscr{F}v(G)$ if and only if every value that doesn't contain g is special. We say that $G \in \mathscr{F}v^2$ if it is an extension of one finite-valued l-groups by another.

Now our main result.

3. Main theorem. For an l-group G the following are equivalent.

(1) Each value of G is either 1-special or special.

(2) G is almost finite-valued.

(3) Each $0 < g \in G$ can be written as a sum of pairwise disjoint 1-special elements.

(4) $G \in \mathscr{F}v^2$.

Proof. The equivalence of (1), (2) and (3) is the global version of Lemma 2.

So suppose any of these three conditions holds. We must prove that $G/\mathscr{F}v(G)$ is finite-valued. Suppose $0 < g \in G \backslash \mathscr{F}v(G)$; as in previous arguments, let $\{Q_1, \ldots, Q_m\}$ be the non-special values of g, and $\{V_\lambda | \lambda \in \Lambda\}$ be its special values. There is at least one such Q_i, and each $Q_i \geqslant \mathscr{F}v(G)$. What might go wrong is that infinitely many of the V_λ contain $\mathscr{F}v(G)$ as well. Recall that $V_\lambda \not\geqslant \mathscr{F}v(G)$ if and only if every value beneath V_λ is special.

So suppose Q is an non-special value lying beneath some special value of g. Following the proof of Lemma 2, select a pairwise disjoint set $g_1, \ldots, g_m$, h such that each $g_i \leqslant g$ and $h \leqslant g$, and Q is a value of h, while Q_i is a value of g_i. According to the remark following the proof of Lemma 2 there is a selection $\lambda_1, \ldots, \lambda_n$ so that if $\lambda \in \Lambda \backslash \{\lambda_1, \ldots, \lambda_n\}$ V_λ is the value of one of the g_i. Since Q is a value of h and $h \wedge g_i = 0$ for each $i = 1, 2, \ldots, m$, Q must lie beneath V_{λ_j} for some $j = 1, \ldots, n$. The selection of $\lambda_1, \ldots, \lambda_n$ does not depend on h, and so we have proved that at most finitely many special values lie over non-special ones. It is then clear that $g + \mathscr{F}v(G)$ has finitely many values in $G/\mathscr{F}v(G)$, and hence that $G/\mathscr{F}v(G)$ in finite-valued.

The proof that (4) implies the other three is straight-forward.

From this theorem we can get several corollaries about particular kinds of extensions of finite-valued l-groups. For example:

Corollary I. G is an extension of a finite-valued l-group by one with a finite basis if and only if there is a natural number n such that each $0 < g \in G$ has at most n non-special values.

Corollary II. G is an extension of a finite-valued l-group by an o-group, if and only if each $0 < g \in G$ is either finite-valued or else 1-special.

The proofs are quite straight-forward. For the pertinent definitions we refer the reader to [1].

Before closing this section we should point out that there is an obvious inductive definition of an α-special element, where α is an ordinal number, leading to a characterization of l-groups in the class $\mathscr{F}v$, we shall defer any discussion of these ideas to another time.

3. Local characteristics of 1-special elements.

We wish to examine 1-special elements, and determine when they can be «approximated» by special ones. Specifically: if $0 < g \in G$ is

1-special, then under what conditions can g be written as a join of pairwise disjoint special elements: If this join is finite then g must be finite-valued (to satisfy such a condition). Since we are dealing with a 1-special element such a join of special elements, when possible, is necessarily infinite. Let Q be the non-special value of g, and $\{V_\lambda | \lambda \in \Lambda\}$ its set of special values. Recall that a convex l-subgroups C of G is *closed* if it is closed under all existing joins and meets of subsets of C. It is well known that if a prime lies over a closed prime then it too is closed. Furthemore, any special value is closed; (see [1]).

Our first result is as follows:

4. Proposition. Suppose $0 < g \in G$ is 1-special. Then g can be written as a join of special elements if and only if Q is not closed.

Proof. Suppose Q is not closed, and select, for each $\lambda \in \Lambda$, a special element $0 < x_\lambda \in G$ with V_λ as value. As in previous arguments we can suppose $g \geqslant x_\lambda$, for each $\lambda \in \Lambda$, and that the x_λ are pairwise disjoint. In this argument we must be a little more careful in our selection of the x_λ. First, we make certain that, modulo V, $g \leqslant x_\lambda$ for each value $V \leqslant V_\lambda$; (this can be done since V_λ is normal in V_λ^* and we can replace x_λ by a suitably large multiple). Then insure that $x_\lambda \leqslant g$ by taking $x_\lambda \wedge g$ in place of x_λ; notice that $g \equiv x_\lambda$ mod V for all values $V \leqslant V_\lambda$. We claim that $g = Vx_\lambda$.

Suppose $0 < h \in G$ and $x_\lambda \leqslant h$ for each $\lambda \in \Lambda$. In order to show that $g \leqslant h$ we must prove that $-h + g$ has no positive values. By way of contradiction, suppose N is a positive value of $-h + g$, that is, $g + N > h + N$. Since $h > 0$ it follows that $g \notin N$, and therefore that N lies under a value of g. If $N \leqslant V_\lambda$ for some $\lambda \in \Lambda$ then $g + N = x_\lambda + N \leqslant h + N$, which contradicts our choice of N. Therefore $N \leqslant Q$. We've proved then that every positive value of $-h + g$ lies beneath Q; putting it differently: every value of $(-h + g) \vee 0$ lies beneath Q. This makes Q an essential value (see [1]) and essential values are closed; this is a contradiction. Hence $g \leqslant h$ and $g = \bigvee x_\lambda$ as promised.

If on the other hand Q is closed then the canonical map $x \to x + Q$ preserves all existing infs and sups. Therefore if g can be expressed as a join of special elements there must be a special element $0 < s \leqslant g$ not in Q. This implies that Q is special, a contradiction. Hence g is not expressible as a join of pairwise disjoint special elements, and our result is proved.

We state some corollaries of Proposition 4.

COROLLARY I. If G is an Archimedean l-group then each positive 1-special element is a join of pairwise-disjoint special elements.

PROOF. In an Archimedean l group a closed convex l subgroup is a polar; (see [1]). Furthermore, a value which is at once a polar is minimal and the value of a basic element; (again, refer to [1]). This is implies that a 1 special value in an Archimedean l group cannot be closed; now apply Proposition 4.

The next corollary may be proved independently, without appealing to Proposition 4.

COROLLARY II. Suppose $G \in \mathscr{F}v^2$; then $\mathscr{F}v(G)$ is closed if and only if each value of G is closed.

If $G \in \mathscr{F}v^2$ then certainly the set $\mathscr{S}$ of special values of G separate points; $(\cap \mathscr{S} = 0)$. In addition, G is normal valued, and so every closed value is essential; (see [1]). It follows that M is a closed value if and only if it lies over a special value. It is well known, (see [3]), that in an l-group G each $0 < g \in G$ is a join of pairwise-disjoint special elements if and only if $\mathscr{S}$ is a *plenary set*, meaning that (1) $\cap \mathscr{S} = 0$, and (2) if $S \in \mathscr{S}$ and M is a value lying over S, then $M \in \mathscr{S}$. Putting together the above remarks we have:

5. PROPOSITION. Suppose G is a normal valued in which the special values separate points. Then each $0 < g \in G$ is a join of pairwise disjoint special elements if and only if every closed value of G is special.

Corollary II to Proposition 4 states when the radical $\mathscr{F}v(G)$ in a $\mathscr{F}v^2$ l-group is closed. Proposition 5 records the other extreme: if $G \in \mathscr{F}v^2$ and every closed value is special then G is the closure of $\mathscr{F}v(G)$. For the intermediate cases we have the following.

COROLLARY. Suppose G is an l-group and $0 < g \in G$ is 1-special. The following are equivalent.

(1) Each closed value of g is special.

(2) g can be expressed as a pairwise-disjoint supremum of special elements.

(3) g belongs to the closure of $\mathscr{F}v(G)$.

PROOF. (1) implies (2) by Proposition 4; (2) implies (3) is clear. Now if (3) holds then every closed, non-special value of G contains g. Thus (1) is satisfied.

We add one comment to the proof; in view of the above equivalences it follows that if g can be written as a join of special elements it can also be done via *pairwise-disjoint* special elements.

4. Extensions from a torsion class by a finite valued l-group.

In the present context a *torsion class* shall be one closed under forming (a) l-homomorphic images, (b) convex l-subgroups and (c) joins of convex l-subgroups. If $\mathscr{T}$ is a torsion class we let $\mathscr{T}(G)$ denote the $\mathscr{T}$-radical of G; this is the supremum of all convex l-subgroups of G belonging to $\mathscr{T}$. Torsion classes were introduced in [4].

In [5] the author introduced the notion of a *prime selector*. Suppose $\mathbb{P}(G)$ stands for the family of prime subgroups of an l-group G. The function $G \to \mathbb{H}(G) \leqslant \mathbb{P}(G)$ is a (hereditary) *prime selector* if (i) for each l-homomorphism $\varphi\colon G \xrightarrow{\text{onto}} H$ and each prime $N \geqslant \operatorname{Ker}\varphi$, $N \in \mathbb{H}(G)$ implies that $N\varphi \in \mathbb{H}(H)$, and (ii) for each convex l-subgroup C of G and each prime $N \not\geqslant C$, $N \in \mathbb{H}(G)$ if and only if $N \cap C \in \mathbb{H}(C)$.

We set $\operatorname{Tor}(\mathbb{H}) = \{G | \mathscr{T}(G) = \mathbb{P}(G)\}$. Then all of the following may be found in [5]: (a) $\operatorname{Tor}(H)$ is a torsion class. If $\mathscr{T} = \operatorname{Tor}(\mathbb{H})$ we say that $\mathbb{H}$ is a *presentation* for $\mathscr{T}$. (b) Each torsion class $\mathscr{T}$ has a presentation $\mathbb{H}$ such that

$$(*) \qquad \cap \{P \in \mathbb{P}(G) | P \notin \mathbb{H}(G)\} = \mathscr{T}(G)\,.$$

Let us look at some familiar examples of prime selectors:

(A) $N \in \mathbb{H}(G)$ if and only if N is a minimal prime. Then $\operatorname{Tor}(\mathbb{H})$ is the class of hyperarchimedean l-groups.

(B) $N \in \mathbb{H}(G)$ if and only if N is not a value, or else N is special.

Then $\operatorname{Tor}(\mathbb{H}) = \mathscr{F}v$

(C) $N \in \mathbb{H}(G)$ if and only if N is not a value, or else N is normal in its cover N^*. $\operatorname{Tor}(\mathbb{H}) = \mathscr{N}$, the class of normal-valued l-groups.

All three of the above selectors satisfy $(*)$.

Now let us suppose that $\mathscr{T}$ is a torsion class with a presentation $\mathbb{H}$ subject to $(*)$. We say that $g \neq 0$ in G is *almost-$\mathscr{T}$* if all but (possibly) finitely many of its values lie in $\mathbb{H}(G)$. If each non-zero element of G is *almost-$\mathscr{T}$* we say that G is *almost-$\mathscr{T}$*. We realize that *almost-$\mathscr{T}$*-ness may depend on the choice of selector; our conjecture below is that it doesn't.

If $G \in \mathscr{T} \cdot \mathscr{F}v$, that is, if $G/\mathscr{T}(G)$ is finite valued, then since our selectors satisfy $(*)$ it follows that every non-zero element of G can have no more than a finite number of values outside $\mathbb{H}(G)$. Hence G is *almost-$\mathscr{T}$*.

On the other hand it follows from the definition of prime selectors that the class of *almost-$\mathscr{T}$* l-groups is a torsion class. In particular then, $G/\mathscr{T}^*(G)$ is *almost-$\mathscr{T}$* if G is *almost-$\mathscr{T}$*. ($\mathscr{T}^*$ denotes the completion of $\mathscr{T}$.) Hence, if G is an *almost-$\mathscr{T}$* l-group we may without loss of generality assume that $\mathscr{T}(G) = 0$. If the selector satisfies the property that $\mathbb{H}(L)$ is an ideal of $\mathbb{P}(L)$ (relative to inclusion), for each l-group L, then we have (by property $(*)$) that G has a plenary set (namely the non-selected values) in which every element $g \neq 0$ has finitely many values. By a result from [2] (Theorem 3.7) this implies that G is finite-valued.

We summarize the above as follows:

6. PROPOSITION. Suppose $\mathscr{T}$ is a torsion-class with a presentation $\mathbb{H}$ satisfying $(*)$, and such that for each l-group L, $\mathbb{H}(L)$ is an ideal of $\mathbb{P}(L)$. Then the class of *almost-$\mathscr{T}$* l-groups is a torsion class and $\mathscr{T} \cdot \mathscr{F}v \leqslant$ *almost-$\mathscr{T}$* $\leqslant \mathscr{T}^* \cdot \mathscr{F}v$, where $\mathscr{T}^*$ denotes the completion of $\mathscr{T}$.

Once again, we should point out that « *almost-$\mathscr{T}$* » depends (a priori) on the selector $\mathbb{H}$. We conjecture though that *almost-$\mathscr{T}$* $= \mathscr{T} \cdot \mathscr{F}v$ regardless of the choice of $\mathbb{H}$. Unfortunately the techniques of Section 2 seem to be difficult to apply, unless the selector $\mathbb{H}(G) = \{N \in \mathbb{P}(G|N \not\supseteq \mathscr{T}(G))\}$. We can prove for this selector only that *almost-$\mathscr{T}$* $= \mathscr{T} \cdot \mathscr{F}v$.

In particular, the selector of minimal primes from (A) above satisfies all the hypotheses of Proposition 6. So if $\mathscr{A}r$ denotes the class of hyper-archimedean l-groups, then $\mathscr{A}r \cdot \mathscr{F}v \leqslant$ *almost-$\mathscr{A}r$* $\leqslant \mathscr{A}r^* \cdot \mathscr{F}v$. (*Almost-$\mathscr{A}r$* here means: for each $g \neq 0$ in G all but finitely many values of g are minimal.) However, this selector may leave a minimal prime that lies above the $\mathscr{A}r$-radical. We do not know whether *almost-$\mathscr{A}r$* $= \mathscr{A}r \cdot \mathscr{F}v$.

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