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Stability and Asymptotic Equivalence of Perturbations of Nonlinear Systems of Differential Equations.

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1. Introduction.

A nonlinear variation of constants method was introduced by Alekseev [1] and applications of this formula to questions of stability and asymptotic equivalence of differential systems was demonstrated by Brauer [2, 3, 4]. In [6] a different approach to the nonlinear variation of constants method is given. This new approach involves determining the solution of the perturbed system by variation of the starting vector in the unperturbed system. Conceptually this is the method used in obtaining the classical variation of constants formula for perturbations of linear systems.

In [6] the method yields two different formulas, one of which is equivalent to the Alekseev formula under the hypothesis which guarantees the Alekseev representation. Also, in [6] some applications to stability and asymptotic equilibrium are given.

The approach introduced in [6] was shown to be applicable for the study of integral and integro-differential systems in [7] and for the study of difference equations in [8]. In this paper some further applications of the nonlinear variation of constants result of [6] are

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obtained for differential equations. The result on asymptotic equivalence is related to that given by Brauer [3] and is shown to complement those results.

2. Preliminaries.

In this section we present some general hypothesis used throughout the paper and give as lemmas some results obtained in [6]. Consider the system

$$x' = f(t, x) , \quad x(t_0) = x_0 , \quad t_0 \geqslant 0 , \; x_0 \in \mathbb{R}^n , \tag{2.1}$$

and its perturbed system

$$y' = f(t, y) + F(t, y) , \qquad y(t_0) = x_0 , \tag{2.2}$$

where $f, F \in C[R^+ \times D, R^n]$ and D is a convex region in $\mathbb{R}^n$. The following lemma is obtained in [Theorem 2.1, 6].

LEMMA 1. Suppose that the system (2.1) admits a unique solution $x(t, t_0, x_0)$. Also, assume that $\Phi(t, t_0, x_0) = \partial x/\partial x_0(t, t_0, x_0)$ exists and is continuous for $t \geqslant t_0$ and that $\Phi^{-1}(t, t_0, x_0)$ exists for $t \geqslant t_0$. If $v(t)$ is a solution of (2.4) or (2.5) then any solution $y(t, t_0, x_0)$ of (2.2) satisfies

$$y(t, t_0, x_0) = x\big(t, t_0, v(t)\big) \tag{2.3}$$

as far as $v(t)$ exists to the right of t_0.

As shown in the proof of Lemma 1 $v(t)$ must satisfy the initial value problem

$$v'(t) = \Phi^{-1}\big(t, t_0, v(t)\big) F\big(t, x(t, t_0, v(t))\big) , \qquad v(t_0) = x_0 . \tag{2.4}$$

Integration of (2.4) yields

$$v(t) = x_0 + \int_{t_0}^{t} \Phi^{-1}\big(s, t_0, v(s)\big) F\big(s, x(s, t_0, v(s))\big)\, ds . \tag{2.5}$$

Formula (2.3) gives one form of the nonlinear variation of constants formula. A second integral form is given in [6] by

LEMMA 2. Under the hypothesis of Lemma 1, the following relation is valid

$$(2.6)\qquad y(t, t_0, x_0) = x(t, t_0, x_0) + \\ + \int_{t_0}^{t} \Phi(t, t_0, v(s))\, \Phi^{-1}(s, t_0, v(s))\, F(s, y(s, t_0, x_0))\, ds ,$$

where $v(t)$ is any solution of (2.4) or (2.5).

The nonlinear variation of constants type results due to Alekseev [1, or p. 74 in 5] is of the form

$$(2.7)\qquad y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, F(s, y(s))\, ds .$$

This formula is obtained under the assumption that $f_x(t, x)$ is continuous for $(t, x) \in R^+ \times D$. The proof of (2.7) uses the facts that solutions $x(t, t_0, x_0)$ of (2.1) are unique, differentiable with respect to initial data and satisfies

$$(2.8)\qquad \begin{aligned} \frac{\partial x}{\partial x_0}(t, t_0, x_0) &= \Phi(t, t_0, x_0) , \\ \frac{\partial x}{\partial t_0}(t, t_0, x_0) &= -\, \Phi(t, t_0, x_0) f(t_0, x_0) , \end{aligned}$$

where $\Phi(t, t_0, x_0)$ is the fundamental matrix solution of the variational equation

$$z' = f_x(t, z(t, t_0, x_0)) , \qquad \Phi(t_0, t_0, x_0) = I .$$

From [6] we have

LEMMA 3. Assume that $f_x(t, x) \in C[R_+ \times D, R^n]$, then formulas (2.6) and (2.7) are equivalent.

The proof of Lemma 3 demonstrates that

$$\Phi(t, s, y(s)) = \Phi(t, t_0, v(s))\,\Phi^{-1}(s, t_0, v(s)) \,, \tag{2.9}$$

where $v(t)$ is a solution of (2.4) or (2.5).

A final preliminary result is given by

LEMMA 4. If $x_0, y_0 \in D$ then any solution of (2.1) satisfies

$$x(t, t_0, y_0) - x(t, t_0, x_0) = \int_0^1 \Phi(t, t_0, sy_0 + (1-s)x_0)\, ds \cdot (y_0 - x_0) \,. \tag{2.10}$$

The proof of this estimate is obtained by integration from $s = 0$ to $s = 1$ of the expression

$$\frac{d}{dx} x(t, t_0, sy_0 + (1-s)x_0) = \Phi(t, t_0, sy_0 + (1-s)x_0)\cdot(y_0 - x_0) \,.$$

In [6] results on uniform stability and asymptotic equilibrium are presented. These results are obtained using the hypothesis of Lemma 1 together with the estimate

$$\|\Phi^{-1}(t, t_0, x_0) F(t, x(t, t_0, x_0))\| \leqslant g(t, x_0) \,, \tag{2.11}$$

where $g \in C[R^+ \times R^+, R^+]$, $g(t, 0) \equiv 0$. By assuming appropriate asymptotic behavior of the comparison equation

$$u' = g(t, u) \,, \qquad u(t_0) = u_0 \geqslant 0 \,, \tag{2.12}$$

corresponding asymptotic behavior of the perturbed system (2.2) can be determined. The remainder of this paper deals with further application of the estimate (2.10). It is also emphasized that these results are obtained using the variation of constants formula (2.3). This form is not available using the Alekseev method.

3. Asymptotic equivalence.

The two systems (2.1) and (2.2) are said to be asymptotically equivalent if given a solution $y(t, t_0, y_0)$ of (2.2) there exists a solution

$x(t, t_0, x_0)$ of (2.1) satisfying

$$\lim_{t \to +\infty} y(t, t_0, y_0) - x(t, t_0, x_0) = 0 , \tag{3.1}$$

and, conversely, given $x(t, t_0, x_0)$ a solution of (2.1) there exists $y(t, t_0, y_0)$ a solution of (2.2) satisfying (3.1). The following theorem relates the asymptotic equivalence of the two systems (2.1) and (2.2).

THEOREM 3.1. Assume the hypothesis of Lemma 1 and $\Phi(t, t_0, x_0)$ is bounded for $t \geqslant t_0$ and $x_0 \in D$. Further, assume the estimate (2.11) where g is monotone nondecreasing in u for each fixed $t \in R^+$ and all solutions of (2.12) are bounded on $t \geqslant t_0$. Then given $y(t, t_0, y_0)$ a solution of (2.2) there exists a solution $x(t, t_0, x_0)$ of (2.1) satisfying (3.1).

PROOF. Now any solution of (2.2) can be represented by

$$y(t, t_0, y_0) = x\big(t, t_0, v(t)\big) ,$$

where $v(t)$ is a solution of (2.4) with $v(t_0) = y_0$. The estimate (3.1) guarantees $v(t)$ has asymptotic equilibrium by a result in [Theorem 2.9.1, 5]. Thus let $v(t)$ denote the solution of (3.1) with $v(t_0) = y_0$ and take

$$x_0 = \lim_{t \to +\infty} v(t) .$$

Now Lemma 4 together with the boundedness of $\Phi(t, t_0, x_0)$ gives

$$\begin{aligned} \|y(t, t_0, y_0) - x(t, t_0, x_0)\| &= \|x\big(t, t_0, v(t)\big) - x(t, t_0, x_0)\| \\ &\leqslant \sup_{t \geqslant t_0} \|\Phi(t, t_0, x_0)\| \, \|v(t) - x_0\| \\ &= K \|v(t) - x_0\| , \end{aligned}$$

from which it follows that $y(t, t_0, x_0) \to x(t, t_0, x_0)$ as $t \to +\infty$.

We remark that this only shows in one direction the condition for asymptotic equivalence. A complete asymptotic equivalence result could be obtained if the boundedness of $\Psi(t, t_0, y_0) = (\partial/\partial y_0) y(t, t_0, y_0)$ is assumed.

The asymptotic equivalence would then follow if (2.1) was considered as a perturbation of (2.2). The question of complete asymptotic equivalence with conditions on $\Phi(t, t_0, x_0)$ are studied by Marlin and Struble [9]. This problem will not be pursued in this paper. We would expect that in most applications, Theorem 1 would cover the useful part of the problem.

Theorem 3.1 is seen as a generalization of a result [Theorem 2.10.2, 5], where the system (2.1) is linear.

THEOREM 3.2. Assume that solutions of

$$x' = A(t)x\,, \qquad x(t_0) = x_0\,, \qquad A \in C[R^+, R^n]\,, \tag{3.2}$$

are bounded as $t \to +\infty$ and

$$\liminf_{t\to\infty} \int_{t_0}^{t} \operatorname{tr} A(s)\,ds > -\infty\,. \tag{3.3}$$

Further, assume

$$\|F(t, x)\| \leqslant \lambda(t)\|x\|\,, \tag{3.4}$$

where $F \in C[R^+ \times R^n, R^n]$ and $\lambda(t) \geqslant 0$ is continuous on R^+ satisfying

$$\int_{t_0}^{t} \lambda(s)\,ds < +\infty\,. \tag{3.5}$$

Then given any solution of $y' = A(t)y + F(t, y)$ existing for $t \geqslant t_0$ there exists a solution of (3.2) such that $y(t, t_0, y_0) - x(t, t_0, x_0) \to 0$ as $t \to \infty$.

PROOF. The estimate (3.3) implies $\|\Phi^{-1}(t, t_0)\|$ is bounded for all $t \geqslant t_0$. Thus the inequality (3.4) gives

$$\begin{aligned} \|\Phi^{-1}(t, t_0)F\big(t, x(t, t_0, x_0)\big)\| &\leqslant K\lambda(t)\|x(t, t_0, x_0)\| \\ &\leqslant K\lambda(t)\|\Phi(t, t_0)\|\,\|x_0\| \\ &\leqslant \lambda(t)\|x_0\|\,, \end{aligned}$$

where $\lambda(t)$ is scaled by a constant factor. The hypothesis (3.5) guarantees solutions of $u' = g(t, u) = \lambda(t)u$ are bounded for all $t \geqslant t_0$. Clearly, $\lambda(t)u$ is monotone nondecreasing in u. Theorem 1 gives the desired result.

Note that in applying Theorem 3.1 just one side of the asymptotic equivalence is obtained.

An asymptotic equivalence (one sided) result is given by Brauer [Theorem 4, 3] under the hypothesis of Lemma 3.

THEOREM 3.3. Suppose for $y(t, t_0, y_0)$ a solution of (2.2) that

$$\lim_{t\to+\infty}\int_t^{+\infty}\Phi(t, s, y(s))F(s, y(s))ds = 0\,. \tag{3.6}$$

Then there exists a solution $x(t, t_0, x_0)$ of (2.1) such that

$$x(t, t_0, x_0) = y(t, t_0, y_0) + \int_t^{+\infty}\Phi(t, s, y(s))F(s, y(s))\,ds\,. \tag{3.7}$$

In particular $y(t, t_0, y_0) - x(t, t_0, x_0) \to 0$ as $t \to +\infty$.

The relationship between Theorem 3.1 and Theorem 3.3 will be considered. It will now be shown that under the hypothesis of Theorem 3.1 a condition similar to (3.5) holds and under the further hypothesis of Lemma 3 this condition agrees with (3.5).

Recall from the proof of Theorem 3.1 that

$$y(t, t_0, y_0) = x(t, t_0, v(t))\,, \qquad v(t_0) = y_0\,, \tag{3.8}$$

for $v(t)$ a solution of (2.4) and $v(t)$ has asymptotic equilibrium. Solving (3.7) for $v(t)$ gives

$$v(t) = x(t_0, t, y(t))\,, \tag{3.9}$$

and let

$$x_0 = \lim_{t\to\infty} v(t) = \lim_{t\to\infty} x(t_0, t, y(t))\,. \tag{3.10}$$

From the boundedness of $\Phi(t, t_0, x_0)$

$$\|x(t, t_0, v(T)) - x(t, t_0, x_0)\| \leqslant K\|v(T) - x_0\|\,. \tag{3.11}$$

Now $x(t, T, y(T)) = x(t, t_0, v(T))$ so that (3.10) and (3.11) imply

$$\lim_{T\to+\infty} x(t, T, y(T)) = \lim_{T\to+\infty} x(t, t_0, v(T)) = x(t, t_0, x_0)\,.$$

Integration from t to T of

$$\frac{d}{ds} x(t, t_0, v(s)) = \Phi(t, t_0, v(s)) v'(s))$$

yields

$$x(t, T, y(T)) = y(t, t_0, y_0) + \int_t^T \Phi(t, t_0, v(s))\, \Phi^{-1}(s, t_0, v(s)) F(s, y(s))\, ds .$$

Taking the limit as $T \to +\infty$ implies

$$(3.12) \qquad x(t, t_0, x_0) =$$

$$= y(t, t_0, y_0) + \int_t^{+\infty} \Phi(t, t_0, v(s))\, \Phi^{-1}(s, t_0, v(s)) F(s, y(s))\, ds .$$

From the proof of Theorem 3.1 $x(t, t_0, x_0) - y(t, t_0, y_0) \to 0$ as $t \to +\infty$, therefore,

$$(3.13) \qquad \lim_{t \to +\infty} \int_t^{+\infty} \Phi(t, t_0, v(s)) \Phi^{-1}(s, t_0, v(s)) F(s, y(s)) ds = 0 .$$

It is clear now that if the conditions of Lemma 3 are satisfied then conditions (3.12) and (3.13) reduce to conditions (3.7) and (3.6), respectively.

In determining asymptotic equivalence Brauer [3] applies a corollary of Theorem 3.3 which assumes boundedness of $\Phi(t, t_0, x_0)$ uniformly in $t \geqslant 0$, $t_0 \geqslant 0$ and $x_0 \in D$, and $\int_0^{\infty} F(s, y(s))\, ds < \infty$. The condition in Theorem 3.1 assumes boundedness of $\Phi(t, t_0, x_0)$ for $t \geqslant t_0$ and $x_0 \in D$, that is, we do not require boundedness with respect to t_0. This provides flexibility in applications as demonstrated by the following example.

Consider $x' = -(x+1) \ln (x+1)$, $x(t_0) = x_0 > -\frac{1}{2}$. The solution is given by

$$x(t, t_0, x_0) = -1 + (x_0 + 1)^{\exp[-(t - t_0)]}, \qquad t \geqslant t_0 .$$

Then

$$\Phi(t, t_0, x_0) = \exp[-(t-t_0)](x_0+1)^{\exp[-(t-t_0)]-1}, \qquad t \geqslant t_0 .$$

$\Phi(t, t_0, x_0)$ has the property that it is uniformly bounded in t and x_0 if x_0 is bounded away from -1, say $x_0 > -\frac{1}{2}$, but $\Phi(t, t_0, x_0)$ is not uniformly bounded in t_0. Now $\Phi^{-1}(t, t_0, x_0)$ exists and Theorem 3.1 applies to any system $y' = -(y+1)\ln(y+1) + F(t, y)$ satisfying

$$\|\Phi^{-1}(t, t_0, x_0)F(t, x(t, t_0, x_0))\| \leqslant g(t, \|x_0\|),$$

where g meets the conditions stated in Theorem 3.1.

4. Stability.

The estimate (2.10) can be used to obtain stability results. The first theorem gives conditions under which stability (uniform stable) are preserved.

THEOREM 4.1. Assume the conditions of Lemma 1 and the estimate

$$\|\Phi^{-1}(t, t_0, x_0)F(t, x(t, t_0, x_0))\| \leqslant g(t, \|x_0\|) \tag{4.1}$$

holds, where $g \in C[R^+ \times R^+, R^+]$, $g(t, 0) \equiv 0$ and the trivial solution of

$$u' = g(t, u), \qquad u(t_0) = u_0 \geqslant 0 \tag{4.2}$$

is stable (uniformly stable). Further, assume the trivial solution of (2.1) is stable (uniformly stable). Then the trivial solution of (2.2) is stable (uniformly stable).

This theorem for the case of uniform stability is given in [6]. Therefore, the proof is omitted. It is given here with the additional result for stability for completeness.

The next theorem is new and its proof is similar to the proof of Theorem 4.1.

THEOREM 4.2. Assume the conditions of Lemma 1 and the estimate (4.1) holds, where the trivial solution of

$$u' = g(t, u), \qquad u(t_0) = u_0 \geqslant 0, \tag{4.2}$$

is stable (uniformly stable). Further, assume the trivial solution of (2.1) is asymptotically stable (uniformly asymptotically stable). Then the trivial solution of (2.2) is asymptotically stable (uniformly asymptotically stable).

PROOF. The stability of (2.1) and (4.2) implies by Theorem 4.1 the stability of (2.2). By Lemma 1 any solution of (2.2) satisfies

$$y(t, t_0, x_0) = x(t, t_0, v(t))$$

where $v(t)$ is a solution of (2.4). The assumption (4.1) then implies, setting $m(t) = \|v(t)\|$, the inequality

$$D^+ m(t) \leqslant g(t, m(t)) ,$$

which by the comparison theorem [Theorem 1.4.1, 5] yields the estimate

$$\|v(t)\| = m(t) \leqslant r(t, t_0, \|x_0\|) , \qquad t \geqslant t_0 ,$$

where $r(t, t_0, u_0)$ is the maximal solution of (4.2). The asymptotic stability of (2.1) implies there is a $\delta_1 > 0$ such that when $\varepsilon > 0$, $t_0 \geqslant 0$ are given there exists a $T \in (\varepsilon, t_0)$ satisfying $\|x(t, t_0, x_0)\| < \varepsilon$ for $t \geqslant T \cdot (\varepsilon, t_0)$ and $\|x_0\| < \delta_1$. Also, the stability of (4.2) implies there is a δ depending on δ_1 and t_0 such that $r(t, t_0, \|x_0\|) < \delta_1$ whenever $\|x_0\| < \delta$. Thus $\|v(t)\| \leqslant r(t, t_0, \|x_0\|) < \delta_1$ whenever $\|x_0\| < \delta$. The solution of (2.2) then satisfies

$$\|y(t, t_0, x_0)\| = \|x(t, t_0, v(t))\| < \varepsilon \qquad \text{for all } t \geqslant T(\varepsilon, t_0) , \text{ when } \|x_0\| < \delta .$$

This completes the proof of the asymptotic stability. The uniform asymptotic stability is similar to the above except T and δ are independent of t_0.

THEOREM 4.3. Assume the conditions of Lemma 1 and the estimate (4.1) holds, where solutions of

$$u' = g(t, u) , \qquad u(t_0) = u_0 > 0 ,$$

are bounded for all $t \geqslant t_0$. Further, assume the zero solution of (2.1) is exponentially asymptotically stable, then the zero solution of (2.2) is exponentially asymptotically stable.

PROOF. Arguing as in the proof of Theorem 4.2 we have

$$\|y(t, t_0, x_0)\| = \|x(t, t_0, v(t))\| \leqslant K\|v(t)\| \exp[-\alpha(t-t_0)], \qquad K, \alpha > 0 .$$

Thus the exponentially asymptotic stability of the zero solution of (2.2) follows since the boundedness of solutions of (4.2) implies the boundedness of $\|v(t)\|$ for all $t \geqslant t_0$.

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