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Carathéodory's Selections for Multifunctions with Non-Separable Range.

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Introduction.

Let $(S, \mathcal{E})$ be a measurable space, X a paracompact topological space, Y a Banach space and F a multifunction from $S\times X$ into Y. Our problem is to find a Carathéodory selection for F, that is a single-valued function f from $S\times X$ into Y such that: i) $f(t, x)\in F(t, x)$ for each $(t, x)\in S\times X$; the function $t\to f(t, x)$ is $\mathcal{E}$-measurable for each $x\in X$; ii) the function $x\to f(t, x)$ is continuous for each $t\in S$.

There are already some papers devoted to this problem (see [1], [2], [3], [4]). Nevertheless, in all of these papers the separability of X is assumed and then, from it, the separability of $F(S\times X)$ follows (see Proposition 2.2).

The aim of the present paper is to establish two theorems in which no hypothesis of separability of X neither of $F(S\times X)$ is needed. Moreover, some particular values of the multifunction F are allowed to be non-convex.

In Section 1 we put notations and basic definitions, while Section 2 contains the main results.

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1. Notations and basic definitions.

Given two sets $A', A'' \neq \emptyset$, we denote by $\mathcal{F}(A', A'')$ the set of all functions from A' into A''. Given a function F from A' into the family of all non-empty subsets of A'', we say that F is a multifunction from A' into A'' and, for each $A \subseteq A'$ and each $\Omega \subseteq A''$, we put: $F(A) = \bigcup_{t\in A} F(t)$ and $F^-(\Omega) = \{t \in A' : F(t) \cap \Omega \neq \emptyset\}$. If A', A'' are topologized, we say that a multifunction F from A' into A'' is lower semicontinuous whenever the set $F^-(\Omega)$ is open in A' for each open set $\Omega \subseteq A''$. Moreover, if $\mathcal{E}$ is a σ-algebra of subsets of A', we say that the multifunction F is $\mathcal{E}$-measurable whenever the set $F^-(\Omega)$ belongs to $\mathcal{E}$ for each open set $\Omega \subseteq A''$. Now let (Σ, d) be a metric space. For each $x \in \Sigma$, $\Omega, \Omega', \Omega'' \subseteq \Sigma$ and $r \in \mathbb{R}^+$, we put: $B_d(\Omega, r) = \{y \in \Sigma : \exists z \in \Omega : d(y, z) < r\}$; $d(x, \Omega'') = \inf\{d(x, y) : y \in \Omega''\}$; $d^*(\Omega', \Omega'') = \sup\{d(z, \Omega'') : z \in \Omega'\}$. If $(\Sigma, \|\cdot\|)$ is a normed space, we will consider Σ with the metric d induced by the norm $\|\cdot\|$. Furthermore, given a set $A \neq \emptyset$, we put $\mathcal{F}_b(A, \Sigma) = \{f \in \mathcal{F}(A, \Sigma) : \sup_{t\in A} \|f(t)\| < +\infty\}$ and we consider $\mathcal{F}_b(A, \Sigma)$ as a normed space with the usual norm: $\|f\|_b = \sup_{t\in A} \|f(t)\|$, $f \in \mathcal{F}_b(A, \Sigma)$. Finally, we denote by d_b the metric induced by the norm $\|\cdot\|_b$.

2. The main results.

We begin this section by proving an useful proposition.

PROPOSITION 2.1. *Let $(S, \mathcal{E})$ be a measurable space, (Σ, d) a metric space and F, G two $\mathcal{E}$-measurable multifunctions from S into Σ such that $F(S)$ and $G(S)$ are separable. If $r \in \mathbb{R}^+$ is such that $I_r(t) = F(t) \cap B_d(G(t), r) \neq \emptyset$ for each $t \in S$, then the multifunction I_r is $\mathcal{E}$-measurable.*

PROOF. Put $P(t) = F(t) \times G(t)$ for each $t \in S$. Thus, for each Ω', $\Omega'' \subseteq \Sigma$, we have

$$(1) \qquad P^-(\Omega' \times \Omega'') = F^-(\Omega') \cap G^-(\Omega'')$$

Moreover, for each $\Omega \subseteq \Sigma$, we have

$$(2) \qquad I_r^-(\Omega) = P^-\big(\{(x, y) \in F(S) \times G(S) : d(x, y) < r\} \cap (\Omega \times \Sigma)\big).$$

Now our claim follows from (2) and (1) since any open set in $F(S)\times G(S)$ is the union of a countable family of sets of type $\Omega'\times\Omega''$, with Ω', Ω'' open sets, respectively, in $F(S)$ and $G(S)$. ▲

We now prove the following selection theorem:

THEOREM 2.1. *Let $(S, \mathcal{E})$, X, Y, F be as in the Introduction and let $\{S_n\}$ be a sequence in $\mathcal{E}$ such that $S=\bigcup_{n=1}^{\infty} S_n$; Z a subset of X, with* $\dim_X(Z)\leqslant 0$ (1); *M an $\mathcal{E}$-measurable function from S into $\mathbb{R}^+$; $\{\lambda_n\}$ a sequence in $\mathcal{F}(X\times X, \mathbb{R}^+)$, with $\lim_{y\to x}\lambda_n(x, y)=0$ for each $x\in X$ and each $n\in N$. Moreover, let the following conditions be satisfied:*

1) *the set $F(t, x)$ is closed for each $(t, x)\in S\times X$ and convex for each $(t, x)\in S\times(X-Z)$;*
2) *the set $F(S_n, x)$ is separable and bounded for each $x\in X$, $n\in N$;*
3) *the multifunction $t\to F(t, x)$ is $\mathcal{E}$-measurable for each $x\in X$;*
4) *we have $d^*(F(t, x), F(t, y))\leqslant M(t)\lambda_n(x, y)$ for each $t\in S_n$, $x, y\in X$, $n\in N$.*

Under such hypotheses, there exists a Carathéodory selection for F.

PROOF. For each $m, n\in N$ put: $S_{m,n}=\{t\in S_n\colon M(t)\leqslant m\}$. Rearrange the non-empty members of the family $\{S_{m,n}\}_{m,n\in N}$ as a single sequence $\{S_k^*\}$. Obviously, $S_k^*\in\mathcal{E}$ for each $k\in N$ and $S=\bigcup_{k=1}^{\infty} S_k^*$. Fix $k\in N$ and denote by $\mathcal{E}_k$ the family of all subsets of S_k^* belonging to $\mathcal{E}$. Furthermore, put $\mathcal{A}_k=\{\varphi\in\mathcal{F}_b(S_k^*, Y)\colon \varphi$ is $\mathcal{E}_k$-measurable and $\varphi(S_k^*)$ is separable$\}$ and observe that $\mathcal{A}_k$ is a closed linear subspace of the Banach space $\mathcal{F}_b(S_k^*, Y)$. For each $x\in X$ put $H_k(x)=\{\varphi\in\mathcal{A}_k\colon \varphi(t)\in F(t, x), \forall t\in S_k^*\}$. From 1), 2), 3) and from the Kuratowski and Ryll-Nardzewski selection theorem, it follows that $H_k(x)\neq\emptyset$, and so we can consider the multifunction $x\to H_k(x)$ from X into $\mathcal{A}_k$. We claim that this multifunction is lower semicontinuous. Therefore, let Ω be an open subset of $\mathcal{A}_k$ and let $\bar{x}\in H_k^-(\Omega)$. We must prove that $\bar{x}$ is an interior point of $H_k^-(\Omega)$. For this purpose, choose $\varphi\in H_k(\bar{x})\cap\Omega$ and $r\in\mathbb{R}^+$ such that $B_{d_b}(\varphi, r)\subseteq\Omega$. Now, let $m^*, n^*\in N$ be such that $S_k^*=S_{m^*,n^*}$. Since $\lim_{x\to\bar{x}}\lambda_{n^*}(\bar{x}, x)=0$, it follows that there exists a

(1) $\dim_X(Z)\leqslant 0$ means that $\dim(T)\leqslant 0$ for each $T\subseteq Z$ which is closed in X, where $\dim(T)$ denotes the covering dimension of T.

neighbourhood V of $\bar{x}$ such that $\lambda_{n^*}(\bar{x}, x) < r/2m^*$ for each $x \in V$, $x \neq \bar{x}$. Therefore, by virtue of 4), for each $t \in S_k^*$ and each $x \in V$, we have

$$d^*(F(t, \bar{x}), F(t, x)) \leqslant m^* \lambda_{n^*}(\bar{x}, x) < m^* \frac{r}{2m^*} = \frac{r}{2} \tag{3}$$

On the other hand, since $\varphi \in H_k(\bar{x})$, we have $\varphi(t) \in F(t, \bar{x})$ for each $t \in S_k^*$, and so, by (3), we have $F(t, x) \cap B_d(\varphi(t), r/2) \neq \emptyset$ for each $t \in S_k^*$ and each $x \in V$. Now fix $x \in V$. For each $t \in S_k^*$, put $I_{r/2}(t) = F(t, x) \cap B_d(\varphi(t), r/2)$ and $\bar{I}_{r/2}(t) = \overline{I_{r/2}(t)}$. From 1), 2), 3), from the fact that $\varphi \in \mathcal{A}_k$ and from Proposition 2.1, it follows that the multifunction $I_{r/2}$ is $\mathcal{E}_k$-measurable and so the multifunction $\bar{I}_{r/2}$, since we have $I^-_{r/2}(W) = \bar{I}^-_{r/2}(W)$ for each open set $W \subseteq Y$. Thus, by virtue of the Kuratowski and Ryll-Nardzewski selection theorem again, there exists $\psi \in \mathcal{A}_k$ such that $\psi(t) \in \bar{I}_{r/2}(t)$ for each $t \in S_k^*$. Given $\varepsilon \in]0, r/2[$, obviously, for each $t \in S_k^*$, we have $\bar{I}_{r/2}(t) \subseteq F(t, x) \cap B_d(\varphi(t), r/2 + \varepsilon)$. Therefore, $\psi \in H_k(x)$ and $d_b(\varphi, \psi) \leqslant r/2 + \varepsilon < r$ and so $\psi \in H_k(x) \cap \Omega$. Hence, the neighbourhood V of $\bar{x}$ is contained in $H_k^-(\Omega)$ and so our claim is proved. Furthermore, from 1) it follows that the set $H_k(x)$ is closed for each $x \in X$ and convex for each $x \in X - Z$. Therefore, by virtue of Theorem 1.1 of [5], there exists a continuous selection h_k for H_k. Now fix $\bar{t} \in S_k^*$ and put $h_{k,\bar{t}}(x) = h_k(x)(\bar{t})$ for each $x \in X$. Moreover, given $\bar{z} \in Y$, for each $x \in X$, define the function $g_{k,x}\colon S_k^* \to Y$ as follows

$$g_{k,x}(t) = \begin{cases} \bar{z} & \text{if } t = \bar{t} \\ h_k(x)(t) & \text{if } t \neq \bar{t},\ t \in S_k^* . \end{cases}$$

Obviously, $g_{k,x} \in \mathcal{F}_b(S_k^*, Y)$. For each $\varrho \in \mathbb{R}^+$, we have

$$h_{k,\bar{t}}^{-1}(B_d(\bar{z}, \varrho)) = \bigcup_{x \in X} h_k^{-1}(B_{d_b}(g_{k,x}, \varrho) \cap \mathcal{A}_k) . \tag{4}$$

From (4) it follows that the function $x \to h_{k,\bar{t}}(x)$ is continuous, since the function $x \to h_k(x)$ is continuous. Now put $f_k(t, x) = h_k(x)(t)$ for each $(t, x) \in S_k^* \times X$. Thus, the function f_k is a Carathéodory selection for $F_{|S_k \times X}$. This holds for any $k \in N$. Finally, put:

$$f(t, x) = \begin{cases} f_1(t, x) & \text{if } (t, x) \in S_1^* \times X \\ f_k(t, x) & \text{if } (t, x) \in \left(S_k^* - \bigcup_{j=1}^{k-1} S_j^*\right) \times X,\ k \geqslant 2. \end{cases}$$

Obviously, the function f is the claimed Carathéodory selection for F. ▲

In a very similar way it is possible to prove the following further result:

THEOREM 2.2. *Let* $(S, \mathcal{E})$ *be a measurable space and* $\{S_n\}$ *a sequence in* $\mathcal{E}$ *such that* $S = \bigcup_{n=1}^{\infty} S_n$ *and* S_n *is finite for each* $n \in N$; *let* X, Y, Z *be as in Theorem* 2.1 *and let* F *be a multifunction from* $S \times X$ *into* Y *which satisfies conditions* 1), 3) *of Theorem* 2.1 *and the following ones*:

2) *the set* $F(S, x)$ *is separable for each* $x \in X$;

4) *the multifunction* $x \to F(t, x)$ *is lower semicontinuous for each* $t \in S$.

Under such hypotheses, there exists a Carathéorodry selection for F.

Finally, we want to justify our assertion, made in the Introduction, that the range of the multifunction F in Theorem 1 of [1] is separable (in fact, in [2], [3], [4] the whole space Y is already assumed to be separable).

Indeed, we have the following

PROPOSITION 2.2. *Let* S *be a non-empty set and* X, Y *two topological spaces, with* X *separable. Furthermore, let* A *be a countable dense subset of* X *and* F *a multifunction from* $S \times X$ *into* Y *such that*:

1) *the set* $F(S, x)$ *is separable for each* $x \in A$;

2) *the multifunction* $x \to F(t, x)$ *is lower semicontinuous for each* $t \in S$.

Under such hypotheses, the set $F(S \times X)$ *is separable.*

PROOF. For each $x \in A$ choose a countable dense subset D_x of $F(S, x)$. We claim that the countable set $\bigcup_{x \in A} D_x$ is dense in $F(S \times X)$. Indeed, let $\bar{y} \in F(S \times X)$ and let W be an open neighbourhood of $\bar{y}$. Let $(\bar{t}, \bar{x}) \in S \times X$ be such that $\bar{y} \in F(\bar{t}, \bar{x})$. From 2) it follows that there exists a neighbourhood V of $\bar{x}$ such that $F(\bar{t}, x) \cap W \neq \emptyset$ for each $x \in V$. Since A is dense in X, there exists $x^* \in A \cap V$. But then, since $F(\bar{t}, x^*) \cap W \neq \emptyset$ and D_{x^*} is dense in $F(S, x^*)$, it follows that $D_{x^*} \cap W \neq \emptyset$, and so our claim is proved. ▲

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