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## **Fiber Products and Morita Duality for Commutative Rings.**

ALBERTO FACCHINI (\*)

The lack of examples of commutative rings with a Morita duality has been noticed by many authors (e. g. P. Vamos [8], B. Ballet [1], etc.). Apart from complete Noetherian local rings and maximal valuation rings, only a few other «sporadic» examples ([8]) seem to exist in the literature. In this paper we show that fiber product is a useful tool for the construction of commutative rings with a Morita duality starting from known examples; with it we are able to construct a number of examples of such rings.

In particular the examples we obtain in this way allow us to give a complete characterization of all the trees with a finite number of maximal chains which are order-isomorphic to spectra of commutative rings with a Morita duality. I. Kaplansky has observed that the spectra  $X$  of commutative rings possess the following two properties (as partially ordered sets):

(K1) Every chain in  $X$  has a least upper bound and a greatest lower bound;

(K2) If  $x, y \in X$  and  $x < y$  then there exist elements  $x_1, y_1 \in X$  with  $x \leq x_1 < y_1 \leq y$  such that there is no element of  $X$  properly between  $x_1$  and  $y_1$ .

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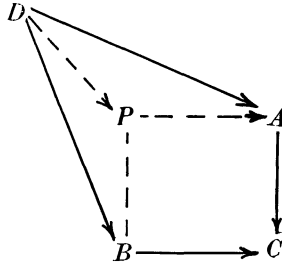
We prove that every tree  $X$  with a finite number of maximal chains and with properties (K1) and (K2) is order-isomorphic to the spectrum of a commutative ring with a Morita duality.

Therefore all the trees with a finite number of maximal chains which are order-isomorphic to spectra of commutative rings are also order-isomorphic to spectra of commutative rings with a Morita duality.

## 1. Notation and elementary results.

All rings in this note are commutative with identity, and ring morphisms respect the identities.

Let  $A, B, C$  be commutative rings, and let  $\alpha: A \rightarrow C$ ,  $\beta: B \rightarrow C$  be ring morphisms. Then there exist a commutative ring  $P$  and two ring morphisms  $\alpha': P \rightarrow B$  and  $\beta': P \rightarrow A$  such that  $\beta\alpha' = \alpha\beta'$ , with the following property: given any ring  $D$  and any two ring morphisms  $\varphi: D \rightarrow A$  and  $\psi: D \rightarrow B$  such that  $\alpha\varphi = \beta\psi$ , there exists a unique morphism  $\omega: D \rightarrow P$  such that  $\varphi = \beta'\omega$  and  $\psi = \alpha'\omega$ .



Disregarding  $\alpha, \beta, \alpha', \beta', P$  is called the *fiber product* of  $A$  and  $B$  over  $C$ , denoted by  $A \times_c B$ . It is unique up to isomorphism in the obvious way. The easiest way to visualize  $A \times_c B$  is as a subring of  $A \times B$  (direct product of the rings  $A$  and  $B$ ), by defining  $A \times_c B = \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$  (the operations are of course by components) and by taking the projections on the first and the second factor as  $\beta'$  and  $\alpha'$  respectively.

It is easy to generalize the above construction to any finite number of rings  $A_1, \dots, A_n$  and morphisms  $\alpha_1, \dots, \alpha_n$ , with  $\alpha_i: A_i \rightarrow C$ .

It is also easy to prove the existence of the following canonical isomorphisms:

- 1)  $A \times_C B \cong B \times_C A$ ;
- 2)  $(A_1 \times_C A_2) \times_C A_3 \cong A_1 \times_C (A_2 \times_C A_3)$ ;
- 3)  $C \times_C A \cong A$ .

The isomorphism in 1) is the exchange; in 3) the morphism  $C \rightarrow C$  is the identity; 2) is improved in the following lemma.

1.1. LEMMA. *Let  $A_1, A_2, A_3, C_1, C_2$  be commutative rings, and  $\alpha_{11}: A_1 \rightarrow C_1, \alpha_{21}: A_2 \rightarrow C_1, \alpha_{22}: A_2 \rightarrow C_2, \alpha_{32}: A_3 \rightarrow C_2$  be ring morphisms. Then  $\alpha_{22}$  canonically induces a morphism  $\tilde{\alpha}_{22}: A_1 \times_{C_1} A_2 \rightarrow C_2$  and  $\alpha_{21}$  canonically induces a morphism  $\tilde{\alpha}_{21}: A_2 \times_{C_2} A_3 \rightarrow C_1$ . Constructing the fiber product with respect to these morphisms, there is a canonical isomorphism  $(A_1 \times_{C_1} A_2) \times_{C_2} A_3 \cong A_1 \times_{C_1} (A_2 \times_{C_2} A_3)$ .*

The proof is standard. Another lemma which plays a fundamental role in the sequel is the following:

1.2. LEMMA. *Let  $A, B, C$  be commutative rings, and  $\alpha: A \rightarrow C, \beta: B \rightarrow C$  be ring morphisms. Let  $C' = \alpha(A) \cap \beta(B), A' = \alpha^{-1}(C'), B' = \beta^{-1}(C')$ . Then  $A', B', C'$  are subrings of  $A, B, C$  respectively, and there is a canonical isomorphism  $A \times_C B \cong A' \times_{C'} B'$ .*

This proof is standard too. Note that in Lemma 1.2 the morphisms  $A' \rightarrow C', B' \rightarrow C'$  used in the construction of the fiber product  $A' \times_{C'} B'$  are the restrictions of  $\alpha$  and  $\beta$ . These restrictions are surjective ring morphisms. Since we shall essentially employ the fiber product to construct new types of rings with a Morita duality, Lemma 1.2 says that we may limit ourselves to the case in which the two morphisms  $\alpha: A \rightarrow C$  and  $\beta: B \rightarrow C$  are surjective. This hypothesis considerably simplifies our approach. Hence from now on whenever we construct the fiber product  $A \times_C B$ , we shall suppose that the two morphisms  $\alpha: A \rightarrow C$  and  $\beta: B \rightarrow C$  are surjective. We conclude this section with three elementary lemmas which will be useful in the sequel.

1.3. LEMMA. *Let  $A, B, C$  be commutative rings,  $\alpha: A \rightarrow C, \beta: B \rightarrow C$  surjective ring morphisms. Then the ring morphisms  $\alpha': A \times_C B \rightarrow B$  and  $\beta': A \times_C B \rightarrow A$  are surjective.*

The proof is trivial (think of  $A \times_C B$  as a subring of  $A \times B$  and of  $\alpha'$  and  $\beta'$  as the restrictions of the canonical projections).

Recall that a ring is *local* if it has a unique maximal ideal.

**1.4. LEMMA.** *Let  $A, B, C$  be commutative rings,  $\alpha: A \rightarrow C, \beta: B \rightarrow C$  surjective ring morphisms. Then  $A \times_c B$  is local if and only if  $A$  and  $B$  are local.*

**PROOF.** The necessity follows from Lemma 1.3. The proof of the sufficiency is standard as soon as  $A \times_c B$  is viewed as a subring of  $A \times B$ ; in this case the maximal ideal of  $A \times_c B$  is  $(A \times_c B) \cap (\mathfrak{M}_A \times \mathfrak{M}_B)$ , where  $\mathfrak{M}_A (\mathfrak{M}_B)$  is the maximal ideal of  $A$  ( $B$ ).

**1.5. LEMMA.** *Let  $A, B, C$  be commutative rings,  $\alpha: A \rightarrow C, \beta: B \rightarrow C$  surjective ring morphisms. Then  $A \times_c B$  is a Noetherian ring if and only if  $A$  and  $B$  are Noetherian rings.*

**PROOF.** The necessity follows from Lemma 1.3. Sufficiency: Consider  $A$  and  $B$  as  $A \times_c B$ -modules via  $\beta'$  and  $\alpha'$ ; then the  $A \times_c B$ -submodules of  $A$  and  $B$  are exactly their ideals. Hence  $A$  and  $B$  are Noetherian  $A \times_c B$ -modules, and therefore  $A \times B$  is a Noetherian  $A \times_c B$ -module. But  $A \times_c B$  is a submodule of  $A \times B$ .

## 2. Spectrum of a fiber product.

In Section 5 we need a description of the spectrum of a fiber product. The fiber product  $A \times_c B$  of two rings  $A$  and  $B$  over a third ring  $C$  may be viewed as a pasting of the two rings  $A$  and  $B$  along  $C$ . It follows that its spectrum  $\text{Spec}(A \times_c B)$  is obtained by pasting together  $\text{Spec}(A)$  and  $\text{Spec}(B)$  along two closed sets homeomorphic to  $\text{Spec}(C)$ . This is better specified in Proposition 2.1.

Let  $A, B, C$  be commutative rings and  $\alpha: A \rightarrow C, \beta: B \rightarrow C$  be surjective ring morphisms. Consider the topological space  $X = \text{Spec}(A) \cup \cup \text{Spec}(B)$ , disjoint union of  $\text{Spec}(A)$  and  $\text{Spec}(B)$  with the topology in which the open sets are exactly the union of an open set of  $\text{Spec}(A)$  and an open set of  $\text{Spec}(B)$ . If  $P \in \text{Spec}(A)$  and  $Q \in \text{Spec}(B)$ , set  $P \sim Q$  if  $P \supseteq \ker \alpha, Q \supseteq \ker \beta$  and  $\alpha(P) = \beta(Q)$ . Consider the equivalence relation in  $X$  generated by  $\sim$ . Call this equivalence relation  $\sim$  too. Then it is possible to consider the quotient topological space  $(\text{Spec}(A) \cup \text{Spec}(B))/\sim$  of the topological space  $X = \text{Spec}(A) \cup \cup \text{Spec}(B)$  modulo the equivalence relation  $\sim$ .

**2.1. PROPOSITION.**  *$\text{Spec}(A \times_c B)$  is canonically homeomorphic to  $(\text{Spec}(A) \cup \text{Spec}(B))/\sim$ .*

PROOF. Apply the functor  $\text{Spec}$  to the commutative diagram

$$\begin{array}{ccc}
 A \times_c B & \xrightarrow{\beta'} & A \\
 \alpha' \downarrow & & \downarrow \alpha \\
 B & \xrightarrow{\beta} & C
 \end{array}$$

of commutative rings and surjective ring morphisms, and get the commutative diagram

$$\begin{array}{ccc}
 \text{Spec}(A \times_c B) & \xleftarrow{\text{Spec}(\beta')} & \text{Spec}(A) \\
 \text{Spec}(\alpha') \uparrow & & \uparrow \text{Spec}(\alpha) \\
 \text{Spec}(B) & \xleftarrow{\text{Spec}(\beta)} & \text{Spec}(C)
 \end{array}$$

of topological spaces and continuous maps.  $\text{Spec}(\alpha)$ ,  $\text{Spec}(\beta)$ ,  $\text{Spec}(\alpha')$  and  $\text{Spec}(\beta')$  are homeomorphisms of  $\text{Spec}(C)$ ,  $\text{Spec}(C)$ ,  $\text{Spec}(B)$ ,  $\text{Spec}(A)$  onto  $V(\ker \alpha)$ ,  $V(\ker \beta)$ ,  $V(\ker(\alpha'))$ ,  $V(\ker \beta')$  respectively (here if  $I$  is an ideal of a ring  $R$ ,  $V(I)$  is the closed set of  $\text{Spec}(R)$  consisting of all prime ideals of  $R$  containing  $I$ ).

Thus if  $X$  is the disjoint union of  $\text{Spec}(A)$  and  $\text{Spec}(B)$  and  $\varphi: X \rightarrow \text{Spec}(A \times_c B)$  is the map which restricted to  $\text{Spec}(A)$  coincides with  $\text{Spec}(\beta')$  and restricted to  $\text{Spec}(B)$  coincides with  $\text{Spec}(\alpha')$ , then  $\varphi$  is a continuous map. Furthermore  $\ker \alpha' \cap \ker \beta' = 0$ , so every prime ideal of  $A \times_c B$  contains either  $\ker \alpha'$  or  $\ker \beta'$ . This means that  $\varphi$  is surjective. Using the fact that the restrictions of  $\varphi$  to  $\text{Spec}(A)$  and  $\text{Spec}(B)$  are injective, it is also easy to check that the kernel of  $\varphi$  is the equivalence relation  $\sim$ . In order to prove the theorem it only remains to show that the continuous map  $\varphi$  is a quotient map, i.e. that the topology of  $\text{Spec}(A \times_c B)$  is the finest for which  $\varphi$  is continuous, i.e. that if  $Y$  is a subset of  $\text{Spec}(A \times_c B)$  and  $\varphi^{-1}(Y)$  is closed in  $X$  then  $Y$  is closed in  $\text{Spec}(A \times_c B)$ . But if  $Y$  is a subset of  $\text{Spec}(A \times_c B)$  and  $\varphi^{-1}(Y) = \text{Spec}(\beta')^{-1}(Y) \cup \text{Spec}(\alpha')^{-1}(Y)$  is closed in  $X$ , then  $\text{Spec}(\beta')^{-1}(Y)$  is closed in  $\text{Spec}(A)$ . But  $\text{Spec}(\beta')$  is a homeomorphism of  $\text{Spec}(A)$  onto  $V(\ker \beta')$ . It follows that  $Y \cap V(\ker \beta')$  is closed in  $V(\ker \beta')$ , hence it is closed in  $\text{Spec}(A \times_c B)$ . Similarly  $Y \cap V(\ker \alpha')$  is closed in  $\text{Spec}(A \times_c B)$ . Thus  $Y = (Y \cap V(\ker \beta')) \cup (Y \cap V(\ker \alpha'))$  is closed in  $\text{Spec}(A \times_c B)$ .

Intuitively Proposition 2.1 says that  $\text{Spec}(A)$  and  $\text{Spec}(B)$  each contains a closed set homeomorphic to  $\text{Spec}(C)$  and that if we paste  $\text{Spec}(A)$  and  $\text{Spec}(B)$  together by identifying these two closed sets, we get  $\text{Spec}(A \times_c B)$  (under the hypothesis that  $\alpha: A \rightarrow C$  and  $\beta: B \rightarrow C$  are surjective, of course).

### 3. Injective envelope of a simple module in a local fiber product.

In Section 2 we studied the spectrum of the fiber product of two rings. This paper is devoted to the study of the commutative rings with a Morita duality. Now a ring with a Morita duality is linearly compact (in the discrete topology) [6, Theorem 1] and every linearly compact commutative ring is the direct product of a finite number of local rings [10, Proposition 14]. Hence without loss of generality we only have to study the local case.

Now by Lemma 1.4, if  $\alpha: A \rightarrow C$  and  $\beta: B \rightarrow C$  are surjective,  $A \times_c B$  is local if and only if both  $A$  and  $B$  are local; in this case  $C$  is local too; moreover it is immediate to see that in this case the unique (up to isomorphism) simple modules over  $A, B, C$  and  $A \times_c B$  are isomorphic (as  $A \times_c B$ -modules). Hence we need to study the injective envelope of the unique (up to isomorphism) simple  $A \times_c B$ -module, i.e. the minimal injective cogenerator in the category of all  $A \times_c B$ -modules. The situation, described in the following theorem, is the best we could hope for: since  $A \times_c B$  is the pull-back of  $A$  and  $B$  over  $C$ , the minimal injective cogenerator in the category of all  $A \times_c B$ -modules is the push-out of the minimal injective cogenerators in the categories of all  $A$ - and  $B$ -modules over the minimal injective cogenerator in the category of all  $C$ -modules.

In the sequel if  $R$  is a ring,  $I$  an ideal of  $R$  and  $M$  an  $R$ -module, then  $\text{Ann}_M I$  denotes the set of all  $x \in M$  such that  $Ix = 0$  and  $E_R(M)$  denotes the injective envelope of  $M$ .

**3.1. THEOREM.** *Let  $A, B, C$  be local rings, and  $\alpha: A \rightarrow C, \beta: B \rightarrow C$  be surjective ring morphisms. Let  $A \times_c B$  be the fiber product of  $A$  and  $B$  over  $C$ , and let  $S$  be the unique (up to isomorphism) simple  $C$ -module.*

$$\begin{array}{ccc}
 A \times_c B & \xrightarrow{\beta'} & A \\
 \alpha' \downarrow & & \downarrow \alpha \\
 B & \xrightarrow{\beta} & C
 \end{array}$$

Then

1)  $S$  is the unique (up to isomorphism) simple  $A$ -,  $B$ - and  $A \times_c B$ -module.

2)  $\alpha$  induces a monomorphism of  $A$ -modules (and therefore of  $A \times_c B$ -modules)  $\alpha^*: E_c(S) \rightarrow E_A(S)$ , whose image is  $\text{Ann}_{E_A(S)} \ker \alpha$ ; similarly for  $\beta$ .

3) If  $P$  is the push-out of the following diagram of  $A \times_c B$ -modules

$$\begin{array}{ccc}
 P & \longleftarrow & E_A(S) \\
 \uparrow & & \uparrow \alpha^* \\
 E_B(S) & \xleftarrow{\beta^*} & E_c(S)
 \end{array}$$

then  $P$  is the injective envelope of the simple  $A \times_c B$ -module  $S$ .

PROOF:

1) is obvious.

2) easily follows from [7, Proposition 2.27]; note that  $\alpha^*(\alpha(a)e) = a\alpha^*(e)$  for  $a \in A, e \in E_c(S)$ .

3). Let us view  $A \times_c B$  as a subring of  $A \times B$ . Then if  $P$  is the push-out of the above diagram, we have  $P = (E_A(S) \oplus E_B(S))/M$ , where  $M = \{(\alpha^*(e), -\beta^*(e)) \mid e \in E_c(S)\}$ ; the multiplication of an element  $(a, b) \in A \times_c B$  by an element  $(\bar{x}, \bar{y}) \in P$ , where  $x \in E_A(S), y \in E_B(S)$  and the bar denotes reduction modulo  $M$ , is defined by  $(a, b)(\bar{x}, \bar{y}) = \overline{(ax, by)}$ . Note that the map  $\varepsilon_A: E_A(S) \rightarrow P$  (dotted in the above diagram) is simply the canonical embedding of  $E_A(S)$  into  $E_A(S) \oplus E_B(S)$  followed by the reduction modulo  $M$ . It is clearly injective. Similarly for  $\varepsilon_B: E_B(S) \rightarrow P$ . We want to prove that  $P = E_{A \times_c B}(S)$ . Let us show that  $P$  is an injective  $A \times_c B$ -module. The proof is divided into four steps.

STEP 1: For each ideal  $I$  in  $A$  every  $A \times_c B$ -morphism  $\varphi: I \rightarrow P$  extends to an  $A \times_c B$ -morphism  $A \rightarrow P$ . Similarly for  $B$ .

In fact if  $I$  is an ideal of  $A$  and  $\varphi: I \rightarrow P$  is an  $A \times_c B$ -morphism, then  $\varphi(I) \subseteq \text{Ann}_P(\ker \beta')$  because  $I \ker \beta' = 0$ . Now



$$\begin{aligned}
\text{Ann}_P(\ker \beta') &= \{\overline{(x, y)} \in P \mid x \in E_A(S), y \in E_B(S), (0 \times \ker \beta)(x, y) \subseteq M\} \\
&= \{\overline{(x, y)} \in P \mid x \in E_A(S), y \in E_B(S), \ker \beta \cdot y = 0\} \\
&= \{\overline{(x, y)} \in P \mid x \in E_A(S), y \in E_B(S), y \in \beta^*(E_C(S))\}.
\end{aligned}$$

But if  $x \in E_A(S)$  and  $y \in \beta^*(E_C(S))$  then  $(x, y) \equiv (x + \alpha^* \beta^{*-1}(y), 0) \pmod{M}$ , and thus  $\text{Ann}_P(\ker \beta') = \varepsilon_A(E_A(S))$ . Hence  $\varphi(I) \subseteq \varepsilon_A(E_A(S))$ , and thus there exists an  $A \times_c B$ -morphism  $\psi: I \rightarrow E_A(S)$  such that  $\varphi = \varepsilon_A \psi$ . But  $\psi$  is also an  $A$ -morphism and  $E_A(S)$  is  $A$ -injective. Therefore there exists an  $A$ -morphism  $\tilde{\psi}: A \rightarrow E_A(S)$  extending  $\psi$ . Thus  $\varepsilon_A \tilde{\psi}: A \rightarrow P$  is an  $A \times_c B$ -morphism extending  $\varphi$ . Similarly for  $B$ .

STEP 2:  $\text{Ext}_{A \times_c B}^1(A, P) = 0$ .

Let  $\varphi: \ker \beta' \rightarrow P$  be an  $A \times_c B$ -morphism. Since  $\ker \beta' = 0 \oplus \ker \beta$ , by the first step  $\varphi$  extends to an  $A \times_c B$ -morphism  $0 \oplus B \rightarrow P$ ; this in turn trivially extends to an  $A \times_c B$ -morphism  $A \times B \rightarrow P$ , whose restriction to  $A \times_c B$  is an  $A \times_c B$ -morphism extending  $\varphi$ . Therefore every  $A \times_c B$ -morphism  $\ker \beta' \rightarrow P$  extends to an  $A \times_c B$ -morphism  $A \times_c B \rightarrow P$ .

Consider the exact sequence  $0 \rightarrow \ker \beta' \rightarrow A \times_c B \rightarrow A \rightarrow 0$ . By applying the functor  $\text{Hom}_{A \times_c B}(-, P)$ , we get an exact sequence

$$\text{Hom}(A \times_c B, P) \rightarrow \text{Hom}(\ker \beta', P) \rightarrow \text{Ext}^1(A, P) \rightarrow \text{Ext}^1(A \times_c B, P).$$

In this sequence the last module is zero and we have just proved that the first morphism is surjective. Hence  $\text{Ext}_{A \times_c B}^1(A, P) = 0$ .

STEP 3: *If  $I$  is an ideal of  $A$ ,  $\text{Ext}_{A \times_c B}^1(A/I, P) = 0$ . Similarly for  $B$ .*

Consider the exact sequence  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ . By applying the functor  $\text{Hom}_{A \times_c B}(-, P)$  we get the exact sequence  $\text{Hom}(A, P) \rightarrow \text{Hom}(I, P) \rightarrow \text{Ext}^1(A/I, P) \rightarrow \text{Ext}^1(A, P)$ . By the first step the first morphism is surjective and by the second step the last module is zero. Thus  $\text{Ext}_{A \times_c B}^1(A/I, P) = 0$ . Similarly for  $B$ .

STEP 4:  *$P$  is an injective  $A \times_c B$ -module.*

We must show that if  $I$  is an ideal of  $A \times_c B$ , every morphism  $I \rightarrow P$  extends to a morphism  $A \times_c B \rightarrow P$ . We shall even show that every morphism  $I \rightarrow P$  extends to a morphism  $A \times B \rightarrow P$ . To do

that it is enough to prove that  $\text{Ext}_{A \times_c B}^1(A \oplus B/I, P) = 0$ . From the exact sequence  $0 \rightarrow (I + B)/I \rightarrow (A \oplus B)/I \rightarrow (A \oplus B)/(I + B) \rightarrow 0$  we get the exact sequence

$$(*) \quad \text{Ext}_{A \times_c B}^1((A \oplus B)/(I + B), P) \rightarrow \text{Ext}_{A \times_c B}^1((A \oplus B)/I, P) \rightarrow \text{Ext}_{A \times_c B}^1((I + B)/I, P).$$

But  $(A \oplus B)/(I + B) \cong A/A \cap (I + B)$  and  $(I + B)/I \cong B/B \cap I$ , where  $A \cap (I + B)$  and  $B \cap I$  are  $A \times_c B$ -submodules of  $A$  and  $B$  respectively, i.e. ideals in  $A$  and  $B$ . By the third step, the first and the last module in  $(*)$  are zero. Hence  $P$  is an injective  $A \times_c B$ -module.

Let us show that  $P$  is an indecomposable  $A \times_c B$ -module. Let  $\varphi$  be a non-zero  $A \times_c B$ -endomorphism of  $P$  such that  $\varphi^2 = \varphi$ . It is enough to prove that  $\varphi$  is the identity morphism. Now  $\varphi(\text{Ann}_P(\ker \beta')) \subseteq \text{Ann}_P(\ker \beta')$ . In the proof of Step 1 we have seen that  $\text{Ann}_P(\ker \beta') = \varepsilon_A(E_A(S))$ . Hence  $\varphi \varepsilon_A E_A(S) \subseteq \varepsilon_A E_A(S)$ . Therefore  $\varphi$  induces an  $A \times_c B$ -endomorphism of  $\varepsilon_A(E_A(S))$ , i.e. an  $A$ -endomorphism of  $E_A(S)$ , which coincides with its own square. Since  $E_A(S)$  is an indecomposable  $A$ -module, it follows that either  $\varphi \varepsilon_A = 0$  or  $\varphi \varepsilon_A = \varepsilon_A$ . Similarly either  $\varphi \varepsilon_B = 0$  or  $\varphi \varepsilon_B = \varepsilon_B$ . Since  $\varphi \neq 0$  and  $\varepsilon_A E_A(S) + \varepsilon_B E_B(S) = P$ , we must have either  $\varphi \varepsilon_A = \varepsilon_A$  or  $\varphi \varepsilon_B = \varepsilon_B$ . If for instance  $\varphi \varepsilon_A = \varepsilon_A$ , then  $\varphi \varepsilon_A(S) = \varepsilon_A(S)$ . But  $\varepsilon_A(S) = \varepsilon_B(S)$ . Hence  $\varphi \varepsilon_B(S) = \varepsilon_B(S) \neq 0$ , from which  $\varphi \varepsilon_B = \varepsilon_B$ . Hence we must have both  $\varphi \varepsilon_A = \varepsilon_A$  and  $\varphi \varepsilon_B = \varepsilon_B$ . But then if  $(\overline{x, y}) \in P$ ,  $\varphi(\overline{x, y}) = \varphi(\varepsilon_A(x) + \varepsilon_B(y)) = \varphi \varepsilon_A(x) + \varphi \varepsilon_B(y) = \varepsilon_A(x) + \varepsilon_B(y) = \overline{x, y}$ . Hence  $\varphi$  is the identity of  $P$  and  $P$  is indecomposable.

Therefore  $P$  is an indecomposable injective module and contains the simple  $A \times_c B$ -module  $\varepsilon_A(S) \cong S$ . Hence  $P = E_{A \times_c B}(S)$ .

#### 4. Fiber products and Morita duality.

We essentially know three operations of commutative rings which preserve the property «to have a Morita duality»:

1) Direct product: If  $A$  and  $B$  are rings with a Morita duality, then  $A \times B$  has a Morita duality. (Note that every commutative ring with a Morita duality is a finite direct product of local rings with a Morita duality.)

2) Homomorphic images: If  $A$  is a ring with a Morita duality and  $I$  is an ideal of  $A$ , then  $A/I$  has a Morita duality.

3) Taking linearly compact extensions and conversely: Let  $A \subseteq B$  be commutative rings, and let  $B$  be a linearly compact  $A$ -module. Then  $A$  has a Morita duality if and only if  $B$  has a Morita duality. This is a theorem due to Peter Vámos [8, Theorem 2.14].

In this section we prove that such a property is also preserved by fiber product; assuming, without loss of generality, that the fiber product is local, we shall prove the following statement:

4) Let  $A, B, C$  be local rings and let  $\alpha: A \rightarrow C$ ,  $\beta: B \rightarrow C$  be surjective ring morphisms. Then  $A \times_c B$  has a Morita duality if and only if  $A$  and  $B$  have a Morita duality.

We must notice that we also know other operations between commutative rings which preserve the property «to have a Morita duality», but all such operations easily follow from the operations 1), 2) and 3). For instance:

a) Taking finitely generated integral extensions (this follows from 3) and is the case of Example 2.4 in [8]);

b) Taking split extensions, by joining a linearly compact module as a nilpotent ideal (see [5, Theorem 10]; this is the case of Example 2.4 in [8], too);

c) Group rings: if  $R$  is a commutative ring with a Morita duality and  $G$  is a finite abelian group, then the group ring  $R[G]$  has a Morita duality (it follows from a)); etc.

The interest in proving 4) lies in the fact that with the operation 4) it is possible to construct a number of examples of commutative rings with a Morita duality. The (local) commutative rings with a Morita duality known up to now are essentially the following:

i) complete Noetherian local rings;

ii) maximal valuation rings;

iii) rings with a prime ideal  $P \neq 0$  such that 1)  $P$  is comparable to every ideal of  $R$ , that is  $I \subseteq P$  or  $P \subseteq I$  for all ideals  $I$  of  $R$ ; 2) the canonical morphism  $R \rightarrow R_P$  is injective; 3)  $R_P$  has a Morita duality; 4)  $R/P$  has a linearly compact field of fractions. This class of rings was discovered by P. Vámos [8]. Müller ([8, page 285]) has noticed

that such rings  $R$  are the fiber product of the rings  $R/P$  and  $R_p$  (but here the morphisms are not both surjective).

All other known examples of commutative rings with a Morita duality are obtained from these rings with operations 1), 2) and 3). This is the case, for instance, for the domains with linearly compact field of fractions ([8]).

The following result has an elementary proof, but since it plays an essential role in the paper, we shall give two different proofs of it.

**4.1. THEOREM.** *Let  $A, B, C$  be local rings, and let  $\alpha: A \rightarrow C$ ,  $\beta: B \rightarrow C$  be surjective ring morphisms. Then  $A \times_c B$  has a Morita duality if and only if  $A$  and  $B$  have a Morita duality.*

**FIRST PROOF.** The necessity is obvious, since  $A$  and  $B$  are homomorphic images of  $A \times_c B$ . The sufficiency will be proved by using Müller's Theorem 1 [6] and our Theorem 3.1. A ring  $R$  has a Morita duality if and only if  $R$  and the minimal injective cogenerator are linearly compact. Thus if  $A$  and  $B$  have a Morita duality,  $A$  and  $B$  are linearly compact  $A \times_c B$ -modules, and therefore  $A \times_c B \subseteq A \oplus B$  is linearly compact. On the other hand the minimal injective cogenerators are linearly compact  $A$ - and  $B$ -modules. By Theorem 3.1 the  $A \times_c B$ -minimal injective cogenerator is linearly compact; hence  $A \times_c B$  has a Morita duality.

**SECOND PROOF.** Let us prove the sufficiency by using Müller's Theorem 1 [6] and Vámos' Theorems 2.14 [8]. If  $A$  and  $B$  have a Morita duality, then  $A$  and  $B$  are linearly compact rings. Hence they are linearly compact  $A \times_c B$ -modules, from which  $A \times B$  is a linearly compact  $A \times_c B$ -module. Since  $A \times B$  has a Morita duality,  $A \times_c B$  has a Morita duality too.

The last part of this paper is devoted to the study of the local rings with a Morita duality which are obtained as fiber products. Note that if  $A, B, C$  are rings and  $\alpha: A \rightarrow C$ ,  $\beta: B \rightarrow C$  are surjective ring morphisms, then  $A \times_c B$  is a complete Noetherian local ring if and only if  $A$  and  $B$  are complete Noetherian local rings (this follows, for instance, from 1.4, 1.5 and 4.1); hence by taking fiber products (via surjective ring morphisms) of Noetherian rings with a Morita duality we still obtain rings of the same type. On the contrary we get new examples of commutative rings with a Morita duality by taking fiber products (via surjective ring morphisms) either of a

Noetherian ring and a valuation ring or of two valuation rings, and by iterating such constructions a finite number of times. This is what we shall do in the next section.

## 5. Particular cases.

Let  $A, B, C$  be commutative rings and let  $\alpha: A \rightarrow C$ ,  $\beta: B \rightarrow C$  be surjective ring morphisms. Assume  $A$  is a complete Noetherian local ring and  $B$  is a maximal valuation ring. Then  $C$  must be a Noetherian maximal valuation ring. Hence  $C$  has to be of one of the following three types: 1) a field; 2) a local artinian principal ideal ring which is not a field; 3) a complete DVR.

In this case  $\text{Spec}(A \times_c B)$  is  $\text{Spec}(A)$  to which the chain of all prime non-maximal ideals of  $B$  (in cases 1) and 2)) or the chain of all prime ideals of coheight  $> 1$  of  $B$  (in case 3)) has been pasted under the maximal ideal  $\mathfrak{M}_A$  of  $A$  (in cases 1) and 2)) or under the prime ideal  $\alpha^{-1}(0)$  of coheight 1 in  $A$  (in the case 3)).

The Noetherianity of  $A$  and the fact that  $B$  is a valuation ring (i.e. every ideal in  $B$  is the union of a chain of principal ideals) « mix » in  $A \times_c B$  in the following sense: every ideal of  $A \times_c B$  is the union of an ascending chain of finitely generated ideals. If we fix an ideal  $I$  of  $A \times_c B$ , this can easily be seen by considering the exact sequence  $0 \rightarrow I \cap \ker \beta' \rightarrow I \xrightarrow{\beta'} \beta'(I) \rightarrow 0$  and by taking a set of generators of  $I$  consisting of a finite number of elements of  $I$  whose  $\beta'$ -images generate  $\beta'(I)$  and of a set of generators of  $I \cap \ker \beta'$ .

**5.1. EXAMPLE.** Let  $C = k$  be a field,  $X_1, \dots, X_n$  indeterminates,  $A = k[[X_1, \dots, X_n]]$  the power-series ring,  $G$  a totally ordered abelian group,  $B = k[[G]]$  the long power-series ring relative to  $k$  and  $G$ . Then  $A \times_c B$ , fiber product of a complete Noetherian local ring and a maximal valuation domain, is a commutative local ring with a Morita duality.

**5.2. EXAMPLE.** Let  $k, A$  and  $G$  be as in the above example, let  $G \oplus \mathbb{Z}$  have the lexicographic order,  $B = k[[G \oplus \mathbb{Z}]]$  be the long power-series ring,  $C = A/(X_2, \dots, X_n) \cong k[[\mathbb{Z}]]$ .

**5.3. EXAMPLE.** Let  $k, A, G, B$  be as in Example 5.2, and let  $n$  be an integer  $> 1$ . Take  $C = A/(X_1^n, X_2, \dots, X_n) \cong B/I$ , where  $I$  is the ideal of  $B$  consisting of all elements of  $B$  with valuation  $\geq (0, n)$ .

5.4. **EXAMPLE.** Let  $p$  be a prime number,  $A$  the ring of  $p$ -adic integers,  $G$  a totally ordered abelian group,  $C = \mathbf{Z}/p\mathbf{Z}$ ,  $B = C[[G]]$  the long power-series ring.

Examples 5.2, 5.3, 5.4, like Example 5.1, are local rings with a Morita duality, which are fiber products of a Noetherian ring and a valuation domain.

Now, on the contrary, let  $A, B, C$  be maximal valuation rings, and let  $\alpha: A \rightarrow C, \beta: B \rightarrow C$  be surjective ring morphisms. By Proposition 2.1 the spectrum of  $A \times_c B$  looks like a reverse  $Y$ . The fact that  $A$  and  $B$  are valuation rings (i.e. every finitely generated ideal is principal) is inherited by  $A \times_c B$  in the following way: every finitely generated ideal of  $A \times_c B$  can be generated by two elements.

5.5. **EXAMPLE.** Let  $k$  be a field,  $G_1, G_2, H$  be totally ordered abelian groups,  $G_1 \oplus H, G_2 \oplus H$  have the lexicographic order. Set  $A = k[[G_1 \oplus H]], B = k[[G_2 \oplus H]], C = k[[G]]$ , long power-series rings. Then  $A \times_c B$  is local, has a Morita duality, is the fiber product of two valuation rings and every finitely generated ideal can be generated by two elements.

Such a construction can be easily iterated to any finite number of maximal valuation rings  $A_1, \dots, A_n, C_1, \dots, C_{n-1}$  and surjective morphisms  $\alpha_i: A_i \rightarrow C_i, \beta_i: A_{i+1} \rightarrow C_i, i = 1, \dots, n-1$ . In this case the spectrum of  $A_1 \times_{C_1} A_2 \times_{C_2} \dots \times_{C_{n-1}} A_n$  is a tree with at most  $n$  maximal chains and every finitely generated ideal can be generated by  $n$  elements.

Let us invert such a result with a construction which is curiously dual to a construction due to S. Wiegand [9]. Recall that a *chain* of a partially ordered set  $X$  is a totally ordered subset of  $X$ . A *tree* is a partially ordered set  $X$  with maximum such that  $\{y \in X | y \geq x\}$  is a chain of  $X$  for all  $x \in X$ .

S. Wiegand [9] proved that if  $X$  is a finite tree, then there exists a Bezout domain  $R$  such that  $\text{Spec}(R)$  is order anti-isomorphic to  $X$  and  $R_P$  is a maximal valuation domain for all  $P \in \text{Spec}(R)$ . We shall prove that if  $X$  is a finite tree, then there exists a local ring  $R$  with a Morita duality such that  $\text{Spec}(R)$  is order isomorphic to  $X$  and  $R/P$  is a maximal valuation domain for all  $P \in \text{Spec}(R)$ . In fact we shall prove much more, allowing  $X$  to be infinite but with a finite number  $n$  of maximal chains, and we shall obtain that the  $R$  constructed will have the property that every finitely generated ideal of  $R$  can be generated by  $n$  elements.

Of course not every tree with a finite number of maximal chains is order-isomorphic to the spectrum of a commutative ring. In fact let  $X$  be a partially ordered set and suppose that  $X$  is order-isomorphic to  $\text{Spec}(R)$  for some commutative ring  $R$ . Then, as noted by I. Kaplansky [3, page 6, Theorems 9 and 11] (see also [4]), the following two properties hold:

(K1) Every chain in  $X$  has a supremum (sup) and an infimum (inf).

(K2) If  $x, y \in X$  and  $x < y$  then there exist elements  $x_1, y_1 \in X$  such that  $x < x_1 < y_1 < y$  and there does not exist an element of  $X$  properly between  $x_1$  and  $y_1$ .

Recall the definition of lexicographic product [2, Chap. III, § 15, Exercise 3]. Let  $\{G_\lambda\}_{\lambda \in A}$  be a family of totally ordered abelian groups, and assume that  $A$  is totally ordered under a relation  $<$ . Let  $G = \prod_{\lambda \in A} G_\lambda$  be the direct product of the groups  $G_\lambda$ . We consider the elements of  $G$  as functions  $f: A \rightarrow \bigcup_{\lambda \in A} G_\lambda$  such that  $f(\lambda) \in G_\lambda$  for each  $\lambda$  in  $A$ . For  $f \in G$ , we define the *support* of  $f$ , denoted  $S(f)$ , to be  $\{\lambda \in A \mid f(\lambda) \neq 0\}$ . Let  $L = \{f \in G \mid S(f) \text{ is a well-ordered subset of } A\}$ . Then  $L$  is a subgroup of  $G$ . We define a relation  $\leq$  on  $L$  as follows. If  $f, g \in L$ ,  $f \neq g$ , and if  $\lambda$  is the first element of  $S(g - f)$ , then  $f \leq g$  if and only if  $f(\lambda) \leq g(\lambda)$ . Then the relation  $\leq$  is a total order compatible with the group operation on  $L$ . The group  $L$ , under the relation  $\leq$ , is called the *lexicographic product* of the groups  $\{G_\lambda\}_{\lambda \in A}$ .

Also recall that a subgroup  $H$  of a totally ordered group  $G$  is *convex* if  $y \in H$  whenever  $x \in H$ ,  $y \in G$  and  $0 \leq y \leq x$ .

First we need a lemma.

**5.6. LEMMA.** *Let  $(X, \leq)$  be a totally ordered set with properties (K1) and (K2). Let  $Y = \{y \in X \mid y \text{ has an immediate predecessor in } X\}$ . Let  $\leq$  be the order relation in  $Y$  defined by  $y_1 \leq y_2$  if  $y_1, y_2 \in Y$  and  $y_1 \geq y_2$  in  $X$ . Let  $L$  be the lexicographic product of a family of totally ordered groups isomorphic to  $\mathbb{Z}$  indexed by  $(Y, \leq)$ . Then the set of all convex subgroups of  $L$  ordered by  $\subseteq$  is order isomorphic to  $(X, \leq)$ .*

**PROOF.** Let  $\mathcal{C}(L)$  be the set of all convex subgroups of  $L$ .

Define a map  $\varphi: \mathcal{C}(L) \rightarrow X$ : if  $C \in \mathcal{C}(L)$ ,  $C \neq 0$ ,  $\varphi(C) = \sup_x \left( \bigcup_{f \in C} S(f) \right)$ ,  $\varphi(0) = \inf_x(X)$ . Note that  $\varphi$  is well-defined because  $X$  is totally ordered and has property (K1). Furthermore if  $C_1, C_2 \in \mathcal{C}(L)$  and  $C_1 \subseteq C_2$ , then  $\varphi(C_1) \leq \varphi(C_2)$ .

Define a map  $\psi: X \rightarrow \mathcal{C}(L)$ : if  $x \in X$ ,  $\psi(x) = \{f \in L \mid \sup_x(S(f)) \leq x\} \cup \{0\}$ . It is easy to check that  $\psi(x)$  is a convex subgroup of  $L$  and that if  $x_1 \leq x_2$ , then  $\psi(x_1) \subseteq \psi(x_2)$ .

Let us show that  $\varphi$  and  $\psi$  are inverses of each other. If  $x \in X$ ,  $\varphi\psi(x) = x$ , i.e.  $\sup_x(\cup \{S(f) \mid f \in L, \sup_x(S(f)) \leq x\}) = x$ , because  $x = \sup_x(\{y \in Y \mid y \leq x\})$  by (K2). Let us show that if  $C \in \mathcal{C}(L)$ , then  $\psi\varphi(C) = C$ . We have to show that for all  $f \in L$ ,  $f \in C$  if and only if  $\sup_x(S(f)) \leq \sup_x(\bigcup_{g \in C} S(g))$ . The «only if» is trivial. Conversely if  $\sup_x(S(f)) \leq \sup_x(\bigcup_{g \in C} S(g))$ , then there exists  $h \in C$  such that  $\sup_x(S(f)) \leq \sup_x(S(h))$ ; otherwise, since  $\sup_x(S(f)) = \inf_x(S(f)) \in Y$ ,  $\sup_x(S(f))$  would have an immediate predecessor  $y$  in  $X$ ; it would follow that  $y \geq \sup_x(S(g))$  for all  $g \in C$ , i.e.  $\sup_x(S(f)) > y \geq \sup_x(\bigcup_{g \in C} S(g))$ , contradiction. Hence  $h$  exists. But then if  $n = |f(\sup_x S(f))| + 1 \in \mathbb{Z}$ ,  $n|h| \in C$  and  $0 \leq |f| \leq n|h|$ . Since  $C$  is convex,  $|f| \in C$  and thus  $f \in C$ .

We are ready for our last theorem.

**5.7. THEOREM.** *Let  $X$  be a tree with a finite number  $n$  of maximal chains. The following are equivalent:*

- (i)  $X$  has properties (K1) and (K2).
- (ii) There exists a commutative ring  $R$  such that  $\text{Spec}(R) \cong X$  (as an ordered set).
- (iii) There exists a commutative ring  $R$  such that:
  - 1)  $\text{Spec}(R) \cong X$  (as an ordered set);
  - 2)  $R$  is local;
  - 3)  $R$  has a Morita duality;
  - 4)  $R/P$  is a maximal valuation domain for all  $P \in \text{Spec}(R)$ ;
  - 5) Every finitely generated ideal of  $R$  can be generated by  $n$  elements.

**PROOF.** (iii)  $\Rightarrow$  (ii) is trivial and (ii)  $\Rightarrow$  (i) is Kaplansky's remark. We only have to prove that (i)  $\Rightarrow$  (iii). Let  $X$  be a tree with a finite number  $n$  of maximal chains and with properties (K1) and (K2). Let  $C_1, \dots, C_n$  be the maximal chains of  $X$ . Then  $C_1, \dots, C_n$ , as ordered subsets of  $X$ , satisfy (K1) and (K2), too. Let  $L_1, \dots, L_n$  be the ordered groups corresponding to  $C_1, \dots, C_n$  via Lemma 5.6 ( $L_i$  is a suitable lexicographic product of copies of  $\mathbb{Z}$ , whose set of convex subgroups is order-isomorphic to  $C_i$ ).



Then  $C_i \cap C_{i+1}$  is a chain in  $X$ , hence it has an inf in  $X$ . Such an inf belongs to both  $C_i$  and  $C_{i+1}$  ( $i = 1, \dots, n-1$ ). Let  $x_i \in C_i \cap C_{i+1}$  be the inf of  $C_i \cap C_{i+1}$ . Now let  $L'_i$  be the convex subgroup of  $L_i$  consisting of all elements  $f \in L_i$  such that  $x \geq x_i$  for all  $x \in \mathcal{S}(f)$ . Then  $L'_i$  is canonically isomorphic to a convex subgroup of  $L_{i+1}$  ( $i = 1, \dots, n-1$ ). Now let  $k$  be any field and let  $A_1 = k[[L_1]], \dots, A_n = k[[L_n]], C_1 = k[[L'_1]], \dots, C_{n-1} = k[[L'_{n-1}]]$  be the long power-series rings over  $L_1, \dots, L_n, L'_1, \dots, L'_{n-1}$  respectively. Let  $\alpha_i: A_i \rightarrow C_i, \beta_i: A_{i+1} \rightarrow C_i, i = 1, \dots, n-1$ , be the surjective canonical ring morphisms induced by the embeddings  $L'_i \rightarrow L_i, L'_i \rightarrow L_{i+1}$  respectively. Then clearly  $R = A_1 \times_{C_1} A_2 \times_{C_2} \dots \times_{C_{n-1}} A_n$  has the properties required in (iii).

REMARK. Of course with the fiber product it is possible to construct examples of rings with a Morita duality by starting from rings which are neither Noetherian nor valuation, too.

5.8. EXAMPLE. Let  $A$  be Vamos' Example (3.3) in [8]. If  $C = k$  is a field and  $B = k[[X_1, \dots, X_n]]$ , where  $X_1, \dots, X_n$  are indeterminates over  $k$ , then  $A \times_C B$  is a local ring with a Morita duality.

5.9. EXAMPLE. Let  $p$  be a prime number,  $C$  be the field of  $p$  elements,  $A$  the ring of  $p$ -adic integers,  $B$  the ring of [8, Example (3.3)].

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