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Ruggero Ferro

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ω-Satisfiability, ω-Consistency Property, and the Downward Lowenheim Skolem Theorem for $L_{k,k}$.

RUGGERO FERRO (*)

SUMMARY - A new notion of consistency property, called ω-consistency property, for $L_{k,k}$ is introduced. This notion has the advantages that not only sets of sentences of $L_{k,k}$ in an ω-consistency property are ω-satisfiable, but also sets of sentences in $L_{k,k}$ that are ω-satisfiable are in an ω-consistency property, and that there is always a sufficient supply of new witnessing constants available. These results easily yield a version of the downward Lowenheim Skolem theorem for $L_{k,k}$.

SUNTO - Viene presentata una nuova nozione di proprietà di consistenza per $L_{k,k}$, detta proprietà di ω-consistenza. Questa nozione ha i vantaggi che non solo insiemi di enunciati di $L_{k,k}$ in una proprietà di ω-consistenza sono ω-soddisfacibili, ma anche insiemi di enunciati di $L_{k,k}$ che sono ω-soddisfacibili sono in una proprietà di ω-consistenza, e che c'è sempre disponibile una scorta sufficiente di nuovi testimoni. Da questi risultati segue facilmente una versione per $L_{k,k}$ del teorema di Lowenheim Skolem discendente.

1. Introduction.

In [3] it was pointed out the need for a new notion of consistency property for $L_{k,k}$ infinitary languages, k an uncountable strong limit cardinal of denumerable cofinality, different from Karp's notion of k-consistency property.

(*) Indirizzo dell'A.: Seminario Matematico, Università di Padova.

Indeed the notion used in [4] has the drawback that a set of ω-satisfiable sentences may not belong to any consistency property.

For instance the set consisting of the ω-satisfiable sentence:

$$
\begin{aligned}
\& \Big\{\Big[& (\forall v_{j+1} - P(v_{j+1}, v_{j+1})) \& \big(\forall v_{j+2} \forall_{j+3} (P(v_{j+2}, v_{j+3}) \\
& \vee P(v_{j+3}, v_{j+2}) \vee v_{j+2} = v_{j+3})\big) \& \big(\forall v_{j+4} \forall v_{j+5} \forall v_{j+6} \\
& (P(v_{j+4}, v_{j+5}) \& P(v_{j+5}, v_{j+6})) \rightarrow P(v_{j+4}, v_{j+6})\big) \\
& \& (\forall v_{j+7} \exists v_{j+8} P(v_{j+7}, v_{j+8})) \\
& \& (\forall \{v_{j+9+n} \colon n \in \omega\} - \& \{P(v_{j+9+n}, v_{j+10+n}) \colon n \in \omega\}) \\
& \qquad \& (\exists \{v_i \colon i \leqslant j\} \& \{P(v_i, v_{i+1}) \colon i < j\})\Big]\ j \in \omega \Big\}
\end{aligned}
$$

does not belong to any consistency property, [3].

In [3] there was also an analysis of the reasons for this drawback.

To overcome this problem in [1] E. Cunningham introduced the notion of chain consistency property. In the same paper she compares her notion and Karp's one remarking that something is gained but something is lost.

Namely, with the notion of chain consistency property she proves that (Theorem 2.2 in [1]) a sentence is ω-satisfied in an ω-chain of models iff the singleton of that sentence belongs to a chain consistency property.

The hardest part of this result is the left to right direction of the implication which is proved obtaining first the downward Lowenheim Skolem theorem for $L_{k,k}$ via the notion of B models for which she derived a completeness theorem with respect to the B axioms and rules (Theorem 2.4 in [1]). More specifically what is needed in the proof is that a sentence not containing B_n's is valid in B models iff it is ω-valid in ω-chains of models.

Indeed the reasons for adopting such a sophisticated machinery are well explained by E. Cunningham herself in [1] when she considers the difference with Karp's notion of k-consistency property in which « there is always a sufficient supply of new witnessing constants available ».

Somehow, recoursing to B models is a way of recapturing the supply of new witnessing constants, and of having to distribute them correctly on the way.

In this paper the author propose a new approach with a different notion of consistency property, call it ω-consistency property, that takes care of the need felt by Karp of a sufficient supply of new witnessing constants and, at the same time, is closer to the notion of ω-satisfiability yielding the same result as Theorem 2.2 in [1] without having to go through B axioms, completeness and downward Lowenheim Skolem theorem. Actually the downward Lowenheim Skolem theorem for $L_{k,k}$ is an easy consequence of the other results. Indeed each set s of sentences in an ω-consistency property is in some language L_n (see below) leaving uncompromised all the witnessing constants in $\cup \{C_i: i > n\}$.

In the clauses of an ω-consistency property dealing with disjunction and existential quantification some technicalities are needed to handle the problem of maintaining the information got from a sentence already analyzed.

In 2. we present the notion of ω-consistency property.

In 3. we prove the model existence theorem going through the details of the construction of the Hintikka set in the Hintikka type proof.

In 4. given an ω-satisfiable set S of sentences, we describe an ω-chain of models that contains S.

In 5. we conclude with the now easy proof of the downward Lowenheim Skolem theorem for $L_{k,k}$.

2. ω-consistency property.

For the notation that we are going to use we refer to [2] from which we depart in that there are no second order variables in what we are doing now.

For the notions of ω-chain of models and of ω-satisfiability we refer to [4] and [2].

We assume, without loss of generality, that our sentences and sets of sentences are such that no variable occurs in more than one set of variables immediately after a quantifier.

Since k is a strong limit cardinal of cofinality ω and it is greater than ω, we may assume that $k = \cup \{k_n: n \in \omega\}$, where $2^{k_n} \leqslant k_{n+1}$. Let $L_{k,k}$ have no individual constants.

Let C_n, $n \in \omega$, be sets such that $|C_n| = k_n$ and for all $m, n \in \omega$ if $m \neq n$ then $C_m \cap C_n = \emptyset$.

Let L_n be the language obtained from $L_{k,k}$ by adding $\cup\{C_i: i < n\}$ as individual constants.

These assumptions and notations will hold throughout this paper.

Let us now define the notion of ω-consistency property for $L_{k,k}$.

Σ is an ω-consistency property for $L_{k,k}$ with respect to $\{C_i: i \in \omega\}$ if Σ is a set of sets s of sentence such that $|s| \leqslant k$ and there is an n (depending on s) such that all the sentences in s are in L_n and all the following clauses are satisfied.

C0) If Z is an atomic sentence then either $Z \notin s$ or $-Z \notin s$, and if Z is of the form $t \neq t$, t a constant, then $Z \notin s$;

C1) Suppose $|I| < k$

a) if $\{c_i = d_i: i \in I\} \subset s \in \Sigma$ with c_i and d_i constants then $s \cup \{d_i = c_i: i \in I\} \in \Sigma$;

b) if $\{Z_i(c_i),\ c_i = d_i: i \in I\} \subset s \in \Sigma$, where $Z_i(c_i)$ are atomic or negated atomic sentences and c_i, d_i are constants, then $s \cup \cup \{(Z_i(d_i): i \in I\} \in \Sigma$;

C2) If $\{-\,-F_i: i \in I\} \subset s \in \Sigma$ and $|I| < k$ then $s \cup \{F_i: i \in I\} \in \Sigma$;

C3) If $\{\& \overline{F}_i: i \in I\} \subset s \in \Sigma$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ $0 < |\overline{F}_i| \leqslant k_m$ then $s \cup \left(\cup \{\overline{F}_i: i \in I\}\right) \in \Sigma$;

C4) If $\{\forall \overline{v}_i F_i: i \in I\} \subset s \in \Sigma$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ $0 < |\overline{v}_i| \leqslant k_m$ then for all $n \in \omega$ we have that $s \cup \left\{F_i(\overline{v}_i/f): f \in {}^{\cup\{\overline{v}_i: i \in I\}} \cup \{C_h: h \leqslant n\},\ i \in I\right\} \in \Sigma$;

C5) If $\{-\& \overline{F}_i: i \in I\} \subset s \in \Sigma$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ $0 < |\overline{F}_i| < k_m$ then there is a function g, $g \in \times\{\overline{F}_i: i \in I\}$ such that for any $\{-\forall \overline{v}_j F_j: j \in J\} \subset s$ such that $[|J| < k$, and there is an $m' \in \omega$ such that for all $j \in J$ $0 < |\overline{v}_j| < k_{m'}$, and there is a partition $\langle J_p: p \in \omega\rangle$ of J such that (for all $p \in \omega$ there is a function f_p, $f_p \in {}^{\cup\{\overline{v}_j: j \in J_p\}} C_p$ with the property that $\{-F_j(\overline{v}_j/f_p): j \in J_p,\ p \leqslant n\} \subset s$ and $s \cup \{-F_j(\overline{v}_j/f_p): j \in J_p\} \in \Sigma$ where n is the least natural number such that all the sentences in s are in L_n)] then for all $p \in \omega$ we have that

$$s \cup \{-g(i): i \in I\} \cup \{-F_j(\overline{v}_j/f_p): j \in J_p\} \in \Sigma;$$

C6) If $\{-\forall \overline{v}_i F_i: i \in I\} \subset s \in \Sigma$ and there is n the least natural number such that $|I| \leqslant k_n$ and for all $i \in I$ $0 < |\overline{v}_i| \leqslant k_n$ and all the sentences in s are in L_n then there is a partition $\langle I_p: p \in \omega\rangle$ of I such that for any $1-1$ function f_p, $f_p \in {}^{\cup\{\bar v_i: i \in I_p\}} C_{n+p}$, if $[\{-\forall \bar v_j F_j: j \in J\} \subset s$, and $|J| < k_n$, and for all $j \in J$ $0 < |\bar v_j| < k_n$, and $J \cap I = \emptyset$, and there is a partition $\langle J_q: q \in \omega\rangle$ of J such that there are $1-1$ functions f'_q, $f'_q \in {}^{\cup\{\bar v_j: j \in J_q\}} C_q$, such that $\{F_j(\bar v_j/f'_q): j \in J_q, q < n\} \subset s$ and $s \cup \{-F_j(\bar v_j/f'_q): j \in J_q\} \in \Sigma$ for all $q \in \omega]$ and if range $(f_p) \cap$ range $(f'_{n+p}) = \emptyset$, then for all $p \in \omega$ $s \cup \{-F_i(\bar v_i/f_p): i \in I_p\} \cup \{-F_j(\bar v_j/f'_{n+p}): j \in J_{n+p}\} \in \Sigma$.

3. Model existence theorem.

MODEL EXISTENCE THEOREM. If S is a set of sentences of $L_{k,k}$, $|S| = k_0 < k$, and $S \in \Sigma$ an ω-consistency property with respect to $\{C_i: i \in \omega\}$, then S is ω-satisfiable in an ω-chain of models. Moreover the n-th structure of the chain has cardinality at most k_n.

PROOF. By a good split of a set s of less than k sentences we shall mean a partition $\langle s_m: m \in \omega\rangle$ of s such that $|s_m| \leqslant k_m$, every sentence of the form either $\& \bar F$ or $-\& \bar F$ in s_m has $|\bar F| \leqslant k_m$, every sentence of the form either $\forall \bar v F$ or $-\forall \bar v F$ in s_m has $|\bar v| \leqslant k_m$.

Let us define, by induction on n, sets $s_n \in \Sigma$ of sentences in L_n, $|s_n| \leqslant k_n$, good splits $\langle s_{n,m}: m \in \omega\rangle$ of each s_n such that $s_{n+1,m} = s_{n,m}$ for $m \leqslant n$ and $s_{n+1,m} \supset s_{n,m}$ for $m > n$, and for all $p \in \omega$ sets s_n^p of existential sentences in s_{n-1} and $1-1$ functions $f_{n,p}$ from $\cup\{\bar v_F: -\forall \bar v_F F \in s_n^p\}$ into C_{n+p-1} such that for i all and j less or equal to n if $i \neq j$ and $i+p = j+q$ then range $(f_{i,p}) \cap$ range $(f_{j,q}) = \emptyset$, such that for all $q \in \omega$

$$s_n \cup \{-F(\bar v_F/\cup\{f_{i,p}: i+p = q, i \leqslant n\}): \\ -\forall \bar v_F F \in \cup\{s_i^p: i+p = q, i \leqslant n\}\} \in \Sigma$$

and

$$\{-F(\bar v_F/\cup\{f_{i,p}: i+p = q, i \leqslant n\}): \\ -\forall \bar v_F F \in \cup\{s_i^p: i+p = q, i \leqslant n\}, q < n\} \subset s_n .$$

Let $s_0 = S$, $\langle s_{0,m}: m \in \omega\rangle$ be any good split of s_0, and for all $p \in \omega$ $s_0^p = \emptyset$ and $f_{0,p} = \emptyset$.

Suppose that s_h, $\langle s_{h,m}: m \in \omega\rangle$, s_h^p, $f_{h,p}$ have been defined for all $p \in \omega$ and for all $k \leqslant n$, with the above mentioned properties.

Let

$$s' = s_{n,n} \cup \{\forall \bar{v}_F F\colon \forall \bar{v}_F F \in \cup \{s_{n,i}\colon i < n\}\} \cup$$
$$\cup \{c = d\colon c = d \in \cup \{s_{n,i}\colon i < n\}\} \cup$$
$$\cup \{Z\colon Z \text{ is an atomic or negated atomic sentence in } s_{n,i},\ i < n\}.$$

Clearly $s'_n \subset s_n$, $|s'_n| \leqslant k_n$, and all conjunction and quantification sets in s'_n have cardinality at most k_n.

Define:

$$s_n^{(1)} = s_n \cup \{(-g(\bar{F})\colon -\&\bar{F} \in s'_n\}, \qquad \text{where } g \in \times \{\bar{F}\colon -\&\bar{F} \in s'_n\}$$

and g is such that for all $q \in \omega$

$$s_n \cup \{-g(\bar{F})\colon -\&\bar{F} \in s'_n\} \cup \{-F(\bar{v}_F/\cup \{f_{i,p}\colon i + p = q, i \leqslant p\})\colon$$
$$-\forall \bar{v}_F F \in \cup \{s_i^p\colon i + p = q, i \leqslant n\}\} \in \Sigma.$$

Such a g exists by C5) since $\cup \{s_i^p\colon i \leqslant n, p \in \omega\}$ is a subset of s_n of cardinality at most k_n of existential sentences of the form $-\forall \bar{v}_F F$ with $|\bar{v}_F| \leqslant k_n$, and $\langle \cup \{s_i^p\colon i + p = q, i \leqslant n\}\colon q \in \omega\rangle$ is a partition of $\cup \{s_i^p\colon i \leqslant n, p \in \omega\}$, and $\cup \{f_{i,p}\colon i + p = q, i \leqslant n\}$ are $1-1$ functions from $\cup \{\bar{v}_F\colon -\forall \bar{v}_F F \in \cup \{s_i^p\colon i + p + q, i \leqslant n\}\}$ into C_{q-1}, and for all $q \in \omega$:

$$s_n \cup \{-F(\bar{v}_F/\cup \{f_{i,p}\colon i + p = q, i \leqslant n\})\colon -$$
$$-\forall \bar{v}_F F \in \cup \{s_i^p\colon i + p = q, i \leqslant n\}\} \in \Sigma$$

and

$$\{-F(\bar{v}_F/\cup \{f_{i,p}\colon i + p = q, i \leqslant n\})\colon$$
$$-\forall \bar{v}_F F \in \cup \{s_i^p\colon i + p = q, i \leqslant n\}, q < n\} \subset s_n$$

by inductive hypothesis.

Let s_{n+1}^p be a partition of the set $\{-\forall \bar{v}_F F\colon -\forall \bar{v}_F F \in s_{n,n}\}$ such that for any $1-1$ function $f_{n+1,p}$ from $\cup \{\bar{v}_F\colon -\forall \bar{v}_F F \in s_{n+1}^p\}$ into C_{n+p} $-$ range $(\cup \{f_{i,p+q}\colon i + q = n + 1, i \leqslant n\})$ we have that for all

$p \in \omega$:

$$s_n^{(1)} \cup \{-F(\bar{v}_F/f_{n+1,p})\colon -\forall \bar{v}_F F \in s_{n+1}^p\} \cup \cup \{-F(\bar{v}_F/\cup \{f_{i,q}\colon i+q = n+1+p, i \leqslant n\})\colon - \forall \bar{v}_F F \in \cup \{s_i^q\colon i+q = n+1+p, i \leqslant n\}\} \in \Sigma .$$

Such partition exists by C6) since $\cup \{s_i^p\colon i \leqslant n, p \in \omega\}$ is a subset of $s_n^{(1)}$ of cardinality at most k_n of existential sentences of the form $-\forall \bar{v}_F F$ with $|\bar{v}_F| \leqslant k_n$, and $\langle \cup \{s_i^p\colon i+p = q, i \leqslant n\}\colon q \in \omega\rangle$ is a partition of $\cup \{s_i^p\colon i \leqslant n, p \in \omega\}$ and $\cup \{f_{i,p}\colon i+p = q, i \leqslant n\}$ are $1-1$ functions from $\cup \{\bar{v}_F\colon -\forall \bar{v}_F F \in \cup \{s_i^p\colon i+p = q, i \leqslant n\}\}$ into C_{q-1} and for all $q \in \omega$

$$s_n^{(1)} \cup \{-F(\bar{v}_F/\cup \{f_{i,p}\colon i+p = q, i \leqslant n\})\colon - \forall \bar{v}_F F \in \cup \{s_i^p\colon i+p = q, i \leqslant n\}\} \in \Sigma$$

and

$$\{-F(\bar{v}_F/\cup \{f_{i,p}\colon i+p = q, i \leqslant n\})\colon - \forall \bar{v}_F F \in \cup \{s_i^p\colon i+p = q, i \leqslant n\}, q < n\} \subset s_n \subset s_n^{(1)}$$

by the results of the previous point.

Thus let s_{n+1}^p be such a partition and $f_{n+1,p}$ a choice of functions as above.

Define

$$s_n^{(2)} = s_n^{(1)} \cup \{-F(\bar{v}_F/f_{n+1,0})\colon -\forall \bar{v}_F F \in s_{n+1}^0\} ;$$

$$s_n^{(3)} = s_n^{(2)} \cup \{d = c\colon c = d \in s_n'\} \cup \{Z(d)\colon Z(c), c = d \in s_n' \text{ and } Z \text{ is an atomic or negated atomic sentence in } s_n'\} ;$$

$$s_n^{(4)} = s_n^{(3)} \cup \{F\colon --F \in s_n'\} ;$$

$$s_n^{(5)} = s_n^{(4)} \cup (\cup \{\bar{F}\colon \& \bar{F} \in s_n'\}) ;$$

$$s_n^{(6)} = s_n^{(5)} \cup \{F(\bar{v}_F/f_F)\colon \forall \bar{v}_F F \in s_n', f_F \in^{\bar{v}_F} \cup \{C_i\colon i \leqslant n\}\} .$$

Define $s_{n+1} = s_n^{(6)}$, $\langle s_{n+1,m}\colon m \in \omega\rangle$ any good split of s_{n+1} such that if $m \leqslant n$ then $s_{n+1,m} = s_{n,m}$ and if $m > n$ then $s_{n+1,m} \supset s_{n,m}$.

To complete the definition by induction we have only to remark that all the conditions on s_n, $\langle s_{n,m}\colon m \in \omega \rangle$, s_n^p, $f_{n,p}$ are preserved.

Indeed it is easy to see that $s_{n+1} \in \Sigma$, due to the conditions of an ω-consistency property, and furthermore the sentences in s_{n+1} are in L_{n+1}, and $|s_{n+1}| \leqslant 2^{k_n} \leqslant k_{n+1}$.

Moreover $\langle s_{n+1,m}\colon m \in \omega \rangle$ satisfies the due condition by definition.

Also notice that not only for all $p \in \omega$

$$s_n^{(1)} \cup \{-F(\bar{v}_F/f_{n+1,p})\colon -\forall \bar{v}_F F \in s_{n+1}^p\} \cup \\ \cup \{-F(\bar{v}_F/\cup \{f_{i,p}\colon i+q = n+1+p\})\colon \\ -\forall \bar{v}_F F \in \cup \{s_i^q\colon i+q = n+1+p\}\} \in \Sigma,$$

but also for all $q \in \omega$

$$s_{n+1} \cup \{-F(\bar{v}_F/\cup \{f_{i,p}\colon i+p = q, i \leqslant n+1\})\colon \\ -\forall \bar{v}_F F \in \cup \{s_i^p\colon i+p = q, i \leqslant n+1\}\} \in \Sigma$$

due to the conditions C1), C2), C3), C4) of an ω-consistency property, and for all $i, j \leqslant n+1$ if $i \neq j$ and $i+p = j+q$ with p and q in ω then range $(f_{i,p}) \cap$ range $(f_{j,q}) = \emptyset$ by construction, and

$$\{-F(\bar{v}_F/\cup \{f_{i,p}\colon i+p = q, i \leqslant n+1\})\colon -\forall_F F \in \\ \cup \{s_i^p\colon i+p = q, i \leqslant n+1\}, q < n+1\} \subset s_n^{(3)} \subset s_n^{(4)} \subset s_n^{(5)} \subset s_{n+1}.$$

Thus for all $p \in \omega$ even s_{n+1}^p and $f_{n+1,p}$ satisfy the conditions.

Remark that for all $n \in \omega$ we have that $s_n \subset s_{n+1}$.

Now let $s_\omega = \cup \{s_n\colon n \in \omega\}$.

Let us point out the properties of s_ω relevant to our purpose.

a) Not both an atomic sentence and its negation are in s_ω, for otherwise they would belong to some s_n which belongs to Σ and this would contradict condition C0). For the same reason, sentences of the form $t \neq t$, t a constant, are not in s_ω.

b) If $c = d$ belongs to s_ω, and hence to s_n for some $n \in \omega$, then $d = c$ belongs to $s_{n'}$, for some $n' > n$, and hence to s_ω. Furthermore, if $Z(c)$ is an atomic or negated atomic sentence which belongs to s_ω and $c = d$ also belongs to s_ω, then both would belong to s_n for $n \in \omega$, and so $Z(d)$ would belong to $s_{n'}$ for some $n' > n$, and hence to s_ω.

c) If a sentence of the form $-\forall F$ is in s_ω, and hence in s_n for some $n \in \omega$, then also F is in s_ω, being in $s_{n'}$ for some $n' > n$.

d) Similarly if a sentence of the form $\& \overline{F}$ belongs to s_ω, and hence to s_n for some $n \in \omega$, then also each F belonging to $\overline{F}$ belongs to s_ω.

e) If a sentence of the form $\forall \bar{v}_F F$ belongs to s_ω, and hence to s_n for some $n \in \omega$, then for any $m \geqslant n$ and for any function f from $\bar{v}_F$ into the constants in $\cup \{C_i\colon i \leqslant m\}$ also the sentence $F(\bar{v}_F/f)$ belongs to s_ω, for $\forall \bar{v}_F F$ would belong to any s'_m with $m \geqslant n'$ for some $n' > n$, and $F(\bar{v}_F/f)$ would belong to s_{m+1} for $m \geqslant n'$ and hence to s_ω.

f) If a sentence of the form $-\& \overline{F}$ belongs to s_ω, and hence to s_n for some $n \in \omega$, then there is an F in $\overline{F}$ such that $-F$ belongs to $s_{n'}$ for some $n' > n$, and hence to s_ω.

g) If a sentence of the form $-\forall \bar{v}_F F$ belongs to s_ω, and hence to s_n for some $n \in \omega$, then it belongs to $s^p_{n'}$ for some $p \in \omega$ and $n' > n$ and therefore there is a function f from $\bar{v}_F$ into $C_{n'+p}$ such that $-F(\bar{v}_F/f)$ belongs to $s_{n'+p+1}$, by the construction of this, and hence to s_ω.

Thanks to these properties s_ω can be used to define an ω-chain of models which will ω-satisfy S.

For any c and d in $\cup \{C_i\colon i \in \omega\}$ let $c \sim d$ if either $c = d \in s_\omega$ or d is c. $\sim$ is an equivalence relation on $\cup \{C_i\colon i \in \omega\}$ as it can be easily deduced from property *b*) of s_ω. Let $\bar{c} = \{d\colon c \sim d\}$. Let $i_{\bar{c}} =$ the least natural number such that there is $d \in \bar{c} \cap C_i$. Let

$$\overline{C}_i = \{\bar{c}\colon i_{\bar{c}} = i\}\,.$$

For any n-ary predicate P in $K_{k,k}$ define the corresponding n-ary relation $\overline{P} = \{\langle \bar{c}_1, \ldots, \bar{c}_n\rangle\colon P(c_1, .., c_n) \in s_\omega\}$. This is well defined since if $c_1 \sim d_1, \ldots, c_n \sim d_n$ and $P(c_1, \ldots, c_n) \in s_\omega$ then also $P(d_1, \ldots, d_n) \in s_\omega$ thanks to n applications of the property *b*) of s_ω.

Pet $\overline{P}_i = P \cap (\overline{C}_i)^n$.

Let $\langle \overline{C}_i, \{\overline{P}_i\colon P \text{ is a predicate in } K_{k,k}\}\rangle$ be the i-th structure of the ω-chain of models $\overline{M}$ adequate for the given language $L_{k,k}$. Eventually expand these structures to the language L_n by interpreting any constant c in $\cup \{C_i\colon i \in \omega\}$ in $\bar{c}$.

Now, using the properties of s_ω, an easy induction on the rank of the sentences in s_ω shows that any such sentence is ω-satisfied in the ω-chain of models $\bar{M}$. We just add, as a remark, that the ω-satisfiability of the sentences of the forms either $\forall\bar{v}_F F$ or $-\forall\bar{v}_F F$ is obtained by assigning $\overline{f(v)}$ to any v in $\bar{v}$, where f is one of the functions mentioned in e) and in g) of the properties of s_ω, and observing that such assignment is bounded because the ranges of such f's are in $\cup\{C_i\colon i\leqslant n\}$ for some $n\in\omega$.

Thus all the sentences in s_ω, and in particular those in S ($\subset s_\omega$), are ω-satisfied in the ω-chain of models $\bar{M}$ that we described, and the universes $\bar{C}_i$ of the structure of $\bar{M}$ have cardinality at most k_i, as it was to be shown.

4. Existence of an ω-consistency property for a set of ω-satisfiable sentences.

By an *immediate subsentence* of a sentence F we mean the sentence:

G if F is $-G$,

G if F is $\&\bar{G}$ and $G\in\bar{G}$,

$G(\bar{v}/f)$ if F is $\forall\bar{v}G$ and f is any function from $\bar{v}$ into the constants.

By a *subsentence* of a sentence we mean either the sentence itself or an immediate subsentence of a sentence which was already proved to be a subsentence of the given sentence.

A *weak subsentence* is either a subsentence or the negation of a subsentence.

THEOREM 1. Let S be a set of ω-satisfiable sentences of $L_{k,k}$ with $|S|<k$. Let C_i be mutually disjoint sets such that $|C_i|=k_i$ for each $i\in\omega$. There is an ω-consistency property Σ with respect to $\{C_i\colon i\in\omega\}$ such that $S\in\Sigma$.

PROOF. Let $|S|=k_0$. Let $\bar{M}$ be an ω-chain of models that ω-satisfies S.

Let us define, by induction on n, sets s_n of weak subsentences of S in L_n with $|s_n|\leqslant k_n$, and ω-chains of models $\bar{M}_n$ which are expansions of $\bar{M}$ to the languages L_n such that, $\bar{M}_n\models^\omega s_n$ as follows

$$s_0=S\,,\qquad \bar{M}_0=\bar{M}\,.$$

Suppose that we have already obtained s_h and $\bar{M}_h$ for $h\leqslant n$.

Let

$s_n' = \{c = d\colon c = d \in s_n\} \cup \{Z\colon Z \in s_n$ and Z is an atomic or negated atomic sentence$\} \cup \{--F\colon --F \in s_n - s_{n-1}\} \cup \{\&\bar{F}\colon$ either $\&\bar{F} \in s_n - s_{n-1}$ and $|\bar{F}| \leqslant k_n$ or $\&\bar{F} \in s_{n-1}$ and $|\bar{F}| = k_n\} \cup \{\forall \bar{v}_F F\colon \forall \bar{v}_F F \in s_n$ and $|\bar{v}_F| \leqslant k_n\} \cup \{-\&\bar{F}\colon$ either $-\&\bar{F} \in s_n - s_{n-1}$ and $|\bar{F}| \leqslant k_n$ or $-\&\bar{F} \in s_{n-1}$ and $|\bar{F}| = k_n\} \cup \{-\forall \bar{v}_F F\colon$ either $-\forall \bar{v}_F F \in s_n - s_{n-1}$ and $|\bar{v}_F| \leqslant k_n$ and there is $\bar{a}_F$, a bounded assignment to $\bar{v}_F$ within the n-th structure of $\bar{M}_n$, that ω-satisfies $-F$ in $\bar{M}_n$, or $-\forall \bar{v}_F F \in s_{n-1}$ and n is the least natural number such that $|\bar{v}_F| \leqslant k_n$ and there is $\bar{a}_F$, a bounded assignment to $\bar{v}_F$ within the n-th structure in $\bar{M}_n$, that ω-satisfies $-F$ in $M_n\}$.

Let f_n be a $1-1$ function from $\cup \{\bar{v}_F\colon -\forall \bar{v}_F F \in s_n'\}$ into C_n, this function exists since $|\cup \{\bar{v}_F\colon -\forall \bar{v}_F F \in s_n'\}| \leqslant k_n$.

Let

$\bar{a}_n = \cup \{\bar{a}_F\colon -\forall \bar{v}_F F \in s_n'$ and $\bar{a}_F$ is a bounded assignment to $\bar{v}_F$ within the n-th structure of $\bar{M}_n$ that ω-satisfies $-F$ in $\bar{M}_n\}$.

Define $\bar{M}_{n+1}$ as the expansion of $\bar{M}_n$ to the language L_{n+1} obtained by interpreting each constant c belonging to $\cup \{f_n(\bar{v}_F)\colon -\forall \bar{v}_F F \in s_n'\}$ (which is a subset of C_n) in $\bar{a}_n(f_n^{-1}(c))$ and each constants in $C_n - \big(\cup \{f_n(\bar{v}_F)\colon -\forall \bar{v}_F F \in s_n'\}\big)$ in any fixed element of the universe of the first structure in $\bar{M}$.

Let

$$g_n \in \times \{\bar{F}\colon -\&\bar{F} \in s_n'\}$$

such that $-g_n(\bar{F})$ is ω-satisfied in $\bar{M}_n$.

Let

$s_{n+1} = s_n \cup \{d = c\colon c = d \in s_n'\} \cup \{Z(d)\colon Z(c) \in s_n',\ c = d \in s_n'$ and Z is an atomic or negated atomic sentence$\} \cup \{F\colon --F \in s_n'\} \cup \cup \{F\colon F \in \bar{F}$ and $\&\bar{F} \in s_n'\} \cup \{F(\bar{v}_F/f_F)\colon \forall \bar{v}_F F \in s_n'$ and $f_F \in {}^{\bar{v}_F}\cup \bar{F}\{C_i\colon i \leqslant n\}\} \cup \{-g_n(\bar{F})\colon -\&\bar{F} \in s_n'\} \cup \{-F(\bar{v}_F/f_n)\colon -\forall \bar{v}_F F \in s_n'\}$.

At this point it should be remarked that all the constants in any sentence in s_n are interpreted within the n-th structure in the ω-chain of models $\bar{M}_n$; thus we can correctly speak of ω-satisfaction and it is easily seen that $\bar{M}_{n+1} \models^{\omega} s_{n+1}$ and that $|s_{n+1}| \leqslant k_{n+1}$. Notice further that $s_n \subset s_{n+1}$ for all $n \in \omega$.

Let

$$\Sigma = \{s: s \subset s_n,\ n \in \omega\}\ .$$

CLAIM. Σ is an ω-consistency property with respect to $\{C_i: i \in \omega\}$.

Indeed if $s \in \Sigma$ then $s \subset s_n$ for some $n \in \omega$. Thus s contains only sentences in L_n and $|s| \leqslant |s_n| \leqslant k_n < k$. Furthermore $\overline{M}_n \models^{\omega} s$ and therefore sentences of the form $t \neq t$ cannot belong to s and also an atomic sentence and its negation cannot both belong to s. Thus C0) is satisfied.

As far as conditions C1), C2), C3), C4) of an ω-consistency property are to be checked, it is easy to see that there is $n' \geqslant n$ such that the set s' which should belong to Σ once $s \in \Sigma$ due to one of these conditions is indeed a subset of $s_{n'}$ and therefore it does belong to Σ

To check C5) remark that the choice of g_n is fixed and, since $s_{n'} \supset s_n$ if $n' \geqslant n$, it works properly even in the rest of the construction.

Finally it suffice to say that the entire construction was explicitly done to satisfy C6).

Thus the proof of the theorem is complete.

Since the union of ω-consistency properties is still an ω-consistency property, we can also state the following:

THEOREM 2. The set of ω-satisfiable sets of sentences in $L_{k,k}$ with cardinality less than k is a subset of an ω-consistency property with respect to $\{C_i: i \in \omega\}$.

5. The downward Lowenheim Skolem theorem for $L_{k,k}$.

In what we have done so far it is implicit an easy proof of the following version of the:

DOWNWARD LOWENHEIM SKOLEM THEOREM FOR $L_{k,k}$. If s is a set of sentences of $L_{k,k}$, $|s| < k$ and s is ω-satisfiable, then there is an ω-chain of models with the universe of the n-th structure of cardinality at most k_n that ω-satisfies s.

PROOF. By Theorem 1 there is an ω-consistency property with respect to $\{C_i: i \in \omega\}$ to which s belongs. By the model existence theorem there is an ω-chain of models that ω-satisfies s such that the cardinality of the n-th structure is at most k_n.

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