RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

BOGDAN RZEPECKI

A note on fixed point theorem of Schauder type with applications

Rendiconti del Seminario Matematico della Università di Padova, tome 66 (1982), p. 1-6
<http://www.numdam.org/item?id=RSMUP_1982__66__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 66 (1982)

A Note on Fixed Point Theorem of Schauder Type with Applications.

BOGDAN RZEPECKI (*)

SUMMARY - In this note we present an axiomatic approach to the measure of noncompactness of sets. A fixed point theorems of Darbo type (cf. [3]) are proved; some applications to the system of differential equations are given.

1. Let $\mathfrak{X}$ be a Banach space. For an arbitrary bounded subset X of $\mathfrak{X}$, the *measure of noncompactness* $\mathfrak{L}(X)$ is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite convering of X by sets of diameter $\leq \varepsilon$. For properties of function $\mathfrak{L}$, see [4] or [6].

Suppose that K is a bounded closed convex subset of $\mathfrak{X}$ and F is a continuous mapping of K into itself such that $\mathfrak{L}(F[X]) \leq k \cdot \mathfrak{L}(X)$ (here $F[X]$ denote the image of X under F) for each subset X of K. G. Darbo [3] proved that if $k < 1$, then F has a fixed point. Obviously, this result is a generalization of the well-known fixed point principle of Schauder.

In the present note we give an axiomatic approach to the measure of noncompactness which is useful in applications to the system of equations. Using this concept we generalize the above Darbo theorem and give applications.

2. Let $\mathfrak{X}$ be a Banach space. For subset X of $\mathfrak{X}$, we shall denote their closure by $\bar{X}$ and their closed convex hull by $\overline{\text{conv}}\, X$.

(*) Indirizzo dell'A.: Institute of Mathematics, A. Mickiewicz University, Matejki 48/49, 61-712 Poznań, Poland.

DEFINITION. Let S be the set of all $(t_1, t_2, \dots, t_n)$ in $\mathbb{R}^n$ ($\mathbb{R}^n$ is n-dimensional Euclidean space with the zero element θ) such that $t_i \geq 0$ for $i = 1, 2, \dots, n$, and let S_∞ be the set of all $(q_1, q_2, \dots, q_n)$ with $0 \leq q_i \leq +\infty$ for $i = 1, 2, \dots, n$. For $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ in S_∞ we write $u \leqq v$ if $u_i \leq v_i$ for every i.

A function $\psi\colon 2^{\mathfrak{X}} \to S_\infty$ ($2^{\mathfrak{X}}$ denote a family of all nonempty subsets of $\mathfrak{X}$) is said to be a *generalized measure of noncompactness* on $\mathfrak{X}$ provided it satisfies for every point x in $\mathfrak{X}$ and every subsets X, Y of $\mathfrak{X}$ the following axioms: (1) $\psi(X \cup \{x\}) = \psi(X)$, (2) if $X \subset Y$ then $\psi(X) \leqq \psi(Y)$, and (3) if $\psi(X) = \theta$ then $\overline{X}$ is compact.

The following theorem is the main result of this paper.

THEOREM 1. *Suppose that $\mathfrak{X}$ is a Banach space, K is a nonempty convex closed subset of $\mathfrak{X}$, and F is a continuous mapping of K into itself. Let ψ be a generalized measure of noncompactness on $\mathfrak{X}$ such that $\psi(K) \in S$ and $\psi(\overline{\mathrm{conv}}\, X) = \psi(X)$ for each subset X of K. Assume, moreover, that $\psi(F[X]) \leqq L(\psi(X))$ for each subset X of K, where L is a bounded linear operator of $\mathbb{R}^n$ into itself with the spectral radius less than one and the property that $L[S] \subset S$. Then the set $\{x \in K\colon Fx = x\}$ is nonempty and compact.*

PROOF. First we prove that if X_0 is a subset of K such that $\psi(F[X_0]) = \psi(X_0)$ then $\overline{X}_0$ is compact. Indeed, $\psi(X_0) \in S$ and $\psi(X_0) \leqq \leqq L(\psi(X_0))$. Since $L[S] \subset S$, we find that $\theta \leqq \psi(X_0) \leqq L^n(\psi(X_0))$ for $n = 1, 2, \dots$. Now, by theorem I.2.2.9 from [7], we obtain $\psi(X_0) = \theta$ and consequently $\overline{X}_0$ is compact.

Let us put $Q = \{x_m\colon m = 0, 1, \dots\}$, where $x_0 \in K$ and $x_n = Fx_{n-1}$ for $n \geq 1$. Then $\psi(Q) = \psi(F[Q])$ and therefore $\overline{Q}$ is compact. Hence there exists a nonempty subset Q_0 of Q such that $F[Q_0] = Q_0$ (see [2, Th. VI.1.8]).

Writing $\mathfrak{U} = \{X \subset K\colon Q_0 \subset X,\ \overline{\mathrm{conv}}\, X = X,\ F[X] \subset X\}$, $V = \bigcap \{X\colon X \in \mathfrak{U}\}$ we have $V \in \mathfrak{U}$ and $\overline{\mathrm{conv}}\, F[X] \in \mathfrak{U}$ whenever $X \in \mathfrak{U}$. Hence $V = \overline{\mathrm{conv}}\, F[V]$. Therefore F is a continuous mapping of the convex compact set V into itself and by Schauder theorem, F has a fixed point in K. This completes the proof.

M. A. Krasnoselskii [5] has given the following theorem: If K is a nonempty bounded closed convex subset of a Banach space, A is a contraction and B is completely continuous on K with $Au + Bv \in K$ for u, v in K, then there exists x in K such that $Ax + Bx = x$. Next we generalize this result.

Throughout the rest of this section, $\mathfrak{X}_i$ ($i = 1, 2, \dots, n$) is a Banach space with the norm $|||\cdot|||_i$ and the measure of noncompactness $\mathfrak{L}_i$, K_i is a nonempty convex closed bounded subset of $\mathfrak{X}_i$, and $K = K_1 \times \times K_2 \times \ldots \times K_n$.

We start with the result stated as follows:

COROLLARY. *Suppose that F_i $(i = 1, 2, \ldots, n)$ is a continuous mapping from K into K_i such that $\mathfrak{L}_i(F_i[X_1 \times X_2 \times \ldots \times X_n]) \leq \sum_{j=1}^{n} c_{ij}\mathfrak{L}_j(X_j^\varphi)$ for each $X_j \subset K_j$ $(j = 1, 2, \ldots, n)$. Assume, moreover, that $[c_{ij}]$ $(i, j = 1, 2, \ldots, n)$ is a matrix with the spectral radius less than one. Then there exists a point $(x_1, x_2, \ldots, x_n)$ in K such that $x_i = F_i(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$.*

PROOF. Let us put $\mathfrak{X} = \mathfrak{X}_1 \times \ldots \times \mathfrak{X}_n$, $F = (F_1, \ldots, F_n)$ and let L denote the linear operator generated by matrix $[c_{ij}]$. Moreover, let us put $\psi(X) = (\tilde{\mathfrak{L}}_1(X_1), \ldots, \tilde{\mathfrak{L}}_n(X_n))$ for each subset $X = X_1 \times \ldots \times X_n$ of $\mathfrak{X}$, where $\tilde{\mathfrak{L}}_i(X_i)$ is equal to $\mathfrak{L}_i(X_i)$ or $+\infty$ if diameter of X_i is finite or infinite, respectively. Then ψ is a generalized measure of noncompactness on $\mathfrak{X}$, and the assertion follows from Theorem 1.

THEOREM 2. *Suppose that G_i $(i = 1, 2, \ldots, n)$ is a mapping from $K \times K$ to K_i satisfying the following conditions:* (i) *for each fixed y in K the function $x \mapsto G_i(x, y)$ is continuous on K,* (ii) $\mathfrak{L}_i(\{G_i(x, y) \colon x \in X\}) \leq \leq \sum_{j=1}^{n} a_{ij}\mathfrak{L}_j(X_j)$ *for each y in K and $X = X_1 \times \ldots \times X_n$ with $X_j \subset K_j$ $(j = 1, 2, \ldots, n)$, and* (iii) $|||G_i(x, u) - G_i(x, v)|||_i \leq \sum_{j=1}^{n} b_{ij}|||u_j - v_j|||_j$ *for all x, $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in K. Assume, moreover, that $[a_{ij} + 2b_{ij}]$ $(i, j = 1, 2, \ldots, n)$ is a matrix with the spectral radius less than one. Then there is a point $x = (x_1, x_2, \ldots, x_n)$ in K such that $G_i(x, x) = x_i$ for $i = 1, 2, \ldots, n$.*

PROOF. Let us put $F_i x = G_i(x, x)$ $(i = 1, 2, \ldots, n)$ for x in K; F_i are continuous. Now, let $X = X_1 \times \ldots \times X_n$ be an arbitrary subset of K with $X_j \subset K_j$ $(j = 1, 2, \ldots, n)$. Modifying the reasoning from [3, p. 91] we obtain

$$\mathfrak{L}_i(F_i[X]) \leq \mathfrak{L}_i(G_i[X \times X]) \leq \sum_{j=1}^{n} (a_{ij} + 2b_{ij})\mathfrak{L}_j(X_j) .$$

Therefore all the conditions of our Corollary are satisfied, and $F = = (F_1, \ldots, F_n)$ has a fixed point in K. This finishes the proof.

Note finally, that a nonnegative matrix $M = [c_{ij}]$ $(i, j = 1, 2, \ldots, n)$ has the spectral radius $r(M)$ less than one if and only if

$$\begin{vmatrix} 1-c_{11} & -c_{12} & \cdots & -c_{1i} \\ -c_{21} & 1-c_{22} & \cdots & -c_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ -c_{i1} & -c_{i2} & \cdots & 1-c_{ii} \end{vmatrix} > 0$$

for all $i = 1, 2, \ldots, n$. Let us remark that there exists a positive constant h_0 such that $r(h\cdot M) < 1$ for every $0 < h \leq h_0$.

3. Now, we are going to consider an application of our result to the theory of differential equations.

Assume that E_i $(i = 1, 2, \ldots, n)$ is a Banach space with the norm $\|\cdot\|_i$ and the measure of noncompactness $\mathfrak{L}_i$. Let us put $I = [0, a]$, $J = [0, h]$ $(0 < h \leq a)$, and $B = B_1 \times B_2 \times \ldots \times B_n$ with $B_i = \{x \in E_i\colon \|x\|_i \leq b\}$ for $i = 1, 2, \ldots, n$.

By (PC) we shall denote the problem of finding the solution of the system of differential equations

$$x_i' = f_i(t, x_1, x_2, \ldots, x_n),\ n_i' = f_i(t, x_1, x_2, \ldots; x_n) \qquad (i = 1, 2, \ldots, n)$$

satisfying the initial conditions $x_i(0) = \theta_i$ (θ_i denotes the zero of E_i) for $i = 1, 2, \ldots, n$, where each f_i is a function from $I \times B \times B$ into E_i and satisfies some regularity conditions (of the Ambrosetti [1] type) with respect to measure $\mathfrak{L}_i$.

PROPOSITION. *Let f_i $(i = 1, 2, \ldots, n)$ be a bounded continuous function from $I \times B \times B$ into E_i satisfying the following conditions:* (a) $\mathfrak{L}_i(f_i[I \times X \times B]) \leq \sum_{j=1}^{n} k_{ij}\mathfrak{L}_j(X_j)$ *for any subset* $X = X_1 \times \ldots \times X_n$ *of* B *with* $X_j \subset B_j$, (b) $\|f_i(t, x, u) - f_i(t, x, v)\|_i \leq \sum_{j=1}^{n} L_{ij}(t)\|u_j - v_j\|_j$ *for all* $t \in I$, $x \in B$ *and* $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$ *in* B, *where each* L_{ij} *is an integrable function on* I. *Let* $\sup\{\|f_i(t, x, y)\|_i\colon (t, x, y) \in I \times B \times B\} \leq M$ *for* $i = 1, 2, \ldots, n$. *Assume, moreover, that* $h \leq \min(a, M^{-1}b)$ *and* $\left[h\cdot k_{ij} + 2\cdot\sup\left\{\int_0^t L_{ij}(s)\,ds\colon 0 \leq t \leq h\right\}\right]$ $(i, j = 1, 2, \ldots, n)$ *is a matrix with the spectral radius less than one. Then there exists a solution of problem* (PC) *defined on* J.

PROOF. Let $i = 1, 2, \ldots, n$. Let us denote by $C(J, E_i)$ the space of all continuous functions from an interval J to E_i, with the usual supremum norm $|||\cdot|||_i$ and the measure of noncompactness $\mathfrak{L}_i^*$. Moreover, let K_i be the set of all functions g in $C(J, E_i)$ such that $g(0) = \theta_i$ and $\|g(t) - g(s)\|_i \leq M|t - s|$ for $t, s \in J$ and let us put $K = K_1 \times \times K_2 \times \ldots \times K_n$.

Define a mapping $G = (G_1, G_2, \ldots, G_n)$ as follows:

$$G_i(x\ y)(t) = \int_0^t f_i(s, x(s), y(s))\, ds \qquad \text{for } x, y \text{ in } K\,.$$

Then for each fixed y in K the function $x \mapsto G_i(x, y)$ is continuous on K, and $|||G_i(x, u) - G_i(x, v)|||_i \leq \sum_{j=1}^{n} \sup_{t \in J} \int_0^t L_{ij}(s)\, ds \cdot |||u_j - v_j|||_j$ for all x, $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in K. Now, assume that $X = X_1 \times \ldots \times X_n$ with $X_i \subset K_i$ and $y \in K$ fixed. By the integral main-value theorem we have $\int_0^t f_i(s, x(s), y(s))\, ds \in t \cdot \overline{\text{conv}}\, (\{f_i(s, x(s), y(s)): 0 \leq s \leq t\})$ for x in X. Hence

$$\sup_{t \in J} \mathfrak{L}_i \left(\left\{ \int_0^t f_i(s, x(s), y(s))\, ds : x \in X \right\}\right) \leq$$

$$\leq h \cdot \mathfrak{L}_i \left(f_i[I \times \cup \{x[J] : x \in X\} \times B]\right) \leq h \cdot \sum_{j=1}^{n} k_{ij}\, \mathfrak{L}_j(\cup \{x_j[J] : x_j \in X_j\})$$

and, using the Ambrosetti Lemma [1, Lemma 2.2°], we obtain

$$\mathfrak{L}_i^*(\{G_i(x, y) : x \in X\}) = \sup_{t \in J} \mathfrak{L}_i \left(\left\{ \int_0^t f_i(s, x(s), y(s))\, ds : x \in X \right\}\right) \leq$$

$$\leq h \cdot \sum_{j=1}^{n} k_{ij}\, \mathfrak{L}_j(\cup \{x_j[J] : x_j \in X_j\}) =$$

$$= h \cdot \sum_{j=1}^{n} k_{ij} \sup_{t \in J} \mathfrak{L}_j(\{x_j(t) : x_j \in X_j\}) = h \cdot \sum_{j=1}^{n} k_{ij}\, \mathfrak{L}_j^*(X_j)\,.$$

Consequently, by Theorem 2 there exists $x^0 = (x_1^0, \ldots, x_n^0)$ in K such that $x_i^0(t) = G_i(x^0, x^0)(t)$ $(i = 1, 2, \ldots, n)$ for $t \in J$.

REFERENCES

[1] A. AMBROSETTI, *Un teorema di esistenza per le equazioni differenziali negli negli spazi di Banach*, Rend. Sem. Mat. Univ. Padova, **39** (1967), pp. 349-360.
[2] C. BERGE, *Topological Spaces*, Oliver and Boyd Ltd, Edinburgh and London, 1963.
[3] G. DARBO, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova, **24** (1955), pp. 84-92.
[4] K. DEIMLING, *Ordinary Differential Equations in Banach Spaces*, Lect. Notes in Math. 596, Springer-Verlag, 1977.
[5] M. A. KRASNOSELSKII, *Two remarks on the method of successive approximations* [in Russian], Uspehi Mat. Nauk, **10** (1955), pp. 123-127.
[6] R. H. MARTIN Jr., *Nonlinear Operators and Differential Equations in Banach Spaces*, John Wiley and Sons, New York, 1976.
[7] J. M. ORTEGA - W. C. RHEINBOLDT, *Iterative Solutions of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

Manoscritto pervenuto in redazione il 9 luglio 1980.