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# Binumerability in a sequence of theories

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# Binumerability in a Sequence of Theories.

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SUMMARY - In this note we answer a question raised by A. Ursini in [2]. In that work he defines a denumerable sequence of arithmetic theories $Q_n$, whose union is complete, and asks a question concerning the binumerability of the $\Delta_{n+1}$ relations in $Q_n$. We show that a relation is in $\Delta_{n+1}$ if and only if it is binumerable in $Q_n$.

We are going to follow the notations and terminology of [1] and [2]. In particular $K_0$ is the language of first order arithmetic, and a $K$-system is a set of sentences in the language $K$. $Prf_T(x, y)$ is the relation « $y$ is a proof of $x$ from axioms in $T$ ». Let's recall that, given a $K$-system, where $K_0 \subseteq K$, a numerical relation $R \subseteq \omega^n$ is called numerable in $T$ is there is a formula $\varphi(x_1, \ldots, x_n)$ in $K$ such that

$$R(k_1, \ldots, k_n) \quad \text{holds if and only if} \quad T \vdash \varphi(\bar{k}t, \ldots, \bar{k}_n) .$$

In that case we say that $\varphi$ numerates $R$ in $T$.

A relation $R$ is binumerable in $T$ if there exists a formula $\varphi$ in $K$ such that $\varphi$ numerates $R$ and $\neg\varphi$ numerates $\omega^n - R$.

The result expressed in the following proposition is obtained as a straigtforward application of the « Rosser trick ».

PROPOSITION 1. Let $T$ be a consistent $K$-system, where $K_0 \subseteq K$, such that $T \vdash x \leqslant \bar{n} \leftrightarrow x = \bar{0} \vee x = \bar{1} \vee \ldots \vee x = \bar{n}$ and $T \vdash x \leqslant \bar{n} \vee \bar{n} \leqslant x$.

Let the relation $Prf_T(x, y)$ be binumerable in $T$ and $R \subseteq \omega^n$.

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**If** both $R$ and $\omega^n - R$ are numerable in $T$ then $R$ is binumerable in $T$.

Proof. For notational convenience let's suppose $R \subseteq \omega$. Let $\varphi(x)$ numerate $R$ and $\psi(x)$ numerate $\omega - R$, and let $P\dot{r}f(x, y)$ binumerate $Prf_T(x, y)$ in $T$.

Consider then the following formula

$$\chi(x) = \exists y (P\dot{r}f(\bar{\varphi}(\dot{\bar{x}}), y) \wedge \forall z \leqslant y \neg P\dot{r}f(\bar{\psi}(\dot{\bar{x}}), z)) .$$

We claim that $\chi(x)$ binumerates $R$ in $T$.

Since $T$ is consistent it is clearly enough to show that

$$\text{(i)} \quad \text{if } k \in R \quad \text{then} \quad T \vdash \chi(\bar{k}) ,$$

and

$$\text{(ii)} \quad \text{if } k \notin R \quad \text{then} \quad T \vdash \neg \chi(\bar{k}) .$$

(i) If $k \in K$ then $T \vdash \varphi(\bar{k})$, since $\varphi$ numerates $R$ in $T$, therefore for some $n \in \omega$, $Prf_T(\varphi(\bar{k}), h)$ holds and therefore

(1) $$T \vdash P\dot{r}f(\overline{\varphi(\bar{k})}, \bar{h}) .$$

On the other hand, since $\psi$ numerates $\omega - R$, we have

$$\forall i \leqslant h , \quad T \vdash \neg P\dot{r}f(\overline{\psi(\bar{k})}, \bar{\imath})$$

and thus

(2) $$T \vdash \forall z \leqslant \bar{h} \neg P\dot{r}f(\bar{\psi}(\bar{k}), z) .$$

From (1) and (2) we have

$$T \vdash \chi(\bar{k}) .$$

(ii) Let's assume that $k \notin R$.

Since $\psi$ numerates $\omega - R$ in $T$, we have $T \vdash \psi(\bar{k})$ and therefore, as above, for some $r \in \omega$

(3) $$T \vdash P\dot{r}f(\overline{\psi(\bar{k})}, \bar{r}) .$$

On the other hand for $i \leqslant r$ $Prf_T(\varphi(\bar{k}), i)$ doesn't hold, hence, as above

$$T \vdash \forall z \leqslant \bar{r} \neg Prf(\overline{\varphi(\bar{k})}, z) .$$

Therefore

(4) $$T \vdash Prf(\overline{\varphi(\bar{k})}, y) \to \neg y \leqslant \bar{r} .$$

From (3) and (4) we have

$$T \vdash Prf(\overline{\varphi(\bar{k})}, y) \to \exists z \leqslant y \, Prf(\overline{\psi(\bar{k})}, z)$$

namely

$$T \vdash \neg \chi(\bar{k}) .$$

In [2] two sequences $\{Q_n : n \in \omega\}$ and $\{R_n : n \in \omega\}$ with the following properties are defined.

1) $Q_n = Pr_{R_n}$.

2) $R_n$ is a $\Sigma_{n+1}$ valid $K_0$-system containing Robinson's Arithmetic $Q$.

3) $R \in \Sigma_{n+1}$ if and only if it is numerable in $R_n$.

4) $R_n$ is binumerable in $Q_n$ via a formula $\alpha_n$ (Proposition 8 in [2]).

Applying Proposition 1 we have the following result,

PROPOSITION 2. $R \in \Delta_{n+1}$ if and only if $R$ is binumerable in $Q_n$.

PROOF. Let's first notice that by 1) numerability (and binumerability) of a relation in $R_n$ or in $Q_n$ are equivalent. Hence, by 4) we have that $R_n$ is binumerable in $R_n$ via $\alpha_n$. Thus the relation $Prf_{R_n}(x, y)$ is binumerable in $R_n$, via, say $Prf_{\alpha_n}(x, y)$, since it is just a primitive recursive combination of $R_n$ and $PR$ relations, which are binumerable in $R_n$ by 2). Also from 2) we have that $R_n$ is consistent and,

$$R_n \vdash x \leqslant \bar{n} \leftrightarrow x = \bar{0} \vee x = \bar{1} \vee \ldots \vee x = \bar{n} \quad \text{and} \quad R_n \vdash x \leqslant \bar{n} \vee \bar{n} \leqslant x .$$

If $R \in \Delta_{n+1}$ we have that both $R$ and $\omega - R$ are numerable in $R_n$.

We can thus apply Proposition 1 to get that $R$ is binumerable in $R_n$. Conversely if $R$ is binumerable in $R_n$ it follows immediately from 3) that both $R$ and its complement are in $\Sigma_{n+1}$, namely $R \in \Delta_{n+1}$.

## REFERENCES

[1] S. Feferman, *Arithmetization of metamathematics in a general setting*, Fundamenta Matematicae, **49** (1960), pp. 35-92.
[2] A. Ursini, *A sequences of theories whose union is complete*, Rend. Sem. Mat. Univ. Padova, **57** (1977), pp. 75-92.

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