RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Blocking sets of maximal type in finite projective planes

Rendiconti del Seminario Matematico della Università di Padova, tome 65 (1981), p. 85-101

<http://www.numdam.org/item?id=RSMUP_1981__65__85_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 65 (1981)

Blocking Sets of Maximal Type in Finite Projective Planes.

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1. Introduction.

Let $(\mathscr{P}, \mathscr{L})$ be a finite projective plane of order n and m the greatest natural number not exceeding $\sqrt{n}$. A « blocking set » is defined as a subset $\mathfrak{S}$ of $\mathscr{P}$ such that every line $l \in \mathscr{L}$ contains at least one point of $\mathfrak{S}$ and no line is completely contained in $\mathfrak{S}$. It has been shown in [4], that $|\mathfrak{S}| \geqslant n + \sqrt{n} + 1$.

If $|\mathfrak{S}| = n + k$, then no more than k points of $\mathfrak{S}$ can be collinear. Let's call a blocking set $\mathfrak{S}$ « of maximal type » provided there is a line in $\mathscr{L}$ which contains k elements of $\mathfrak{S}$ ($\mathfrak{S}$ is called a blocking set « of type (n, k) » in the terminology of [5]).

Then obviously $|\mathfrak{S}| \leqslant 2n$. Assume n is not a square. Then $|\mathfrak{S}| \geqslant n + m + 2$ for every blocking set $\mathfrak{S}$ and Bruen has shown in [4], that for $|\mathfrak{S}| = n + m + 2$, the blocking set $\mathfrak{S}$ is of maximal type. The author showed in [2], that such blocking sets exist only in the projective planes of orders 3 and 5.

First some elementary results about the ocurrence of blocking sets of maximal type in finite projective planes. It is trivial to see, that for $n > 2$ a projective plane of order n always contains a blocking set $\mathfrak{S}$ of maximal type with $|\mathfrak{S}| = 2n$.

LEMMA 1. Let $(\mathscr{P}, \mathscr{L})$ be a finite projective plane of order n. If $n \geqslant 4$, the plane $(\mathscr{P}, \mathscr{L})$ does contain a blocking set $\mathfrak{S}$ of maximal type with $|\mathfrak{S}| = 2n - 1$.

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More precisely: Let $1 \in \mathscr{L}$, $P_1, P_2 \in l$, $P_1 \neq P_2$. Then the number of blocking sets containing the point set $l \setminus \{P_1, P_2\}$ is exactly $n! - n^2 + n$.

PROOF. Give the lines different from l through P_1 resp. P_2 names $h_1, \ldots, h_n$ resp. $v_1, \ldots, v_n$. Then every point $P \in \mathscr{P} - l$ has a unique representation $P = h_i \cap v_j$. So these « affine points » are ordered in a natural way in a $n \times n$-square with rows $h_1, \ldots, h_n$ and columns $v_1, \ldots, v_n$. There are exactly $n!$ sets of n affine points from different rows and columns. Of these $n(n-1)$ correspond to lines in the plane. Let $\mathfrak{S}_0$ be one of the remaining $n! - n^2 + n$ sets of n affine points from different rows and columns. Then $\mathfrak{S} = \mathfrak{S}_0 \cup \{X | X \in l, X \notin \{P_1, P_2\}\}$ is a blocking set (of maximal type) with $|\mathfrak{S}| = 2n - 1$.

LEMMA 2. Let $(\mathscr{P}, \mathscr{L})$ be a finite project ve plane of order n, $l \in \mathscr{L}$, P_1, P_2 and P_3 different points from l. Order the lines through P_1 and P_2 in the same way as in the proof of Lemma 1 and consider the latin square corresponding to the lines through P_3, which are different from l.

Exactly then is there no blocking set of $2n - 2$ elements containing the point set $l - \{P_1, P_2, P_3\}$ if the latin square determined by P_3 has the following property

(T) Given two places in the latin square, which are in different rows, in different columns and have different entries, there is exactly one transversal containing these two places.

PROOF. This is immediate as every transversal of the latin square determined by P_3 either consists of collinear points or leads to a blocking set of maximal type of $2n - 2$ points. For the notion of « latin square » and « transversal » see [1].

The main object of this paper is the proof of the following

THEOREM. Let $(\mathscr{P}, \mathscr{L})$ be a finite projective plane of order n, where n is not a square, $n = m^2 + q$, $1 \leqslant q \leqslant 2m$. Assume $\mathfrak{S}$ is a blocking set of maximal type of $(\mathscr{P}, \mathscr{L})$, where $|\mathfrak{S}| = n + m + 3$.

Further assume, that there are at least two lines containing $m + 3$ elements of $\mathfrak{S}$. Then one of the following holds:

(i) $n \leqslant 7$.

(ii) $n = 8$, $|\mathfrak{S}| = 13$. The points of $\mathfrak{S}$ are ordered like given in fig. 1. We have $(\mathscr{P}, \mathscr{L}) \simeq PG(2, 8)$ and $PG(2, 8)$ does contain such a blocking set of maximal type with 13 elements.

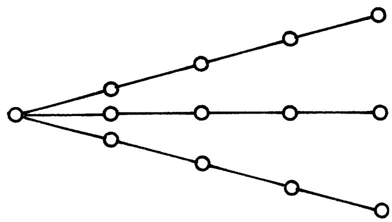

Figure 1

(iii) $n = 10$, $|\mathfrak{S}| = 16$. The incidence structure of $\mathfrak{S}$ as induced from $\mathscr{L}$ is uniquely determined (see fig. 2).

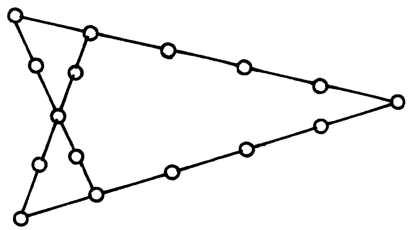

Figure 2

REMARKS. (1) The case $n \leqslant 7$ is not very interesting. It follows from Lemma 2, that $PG(2, 7)$ does contain blocking sets of maximal type of 12 points.

(2) As for the case $n = 8$, it suffices to invoke [6], where the uniqueness of the projective plane of order 8 has been shown.

It is easy to see, that $PG(2, 8)$ does contain a blocking set as in (ii), although this case is missing in the list of « Sylvester-Gallai »-designs embeddable in a desarguesian projective plane as given in [7].

In fact, the author constructed $PG(2, 8)$ starting from the above blocking set, but this has not been included in the present paper.

(3) In case (iii) the methods of this paper don't lead to a contradiction. The author hopes to settle this case with the help of a computer program.

(4) If there is only one line containing $m + 3$ points of $\mathfrak{S}$, somewhat different methods are needed. This case will be the subject of a subsequent paper.

2. Proof of the theorem.

Let $\mathscr{P}$, $\mathscr{L}$, n, m, q, $\mathfrak{S}$ like in the statement of the theorem and assume $n \geqslant 8$. In the sequel set theoretic symbols like «$\in$» and «$\subset$» are used in the set theoretic sense as well as with respect to incidences in $(\mathscr{P}, \mathscr{L})$. Hopefully no confusion will occur. The join of points X and Y is denoted by XY.

We introduce some further notation:

$$\mathscr{L}_i = \{l | l \in \mathscr{L}, |l \cap \mathfrak{S}| = i\}, \qquad a_i = |\mathscr{L}_i| \qquad \text{for } i = 1, 2, \ldots, m+3,$$

so that $\mathscr{L} = \bigcup_{i=1}^{m+3} \mathscr{L}_i$ and by assumption of the theorem $a_{m+3} \geqslant 2$.

Elements of $\mathscr{L}_i$ are called i-lines, elements of $\mathscr{L}_1$ are tangents, elements of $\mathscr{L} - \mathscr{L}_1$ are «lines of $\mathfrak{S}$».

For $P \in \mathscr{P}$ set $\mathscr{L}_i(P) = \{l | P \in l \in \mathscr{L}_i\}$, $a_i(P) = |\mathscr{L}_i(P)|$.

For every $l \in \mathscr{L}$ set $\mathrm{st}(l) = |l \cap \mathfrak{S}|$, the «strength» of l and $l^* = \{P | P \in l, P \notin \mathfrak{S}\}$, so that $|l^*| = n + 1 - \mathrm{st}(l)$.

Like in [2] we speak of a «(P, l)-argument» whenever $P \in \mathfrak{S}$, $P \notin l \in \mathscr{L}_{m+3}$ and when we count $|\mathfrak{S}|$ by considering the $m+3$ lines of $\mathfrak{S}$ joining P to the points of $\mathfrak{S} \cap l$.

LEMMA 3. Let $l_1, l_2 \in \mathscr{L}$, $l_1 \neq l_2$, $\mathrm{st}(l_1) + \mathrm{st}(l_2) > m + 4$. Then $l_1 \cap l_2 \in \mathfrak{S}$.

PROOF. This follows from $|\mathfrak{S}| = n + m + 3$.

Let $1 < \mathrm{st}(l) < m + 3$. Then we set

$$\mathscr{L}(l^*) = \{k | k \in \mathscr{L} - \mathscr{L}_1, k \neq l, k \cap l \notin \mathfrak{S}\}, \qquad \mathscr{L}_i(l^*) = \mathscr{L}(l^*) \cap \mathscr{L}_i,$$

$$i = 2, 3, \ldots, m+2,$$

$$z(l^*) = \sum_{i=2}^{m+3} (i-1) |\mathscr{L}_i(l^*)|, \qquad \mathscr{L}(l^*, X) = \{k | k \in \mathscr{L}(l^*), X \in k\}$$

$$\text{for } X \in \mathscr{P},$$

$$z(l^*, X) = \sum_{i=2}^{m+2} (i-1) |\mathscr{L}_i(l^*, X)|.$$

We have then

LEMMA 4. (l^*-argument.)
Let $1 < \mathrm{st}(l) < m + 3$, $l_1 \in \mathscr{L}_{m+3}$, $P = l \cap l_1$.

Then $z(l^*) = (n + 1 - \mathrm{st}\,(l))(m + 3 - \mathrm{st}\,(l))$.
If $\mathscr{L}_i(l^*) \neq \emptyset$, then $i + \mathrm{st}\,(l) \leqslant m + 4$.

(i) Assume in addition, there exists a triangle of $(m + 3)$-lines. Then $|\mathscr{L}(l^*, X)| = m + 3 - \mathrm{st}\,(l)$ for every $X \in (\mathfrak{S} \cap l_1) - \{P\}$ and thus $|\mathscr{L}(l^*)| = (m + 2)(m + 3 - \mathrm{st}\,(l))$.

(ii) Assume there is no triangle of $(m + 3)$-lines, but $a_{m+3} \geqslant 2$ and thus all the $(m + 3)$-lines meet in a common point $P_0 \in \mathfrak{S}$.

If $P = P_0$, then $|\mathscr{L}(l^*, X)| = m + 3 - \mathrm{st}\,(l)$ for every $X \in (\mathfrak{S} \cap l_1) - \{P_0\}$ and thus $|\mathscr{L}(l^*)| = (m + 2)(m + 3 - \mathrm{st}\,(l))$.

If $P \neq P_0$, then $|\mathscr{L}(l^*, X)| = m + 3 - \mathrm{st}\,(l)$ for $X \in (\mathfrak{S} \cap l_1) - \{P_0, P\}$ and $|\mathscr{L}(l^*)| = (m + 1)(m + 3 - \mathrm{st}\,(l)) + |\mathscr{L}(l^*, P_0)|$.

COROLLARY. For $1 < \mathrm{st}\,(l) < m + 3$ we have $|l^*| \leqslant |\mathscr{L}(l^*)|$.

PROOF. As $|\mathfrak{S}| = n + m + 3$, we have for every $X \in l^*$ that $z(l^*, X) = |\mathfrak{S}| - (n + \mathrm{st}\,(l)) = m + 3 - \mathrm{st}\,(l)$ and thus $z(l^*)$ is like given in the lemma. By lemma 3 we have $i + \mathrm{st}\,(l) \leqslant m + 4$ whenever $\mathscr{L}_i(l^*) \neq \emptyset$. Observe that $l_1 \cap k \in \mathfrak{S}$ if $\mathrm{st}\,(k) > 1$.

Assume there is a triangle of $(m + 3)$-lines. For every $X \in (\mathfrak{S} \cap l_1) - (l \cap l_1)$ we have exactly $m + 3$ lines of $\mathfrak{S}$ passing through X. Exactly $\mathrm{st}\,(l)$ of these don't belong to $\mathscr{L}(l^*)$. This proves (i). The proof of (ii) is analogous.

The proof of the theorem will consist of an examination of the incidence structure $(\mathfrak{S}, \mathscr{L} - \mathscr{L}_1)$ and its embedding in $(\mathscr{P}, \mathscr{L})$. The interested reader is advised to illustrate most of our proofs with diagrams.

LEMMA 5. $\mathscr{L} \neq \mathscr{L}_1 \cup \mathscr{L}_{m+3}$.

PROOF. Assume $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_{m+3}$. Then $(\mathfrak{S}, \mathscr{L}_{m+3})$ is a subplane of $(\mathscr{P}, \mathscr{L})$ of order $m + 2$. Thus $|\mathfrak{S}| = (m + 2)^2 + m + 3$ and $n = (m + 2)^2$, a contradiction.

In the following, notation is chosen so that for example hypothesis (1.1) is meant to include hypothesis 1. Consider first

HYPOTHESIS 1. There is a triangle of $(m + 3)$-lines.
Let $\{l_1, l_2, l_3\}$ be a triangle of $(m + 3)$-lines and set

$$P_i = l_j \cap l_k \quad \text{for } \{i, j, k\} = \{1, 2, 3\}.$$

By lemma 5 there exists $l \in \mathscr{L}_t$ where $1 < t < m + 3$. From the corollary of lemma 4 we get $|l^*| \leqslant |\mathscr{L}(l^*)|$. Together with lemma 4 $m^2 + 2 - t \leqslant n + 1 - t \leqslant (m + 2)(m + 3 - t) = m^2 + (5 - t)m - 2t + 6$. It follows $t - 4 \leqslant (5 - t)m$ and thus $t \leqslant 4$. So under Hyp. 1 we get

$$(1) \qquad \mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2 \cup \mathscr{L}_3 \cup \mathscr{L}_4 \cup \mathscr{L}_{m+3} .$$

Let $l \in \mathscr{L}_t$ like before, so that $t \leqslant 4$. We can choose $P_2 \notin l$.

Then a (P_2, l_2)-argument yields $m^2 + m + 4 \leqslant |\mathfrak{S}| \leqslant 3(m + 2) + 2(m + 1) + (m - 1)2 = 7m + 6$, $m(m - 6) \leqslant 2$ and thus

$$(2) \qquad m \leqslant 6 .$$

Hypothesis (1.1). There is a quadrangle of $(m + 3)$-lines.
Choose $l_4 \in \mathscr{L}_{m+3}$, $l_4 \notin \{l_1, l_2, l_3\}$, $l_4 \cap \{P_1, P_2, P_3\} = \emptyset$.
Set $X_i = l_4 \cap l_i$, $i = 1, 2, 3$.

Obviously $m \geqslant 3$. Further $\mathscr{L}_2 \subseteq \{P_i X_i | i = 1, 2, 3\}$ and $a_2 \leqslant 3$.
Assume $m = 3$. First let $l \in \mathscr{L}_4$. By lemma 4 (i) we have $|\mathscr{L}(l^*)| = 10$.
Further $\mathscr{L}(l^*) = \mathscr{L}_2(l^*) \cup \mathscr{L}_3(l^*)$ and

$$(*) \qquad |\mathscr{L}(l^*)| = \begin{cases} |l^*| & \text{if} \quad \mathscr{L}(l^*) = \mathscr{L}_3(l^*) , \\ |l^*| + 1 & \text{if} \quad \mathscr{L}(l^*) \neq \mathscr{L}_3(l^*) . \end{cases}$$

As $|l^*| = n - 3$ we have $n \in \{13, 12\}$, $|\mathfrak{S}| \in \{19, 18\}$.
Especially $a_6 = 4$ as otherwise $|\mathfrak{S}| \geqslant 20$.
Assume first $n = 13$, $|\mathfrak{S}| = 19$. There exists $M \in \mathfrak{S} - (l_1 \cup l_2 \cup l_3 \cup l_4)$.
Let $(l_4 \cap \mathfrak{S}) - (l_1 \cup l_2 \cup l_3) = \{M_1, M_2, M_3\}$. As $MM_i \notin \mathscr{L}_6$, $i = 1, 2, 3$, we can choose notation so that $P_i \in MM_i = g_i \in \mathscr{L}_4$ for $i = 1, 2, 3$.
It follows from $(*)$ that $\mathscr{L}(g_1^*) = \mathscr{L}_3(g_1^*)$. Thus $a_2(P_2) = 0$.
As $a_4(P_2) \neq 0$, a (P_2, l_2)-argument yields the contradiction $|\mathfrak{S}| \geqslant 20$.
Let $n = 12$, $|\mathfrak{S}| = 18$. A (P_i, l_i)-argument gives $a_2(P_i) \neq 0$ for $i = 1, 2, 3$ and thus $a_4(P_i) = 0$, $a_2(P_i) = 1$, $a_2 = 3$. Choose M_1 like above. Then $a_4(M_1) \neq 0$. Let $M_1 \in g \in \mathscr{L}_4$. Then $g \cap \{P_1, P_2, P_3, X_1, X_2, X_3\} = \emptyset$. Thus $|\mathscr{L}_2(g^*)| = 3$, a contradiction.
We have shown $a_4 = 0$ in case $m = 3$.
Assume next $m = 3$, $a_2 \neq 0$. We can choose $P_1 X_1 \in \mathscr{L}_2$. A (P_2, l_2)-argument shows $|\mathfrak{S}| \leqslant 19$ and thus $a_6 = 4$. It is however immediate that $a_4 \neq 0$, a contradiction. Thus by (1) we have $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_3 \cup$

$\cup\ \mathscr{L}_{m+3}$ and $a_3 \neq 0$ by lemma 5. Let $l \in \mathscr{L}_3$. A l^*-argument gives an immediate contradiction.

We have $4 \leqslant m \leqslant 6$ under Hyp. (1.1).

Assume $a_2 \neq 0$. Choose $P_1X_1 \in \mathscr{L}_2$. A (P_2, l_2)-argument shows $m^2 + m + 4 \leqslant |\mathfrak{S}| \leqslant 3(m+2) + (m+1)2 = 5m+8$, $m(m-4) \leqslant 4$. It follows $m \leqslant 4$ and thus $m = 4$. Assume $a_4(P_2) = 0$. The same (P_2, l_2)-argument gives then the contradiction $|\mathfrak{S}| \leqslant 18 + 5 = 23$, $n \leqslant 16$.

So let $P_2 \in l \in \mathscr{L}_4$. By lemma 4 $|\mathscr{L}(l^*)| = (m+2)(m+3-4) = 18$. Further $|l^*| = n-3$. On the other hand $|\mathscr{L}_2(l^*)| = |\mathscr{L}_3(l^*)| \in \{1, 2\}$, $\mathscr{L}(l^*) = \mathscr{L}_2(l^*) \cup \mathscr{L}_3(l^*) \cup \mathscr{L}_4(l^*)$.

Case 1: Let $|\mathscr{L}_2(l^*)| = 1 = |\mathscr{L}_3(l^*)|$. Then $|\mathscr{L}_4(l^*)| = 16$, $18 = |\mathscr{L}(l^*)| = |l^*| + 1 = n - 2$. Thus $n = 20$, $|\mathfrak{S}| = 27$.

A (P_j, l_j)-argument for $j = 2, 3$ shows $a_2 = 1$. A (P_1, l_1)-argument shows $a_4(P_1) \neq 0$. Let $P_1 \in g \in \mathscr{L}_4$. Then $\mathscr{L}_2(g^*) = \emptyset$, thus $\mathscr{L}(g^*) = \mathscr{L}_4(g^*)$ and $17 = |g^*| = |\mathscr{L}(g^*)| = 18$, a contradiction.

Case 2: Let $|\mathscr{L}_2(l^*)| = 2 = |\mathscr{L}_3(l^*)|$, $|\mathscr{L}_4(l^*)| = 14$. Then $18 = |\mathscr{L}(l^*)| = |l^*| + 2 = n - 1$ and $n = 19$, $|\mathfrak{S}| = 26$. For $j = 2, 3$ a (P_j, l_j)-argument gives $a_4(P_j) = 4$ and $a_3(P_j) = 0$. Clearly $a_7(X_i) = 2$, $i = 1, 2, 3$ and thus $a_4(X_i) = 4$, $a_3(X_i) = 0$. As every 3-line has to pass through one of the points P_i or X_i, we have $\mathscr{L}_3 = \mathscr{L}_3(P_1)$. Let $X = l \cap P_1X_1$. Then $a_3(X) \neq 0$, a contradiction.

We have shown $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_3 \cup \mathscr{L}_4 \cup \mathscr{L}_{m+3}$ under Hyp. (1.1).

If $m = 6$, lemma 4 shows $a_3 \neq 0$ and for $l \in \mathscr{L}_3$ we get $48 = |\mathscr{L}(l^*)| \geqslant 2|l^*| = 2(n-2)$ and $n \leqslant 26$, a contradiction.

If $m = 5$, then $a_4 \neq 0$ and for $l \in \mathscr{L}_4$ we have $28 = |\mathscr{L}(l^*)| = 2|l^*| = 2(n-3)$ and $n = 17$, a contradiction.

So $m = 4$. Let $l \in \mathscr{L}_4$. Then $18 = |\mathscr{L}(l^*)| = |l^*| = n - 3$ and $n = 21$. The Bruck-Ryser theorem [3] gives a contradiction.

We have $a_4 = 0$. By lemma 5 then $a_3 \neq 0$. Let $l \in \mathscr{L}_3$: Then $24 = |\mathscr{L}(l^*)| = 2|l^*| = 2(n-2)$ and $n = 14$. Again we get a contradiction by [3]. We have proved.

LEMMA 6. Under Hyp. 1 we have $\mathscr{L}_{m+3} = \mathscr{L}_{m+3}(P_1) \cup \mathscr{L}_{m+3}(P_2) \cup \mathscr{L}_{m+3}(P_3)$.

Assume $a_{m+3} > 3$. We can choose $P_1 \in l_4 \in \mathscr{L}_{m+3}$, $l_4 \notin \{l_2, l_3\}$.

Assume further $\mathfrak{S} \subset l_1 \cup l_2 \cup l_3 \cup l_4$. Then $m^2 + m + 4 \leqslant n + m + 3 = |\mathfrak{S}| = 3(m+2) + m + 1 = 4m + 7$. It follows $m(m-3) \leqslant 3$, $m \leqslant 3$ and $n = 3m + 4$. If $m = 2$, then $n = 10$, a contradiction.

Thus $m = 3$, $n = 13$. It follows $\mathscr{L}_2 = \mathscr{L}_2(P_1)$, $a_2 = 3$. Let $g \in \mathscr{L}_2$.

For every $X \in g^*$ we have $a_2(X) = 1$ and thus $a_3(X) = 2$. It follows $20 = |\mathscr{L}(l^*)| = 2|l^*| = 24$, a contradiction.

Hence there exist s $M \in \mathfrak{S} - (l_1 \cup l_2 \cup l_3 \cup l_4)$. Take $X \in l_4 \cap \mathfrak{S}$, $X \notin \{P_1, l_4 \cap l_1\}$ such that $MX \cap \{P_1, P_2, P_3\} = \emptyset$. Then st $(MX) \geqslant 5$ and thus $MX \in \mathscr{L}_{m+3}$. This contradicts lemma 6. We have proved

Lemma 7. $a_{m+3} = 3$ under Hyp. 1.

A (P_1, l_1)-argument shows now $m^2 + m + 4 \leqslant |\mathfrak{S}| \leqslant 3(m + 2) + 2(m + 1) = 5m + 8$, $m(m - 4) \leqslant 4$ and thus

Lemma 8. $m \leqslant 4$ under Hyp. 1.

Assume $n = 8$, $|\mathfrak{S}| = 13$. Let $l \in \mathscr{L}_4$. Then $4 = |\mathscr{L}(l^*)| = |l^*| = 5$, a contradiction.

Assume next $m = 3$. First let $P_1 \in l \in \mathscr{L}_4$ and assume there is a $M \in \mathfrak{S} - (l_1 \cup l_2 \cup l_3 \cup l)$. Set $l \cap \mathfrak{S} = \{P_1, P_1 \cap l_1, X_1, X_2\}$.

As $a_5 = 0$, $a_6 = 3$ we can choose $P_2 \in MX_1$, $P_3 \in MX_2$. Thus there exists $g \in \mathscr{L}_4$ with $M \in g$ and $g \cap l \notin \mathfrak{S}$, a contradiction to lemma 3. Thus $\mathfrak{S} \subset l_1 \cup l_2 \cup l_3 \cup l$, $|\mathfrak{S}| = 17$, $n = 11$. Set $P_3X_i \cap l_3 = Y_i$, $i = 1, 2$. Then $g = Y_1X_2 \in \mathscr{L}_4$, $a_2(Y_1) = a_2(X_2) = 0$, $a_2(g \cap l_1) = a_2(g \cap l_2) = 1$.

Obviously $a_2 = 7$. It follows $|\mathscr{L}_2(g^*)| = 7 - 2 = 5$. A g^*-argument yields a contradiction. We have $a_4(P_i) = 0$, $i = 1, 2, 3$.

Assume $n > 10$. There are then two elements $X, Y \in \mathfrak{S} - (l_1 \cup l_2 \cup l_t)$, $X \neq Y$. As $a_4(P_i) = 0$ we get $XY \in \mathscr{L}_6$, a contradiction.

Thus $n = 10$, $|\mathfrak{S}| = 16$. Set $\{M\} = \mathfrak{S} - (l_1 \cup l_2 \cup l_3)$, $l = P_1 M \in \mathscr{L}_3$.

Obviously $\mathscr{L}_4 = \mathscr{L}_4(M)$. Thus $\mathscr{L}(l^*) = \mathscr{L}_2(l^*) \cup \mathscr{L}_3(l^*)$ and even $|\mathscr{L}_2(l^*)| \geqslant |l^*| = 8$. On the other hand $\mathscr{L}_2(l^*) = \mathscr{L}_2(P_2) \cup \mathscr{L}_2(P_3)$ and $|\mathscr{L}_2(l^*)| \leqslant 6$, a contradiction.

Finally let $m = 4$. Set $\mathscr{M} = \mathfrak{S} - (l_1 \cup l_2 \cup l_3)$. For $X, Y \in \mathscr{M}$, $X \neq Y$ we have $XY \ni P_i$, $i \in \{1, 2, 3\}$ as otherwise st $(XY) \geqslant 5$, thus $XY \in \mathscr{L}_7$ by (1), a contradiction to lemma 7. It follows $\binom{|\mathscr{M}|}{2} \leqslant 3 \cdot 5 = 15$ and thus $|\mathscr{M}| \leqslant 6$, $|\mathfrak{S}| \leqslant 18 + 6 = 24$, $n \leqslant 17$. We have $n = 17$, $a_4(P_i) = 5$ for every $i \in \{1, 2, 3\}$. A (P_1, l_1)-argument shows $|\mathscr{M}| = 10$, a contradiction.

We have shown

Lemma 9. $n \leqslant 7$ in case of Hyp. 1.

From now on consider

Hypothesis 2. $a_{m+3} > 1$, $\mathscr{L}_{m+3} = \mathscr{L}_{m+3}(P_0)$, where $P_0 \in \mathfrak{S}$.

Set $l_1, l_2 \in \mathscr{L}_{m+3}$, $P_0 = l_1 \cap l_2$, $l_1 \cap \mathfrak{S} = \{P_0, P_1, \ldots, P_{m+2}\}$, $l_2 \cap \mathfrak{S} = \{P_0, Q_1, \ldots, Q_{m+2}\}$. Consider first

HYPOTHESIS (2.1). $\mathscr{L}_{m+2} \neq \mathscr{L}_{m+2}(P_0)$.

Set $l_3 \in \mathscr{L}_{m+2} - \mathscr{L}_{m+2}(P_0)$, $P_1 = l_1 \cap l_3$, $Q_1 = l_2 \cap l_3$. Obviously $\mathscr{L}(l_3^*) = \mathscr{L}_2(l_3^*)$ and $|\mathscr{L}(l_3^*)| = |l_3^*| = n - m - 1$. As $|\mathscr{L}(l_3^*, P_i)| = 1$ for every $i = 2, 3, \dots, m+2$, we have by lemma 4 (ii) $|\mathscr{L}(l_3^*, P_0)| = n - m - 1 - (m+1) = n - 2m - 2$. On the other hand $|\mathfrak{S} - (l_1 \cup l_2 \cup l_3)| = n - m - 2$. It follows

LEMMA 10. Under Hyp. (2.1) we have $\mathscr{L}(P_0) = \mathscr{L}_2(P_0) \cup \mathscr{L}_{m+3}$, $a_{m+3} = 2$, $a_2(P_0) = n - m - 2$, $\mathscr{L}(l_3^*, X) = \mathscr{L}_2(l_3^*, X)$, $|\mathscr{L}(l_3^*, X)| = 1$ for every $X \in \{P_2, \dots, P_{m+2}, Q_2, \dots, Q_{m+2}\}$.

We set $P_iQ_i \in \mathscr{L}_2$, $i = 2, 3, \dots, m+2$.

Assume $a_{m+2}(Q_1) > 2$. Let $\{Q_1P_2, Q_1P_3\} \subseteq \mathscr{L}_{m+2}$. Apply lemma 10 to these $(m+2)$-lines. It follows $\{P_1Q_2, P_1Q_3\} \subseteq \mathscr{L}_2$. But $\{P_1Q_2, P_1Q_3\} \subseteq \mathscr{L}((Q_1P_2)^*)$, a contradiction to lemma 10.

LEMMA 11. $a_{m+2}(X) \leqslant 2$ for every $X \in \{P_1, \dots, P_{m+2}, Q_1, \dots, Q_{m+2}\}$.

Assume $a_{m+2}(Q_1) = 2$. Set $l_4 = Q_1P_2 \in \mathscr{L}_{m+2}$, so that $P_1Q_2 \in \mathscr{L}_2$.

Let first $n = 8$, set $l_3 \cap \mathfrak{S} = \{P_1, Q_1, R_1, R_2\}$, $l_4 \cap \mathfrak{S} = \{P_2, Q_1, S_1, S_2\}$. Then $\mathfrak{S} \subseteq l_1 \cup l_2 \cup l_3 \cup l_4$. We have $P_1S_1 \in \mathscr{L}_3$, $\{S_2R_1, S_2R_2\} \subseteq \mathscr{L}_4$. So there is a 4-line, which doesn't intersect P_1S_1 in $\mathfrak{S}$. This contradicts lemma 3.

We have $m \geqslant 3$. Assume $Q_1 \in g \in \mathscr{L}_{m+1}$, set $P_3 = g \cap l_1$.

Then $\mathscr{L}(g^*) = \mathscr{L}_2(g^*) \cup \mathscr{L}_3(g^*)$, $|\mathscr{L}_2(g^*, P_0)| = n - m - 2 - (m-1) = n - 2m - 1$, $|\mathscr{L}_2(g^*, P_j)| = 1$ for $j \notin \{0, 3\}$. Thus $|\mathscr{L}_2(g^*)| = n - 2m - 1 + m + 1 = n - m$, $|\mathscr{L}_3(g^*)| = m + 1$. A g^*-argument gives

$$2|g^*| = |\mathscr{L}_2(g^*)| + 2|\mathscr{L}_3(g^*)|, \qquad 2(n-m) = n - m + 2(m+1)$$

and so

$$(*) \qquad n = 3m + 2 .$$

Thus $m^2 + 1 \leqslant n \leqslant 3m + 2$, $m(m-3) \leqslant 1$ and thus $m = 3$.

From $(*)$ we get $n = 11$, $|\mathfrak{S}| = 17$. On the other hand $|\mathfrak{S}| \geqslant 11 + 6 + 2 = 19$, a contradiction. We have $a_{m+1}(Q_1) = 0$.

Assume $Q_1 \in g \in \mathscr{L}_m$. Use a (Q_2, l_1)-argument. We know $a_2(Q_2) = 2$. As $a_2(P_1) = 1$ and $P_1Q_2 \in \mathscr{L}_2$, we have $a_{m+2}(Q_2) = 0$ because of lemma 10.

Further $a_3(Q_2) + a_4(Q_2) \geqslant 1$ as $|\mathscr{L}(g^*, Q_2)| = 3$. It follows

$$(**) \qquad m^2 + m + 4 \leqslant |\mathfrak{S}| \leqslant 2m + 5 + 2 + (m-1)(m-1) = m^2 + 8$$

and $m \leqslant 4$. Let first $m = 3$, set $g \cap \mathfrak{S} = \{Q_1, P_3, M\}$.

As $\{Q_2P_1, Q_2P_2\} \subseteq \mathscr{L}_2$, we get $Q_2M \in \mathscr{L}_5$, which contradicts $a_5(Q_2) = 0$. So $m = 4$. We have equality in $(**)$. Thus $n = 17$, $|\mathfrak{S}| = 24$.

Use a g^*-argument: $|\mathscr{L}_2(g^*, P_0)| = 9$, $|\mathscr{L}_2(g^*, P_i)| = 1$, $i = 1, 2, 4, 5, 6$ and so $|\mathscr{L}_2(g^*)| = 14$. Further $\mathscr{L}_3(g^*, P_i) = \emptyset$ for $i = 4, 5, 6$ and thus $|\mathscr{L}_4(g^*, P_i)| = 2$ for $i = 4, 5, 6$. Set $v = |\mathscr{L}_4(g^*, P_1) \cup \mathscr{L}_4(g^*, P_2)|$.

Then $0 \leqslant v \leqslant 4$ and the g^*-argument gives

$$3(18 - 4) = z(g^*) = 14 + 3\cdot 2\cdot 3 + 3v + 2(4 - v)\,,$$

so $v = 2$. Thus $\{P_1N, P_2N\} \subseteq \mathscr{L}_4$, where $Q_1 \in h \in \mathscr{L}_3$, $h \cap \mathfrak{S} = \{Q_1, P_4, N\}$ (see fig. 3).

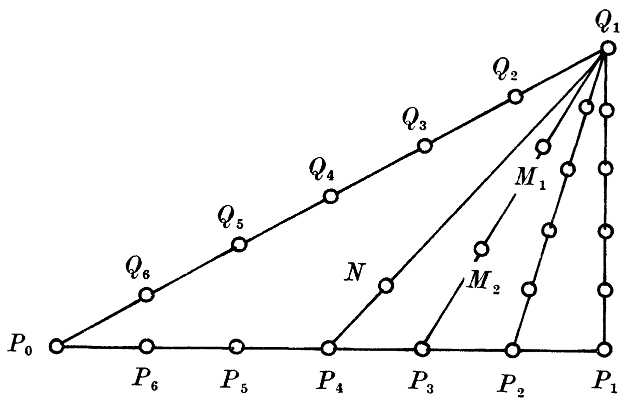

Figure 3

Set $g \cap \mathfrak{S} = \{Q_1, P_3, M_1, M_2\}$. Then $k = NM_1 \in \mathscr{L}_6$ as $k \cap \{P_1, P_2\} = \emptyset$.

We can assume $P_5 \in k$. Then $Q_5 \notin k$ and because $a_2(Q_5) = 1$, $P_5Q_5 \in \mathscr{L}_2$, we get $\mathscr{L}_2(k^*, Q_5) = \mathscr{L}(k^*, Q_5) = \emptyset$, a contradiction.

We have $a_{m+1}(Q_1) = a_m(Q_1) = 0$. A (Q_1, l_1)-argument gives

$$(***)\quad m^2 + m + 4 \leqslant |\mathfrak{S}| \leqslant 2m + 5 + 2m + m(m - 3) = m^2 + m + 5\,.$$

It follows $n \in \{m^2 + 1, m^2 + 2\}$ and $a_{m-1}(Q_1) \neq 0$. Thus $a_3(Q_2) + a_4(Q_2) + a_5(Q_2) \geqslant 2$ and our (Q_2, l_1)-argument gives $m^2 + m + 4 \leqslant 2m + 5 + 6 + (m - 2)(m - 1) = m^2 - m + 13$ and thus $m \in \{3, 4\}$.

Let $m = 3$. Then $|\mathfrak{S}| \geqslant 17$, $n \geqslant 11$ and so $n = 11$ because of $(***)$.

Set $l_3 \cap \mathfrak{S} = \{P_1, Q_1, R_1, R_2, R_3\}$, $l_4 \cap \mathfrak{S} = \{P_2, Q_1, S_1, S_2, S_3\}$.

Then $\mathscr{L}_4 = \{R_iS_j | i, j = 1, 2, 3\}$, $a_4 = 9$, $a_4(P_0) = a_4(P_1) = a_4(P_2) =$

$= a_4(Q_1) = 0$, $a_4(Q_2) \leqslant 3$, $a_4(Q_j) \leqslant 2$, $a_4(P_j) \leqslant 3$ for $j \in \{3, 4, 5\}$. It follows $a_4(Q_2) = 3$, $a_4(Q_j) = 2$, $a_4(P_j) = 3$ for $j = 3, 4, 5$. Let $v \in \mathscr{L}_4$, $Q_2 \notin v$.

As by lemma 3 every 4-line has to intersect v in points of $\mathfrak{S}$ we get $a_4 = 1 + 1 + 2 + 2 + 2 = 8$, a contradiction.

We have $m = 4$, $n \in \{17, 18\}$, $|\mathfrak{S}| \in \{24, 25\}$.

Let $M \in \mathfrak{S} - (l_1 \cup l_2 \cup l_3 \cup l_4)$. As $Q_2 M \cap \{P_1, P_2\} = \emptyset$, $a_6(Q_2) = 0$, we get $g = Q_2 M \in \mathscr{L}_5$. We have $\mathscr{L}(g^*) = \mathscr{L}_2(g^*) \cup \mathscr{L}_3(g^*)$, $\mathscr{L}(g^*, P_0) = \mathscr{L}_2(g^*, P_0)$, $|\mathscr{L}(g^*, P_0)| = 8$ resp. 9 if $n = 17$ resp. $n = 18$, $|\mathscr{L}(g^*)| = |\mathscr{L}(g^*, P_0)| + 10 = 18$ resp. 19 for $n = 17$ resp. $n = 18$.

A g^*-argument shows

$$(+) \qquad 2|g^*| = |\mathscr{L}_2(g^*)| + 2|\mathscr{L}_3(g^*)| .$$

Let first $n = 17$. Then $|g^*| = 13$ and $(+)$ shows $26 = |\mathscr{L}_2(g^*)| + 2(18 - |\mathscr{L}_2(g^*)|)$ and so $|\mathscr{L}_2(g^*)| = 10$. If $n = 18$, we get $|\mathscr{L}_2(g^*)| = 10$ from $(+)$ again. However $|\mathscr{L}_2(g^*, P_0)| \geqslant 8$ and $|\mathscr{L}_2(g^*, X)| \geqslant 1$ for every $X \in (\mathfrak{S} \cap l_1) - \{P_0, P_1, P_2, g \cap l_1\}$. Thus $|\mathscr{L}_2(g^*)| \geqslant 11$, a contradiction. We have

LEMMA 12. $a_{m+2}(X) \leqslant 1$ for every $X \in \{P_1, \ldots, P_{m+2}, Q_1, \ldots, Q_{m+2}\}$.

We have $a_{m+2}(Q_1) = 1$. Assume $a_{m+1}(Q_1) \neq 0$. Let $Q_1 \in l_4 \in \mathscr{L}_{m+1}$, $l_4 \cap l_1 = P_2$. Use a l_4^*-argument: $|\mathscr{L}_2(l_4^*, P_0)| = n + m + 3 - (3m + 4) = n - 2m - 1$.

For $i \in \{3, \ldots, m + 2\}$ we have $|\mathscr{L}_2(l_4^*, P_i)| = |\mathscr{L}_3(l_4^*, P_i)| = 1$. Hence $2(n - m) = |\mathscr{L}_2(l_4^*)| + 2|\mathscr{L}_3(l_4^*)| = n - m - 1 + 2m + a_2(P_1) + 2(2 - a_2(P_1))$, $a_2(P_1) \leqslant 2$.

It follows

$$(*) \qquad n = 3m + 3 - a_2(P_1) .$$

Thus $m^2 + 1 \leqslant 3m + 3$, $m(m - 3) \leqslant 2$ and $m \leqslant 3$.

First let $n = 8$, set $\mathscr{L}_3(Q_1) = \{l_4, l_5\}$, $l_4 \cap \mathfrak{S} = \{Q_1, P_2, S\}$, $l_5 \cap \mathfrak{S} = \{Q_1, P_3, T\}$. From $(*)$ $a_2(P_1) = 1$ and so $ST \notin P_1$. We get the contradiction $ST \in \mathscr{L}_5$.

Let $m = 3$. $(*)$ shows $n = 12 - a_2(P_1)$, $a_2(P_1) \leqslant 2$. Set

$$l_3 \cap \mathfrak{S} = \{P_1, Q_1, R_1, R_2, R_3\}, \qquad l_4 \cap \mathfrak{S} = \{P_2, Q_1, S_1, S_2\} .$$

If $n = 10$, then $\mathscr{L}_4 = \{l_4\} \cup \{S_i R_j | i = 1, 2;\ j = 1, 2, 3\}$, $a_4 = 7$.

Let $r \in \{2, 3, 4, 5\}$. Then $a_2(Q_r) \neq 0$, hence $a_4(Q_r) \neq 0$ by a (Q_r, l_1)-argument. Let $Q_r \in v = S_i R_j \in \mathscr{L}_4$. There exists $v' = S_a R_b \in \mathscr{L}_4$ such that $a \neq i$, $b \neq j$ and $v \cap l_1 \neq v' \cap l_1$. As $v \cap v' \in \mathfrak{S}$, we have $Q_r \in v'$.

Thus $a_4(Q_r) \geqslant 2$ for every $r \in \{2, 3, 4, 5\}$ and $a_4 \geqslant 9$, a contradiction.

If $n = 11$, set $Q_1 \in l_5 \in \mathscr{L}_3$, $l_5 \cap \mathfrak{S} = \{Q_1, P_3, T\}$. Then $TS_i \not\ni P_1$ as $a_2(P_1) = 1$. It follows $\{TS_1, TS_2\} \in \mathscr{L}_5$ and we can choose $P_4 \in TS_1$, $P_5 \in TS_2$. Thus $\{Q_4P_1, Q_5P_1\} \subseteq \mathscr{L}_2$, $a_2(P_1) > 1$, a contradiction. In case $n = 12$ we get the same type of contradiction. We have

LEMMA 13. $a_{m+1}(Q_1) = a_{m+1}(P_1) = 0$ under Hyp. (2.1). Further $m \geqslant 3$.

PROOF. The case $n = 8$ is clearly impossible under Hyp. (2.1).

HYPOTHESIS (2.1.1). $a_m(Q_1) \neq 0$.

Choose $Q_1 \in l_4 \in \mathscr{L}_m$, $P_2 \in l_4$. Use a l_4^*-argument: $|\mathscr{L}_2(l_4^*, P_0)| = n - 2m$. Further $|\mathscr{L}_2(l_4^*, P_i)| = 1$ for $i = 3, 4, \dots, m+2$.

We get the inequality $3(n + 1 - m) \leqslant n - 2m + m + 2m \cdot 3 + 9$, $2n \leqslant 8m + 6$ or

$$(*) \qquad n \leqslant 4m + 3\,.$$

Thus $m^2 + 1 \leqslant 4m + 3$, $m(m-4) \leqslant 2$ and $m \in \{3, 4\}$.

Let $m = 4$ first. Then $n \leqslant 19$ by $(*)$. Obviously $a_4(Q_1) \geqslant 2$.

Let $l_4 = Q_1P_2 \in \mathscr{L}_4$, $l_5 = Q_1P_3 \in \mathscr{L}_4$.

Assume $a_4(Q_1) = 2$. Because of $|\mathfrak{S}| \geqslant 24$ then $Q_1P_i \in \mathscr{L}_3$ for $i = 4, 5, 6$.

Set $Q_1P_i \cap \mathfrak{S} = \{Q_1, P_i, M_i\}$, $i = 4, 5, 6$. The points M_4, M_5, M_6 cannot be collinear as $a_7 = 2$. So we can choose $P_1 \notin M_4M_5$. Then $M_4M_5 \in \mathscr{L}_6$.

Let $Q_i = M_4M_5 \cap l_2$. Then $P_iQ_1 \in \mathscr{L}_2$ by lemma 10, a contradiction. So we have $a_4(Q_1) \geqslant 3$. Let $l_6 = Q_1P_4 \in \mathscr{L}_4$, choose $M \in \mathfrak{S}$, $M \notin l_i$, $1 \leqslant i \leqslant 6$, set $l_i \cap \mathfrak{S} = \{Q_1, P_{i-2}, R_{i-2}, S_{i-2}\}$, $i = 4, 5, 6$.

Choose $g = MR_4 \not\ni P_1$. Then $g \in \mathscr{L}_6$ and thus $g \cap \{P_2, P_3\} \neq \emptyset$. We can choose $P_3 \in g$. Set $g \cap l_2 = Q_j$. Then $P_jQ_1 \in \mathscr{L}_2$ by lemma 10. It follows $Q_1P_6 \in \mathscr{L}_2$, $Q_6 \in g$. Choose R_3 on l_5 such that $MR_3 \not\ni P_1$. Then $MR_3 \in \mathscr{L}_6$ and like above we get $Q_6 \in MR_3$, which is a contradiction.

So we have $m = 3$. As $|\mathfrak{S}| \geqslant 16$, we have $a_3(Q_1) \geqslant 2$. Set $l_4 = Q_1P_2 \in \mathscr{L}_3$, $l_5 = Q_1P_3 \in \mathscr{L}_3$, $l_4 \cap \mathfrak{S} = \{Q_1, P_2, M_1\}$, $l_5 \cap \mathfrak{S} = \{Q_1, P_3, M_2\}$.

Assume $n > 10$. Then we can choose $l_6 = Q_1P_4 \in \mathscr{L}_3$, $l_6 \cap \mathfrak{S} = \{Q_1, P_4, M_3\}$. The points M_1, M_2, M_3 are not collinear because $a_6 = 2$, $a_5(P_1) = 1$.

Set $g_1 = M_3M_1$, $g_2 = M_3M_2$. As $a_4(P_1) = a_4(Q_1) = 0$, we have $g_i \cap \{P_1, Q_1\} = \emptyset$, $i = 1, 2$ and thus $\{g_1, g_2\} \subseteq \mathscr{L}_5$. Set $g_i \cap l_2 = Q_j$. By lemma 10 then $P_jQ_1 \in \mathscr{L}_2$, thus $j = 5$. We have $g_i \ni Q_5$, $i = 1, 2$, a contradiction.

Finally $n = 10$, $|\mathfrak{S}| = 16$. Choose $P_4 \in M_1M_2$. Clearly $M_1M_2 \in \mathscr{L}_5$ and $Q_4P_1 \in \mathscr{L}_2$ by lemma 10. Further $Q_5 \in M_1M_2$ and consequently $P_1Q_5 \in \mathscr{L}_2$. This is case (iii) of the theorem. It is easy to see, that $(\mathfrak{S}, \mathscr{L} - \mathscr{L}_1)$ is uniquely determined. We have

LEMMA 14. Under Hyp. (2.1.1), case (i) or (iii) of the theorem holds.

We can assume $a_m(Q_1) = a_{m+1}(Q_1) = 0$. A (Q_1, l_1)-argument yields the contradiction $m^2 + m + 4 \leqslant |\mathfrak{S}| \leqslant 3m + 5 + (m + 1)(m - 3) = m^2 + m + 2$.

Thus we can assume, that Hyp. (2.1) is not satisfied.

HYPOTHESIS (2.2). $\mathscr{L}_{m+2} = \mathscr{L}_{m+2}(P_0)$, $\mathscr{L}_{m+1} \neq \mathscr{L}_{m+1}(P_0)$.

Set $l_3 = P_1Q_1 \in \mathscr{L}_{m+1}$, use a l_3^*-argument: $z(l_3^*) = 2(n - m)$, $z(l_3^*, P_0) \leqslant n + m + 3 - (3m + 4) = n - 2m - 1$. For $i \in \{2, ..., m + 2\}$ we have $|\mathscr{L}(l_3^*, P_i)| = 2$, thus $z(l_3^*, P_i) \leqslant 4$. It follows $2(n - m) \leqslant n - 2m - 1 + 4(m + 1)$,

$$(*) \qquad n \leqslant 4m + 3 .$$

Thus $m^2 + 1 \leqslant 4m + 3$, $m(m - 4) \leqslant 2$ and $m \leqslant 4$.

Let first $n = 8$. It is then immediate, that we are in case (ii) of the theorem.

Let $m = 3$, set $l_3 \cap \mathfrak{S} = \{P_1, Q_1, R_1, R_2\}$. Assume first $a_4(Q_1) \geqslant 3$, so that $n \geqslant 11$. Set $l_4 = Q_1P_2 \in \mathscr{L}_4$, $l_5 = Q_1P_3 \in \mathscr{L}_4$, $l_4 \cap \mathfrak{S} = \{Q_1, P_2, S_1, S_2\}$, $l_5 \cap \mathfrak{S} = \{Q_1, P_3, T_1, T_2\}$. For $X, Y \in \mathfrak{S} - (l_1 \cup l_2 \cup l_3)$, $X \neq Y$, have $XY \cap \{P_0, P_1, Q_1\} \neq \emptyset$ because of lemma 3 and $\mathscr{L}_5 = \mathscr{L}_5(P_0)$.

So we can choose $\{P_0, R_1, S_1, T_1\} \subseteq g_1$, $\{P_0, R_2, S_2, T_2\} \subseteq g_2$.

Then $z(l_3^*, P_0) \leqslant 11$ and instead of $(*)$ we get $n \leqslant 11$, thus $n = 11$.

We have $P_1 \in S_1T_2 = l \in \mathscr{L}_4$, $z(l^*) = 16$, $z(l^*, P_0) = 0$, $z(l^*, P_i) \leqslant 3$ for $i = 4, 5$, $z(l^*, P_j) \leqslant 4$ for $j = 2, 3$. The contradiction $16 = z(l^*) \leqslant 2 \cdot 3 + 2 \cdot 4 = 14$ follows.

Assume next $a_4(Q_1) = 2$. Define l_4 like above. As $|\mathfrak{S}| \geqslant 16$, there is $T \in \mathfrak{S} - (l_1 \cup l_2 \cup l_3 \cup l_4)$. Like above we can choose l such that $l \supseteq \{P_0, T, S_1, R_1\}$. As $S_2X \in P_1$ for every $X \in (l \cap \mathfrak{S}) - \{P_0, S_1, R_1\}$ we get $l \in \mathscr{L}_4$, $S_2T \ni P_1$.

Further $S_2R_2 \ni P_0$ as otherwise $S_2R_2 \in \mathscr{L}_4$, but $S_2R_2 \cap l \notin \mathfrak{S}$.

Clearly now $|\mathfrak{S}| = 16$, $n = 10$. We have $z(l^*) = 14$, $z(l^*, P_0) = z(l^*, P_1) = z(l^*, P_2) = 0$. But $z(l^*, P_i) \leqslant 4$ for $i \in \{3, 4, 5\}$ and so we have the contradiction $14 = z(l^*) \leqslant 3 \cdot 4 = 12$.

Hence $a_4(Q_1) = a_4(P_1) = 1$, $n \in \{10, 11\}$. Let first $n = 11$, so that $a_2(Q_1) = 0$. Like above we see, that all the points in $\mathfrak{S} - (l_1 \cup l_2 \cup l_3)$ are collinear with P_0 and thus $a_6 = 3$, $a_2 = 1$.

We can choose $P_0R_2 \in \mathcal{L}_2$. Further $a_5 = 0$, $a_4 = 5$, $a_3 = 20$ (see fig. 4).

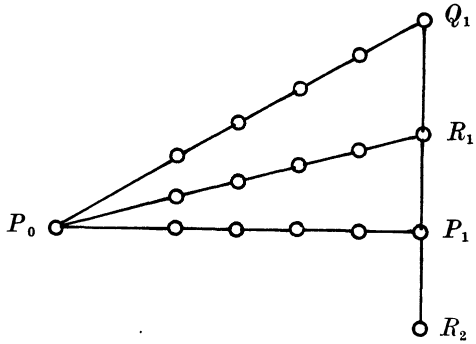

Figure 4

Let $l \in \mathcal{L}_3$. Then $|l^*| = 9$. There are at least four points $X \in l^*$ with $a_4(X) = 0$. But then a l^*-argument shows $a_2(X) \neq 0$ and so $a_2 \geqslant 4$, a contradiction.

Let $n = 10$. Like above we see $a_5(P_0) = 1$, $a_2(P_0) = 1$, $P_0R_2 \in \mathcal{L}_2$.

Let $P_0 \in l \in \mathcal{L}_5$. Then $|l^*| = |\mathcal{L}_2(l^*)| = 6$. On the other hand $|\mathcal{L}(l^*, P_i)| = 1$ for $1 \leqslant i \leqslant 5$ and thus $|\mathcal{L}(l^*)| = 5$, a contradiction.

Let now $m = 4$. By $(*)$ $17 \leqslant n \leqslant 19$. Set $\mathcal{M} = \mathfrak{S} - (l_1 \cup l_2 \cup l_3)$, let $\mathcal{N}$ consist of the pairs of different elements of $\mathcal{M}$ and for $X \in \mathcal{P}$ set $n_x = \left|\{\{A, B\} \in \mathcal{N} | AB \ni X\}\right|$. Clearly $8 \leqslant |\mathcal{M}| \leqslant 10$.

Let $f = |\mathcal{N}| - (n_{P_0} + n_{P_1} + n_{Q_1})$. Then $f = \left|\{l | l \in \mathcal{L}_5, l \cap \{P_0, P_1, Q_1\} = \emptyset\}\right|$ and we have $n_X \leqslant 3$ for $X \in \{P_2, P_3, \ldots, P_6\}$.

Assume $P_0 \in l$, $l \notin \{l_1, l_2\}$, st $(l) \geqslant 5$. Then $l \cap l_3 \in \mathfrak{S}$ by lemma 3 and instead of $(*)$ we get $n \leqslant 16$, which is impossible. If st $(l) = 4$, again $1 \cap l_3 \in \mathfrak{S}$. So we have $n_{P_0} \leqslant \frac{1}{2}|\mathcal{M}|$.

Let $n = 19$. We have equality in $(*)$, $|\mathcal{M}| = 10$, $|\mathcal{N}| = \binom{10}{2} = 45$, $n_{P_0} \leqslant 5$, $n_{P_1} \leqslant 9$, $n_{P_2} \leqslant 9$, $f \geqslant 45 - 23 = 22$. So there is $i \in \{2, 3, \ldots, 6\}$ with $n_{P_i} \geqslant 5$, a contradiction.

Let $n = 18$, so $|\mathcal{M}| = 9$, $|\mathcal{N}| = 36$. We have $n_{P_0} \leqslant 4$.

Assume $a_5(Q_1) = 4$. Then clearly $a_5(P_1) > 1$. Let $P_1 \in k \in \mathcal{L}_5$, $k \neq l_3$. As $a_2(Q_1) = 2$, a k^*-argument analagous to the l_3^*-argument at the beginning of the paragraph leads to a contradiction.

Thus $a_5(Q_1) \leqslant 3$, $a_5(P_1) \leqslant 3$, $n_{P_1} \leqslant 7$, $n_{Q_1} \leqslant 7$ and $f \geqslant 36 - (4 + 7 + 7) = 18$. There exists then a $X \in \{P_2, \ldots, P_6\}$ with $n_X \geqslant 4$, which is impossible.

Finally $n = 17$. Assume first $a_5(Q_1) \leqslant 2 \geqslant a_5(P_1)$. Then $n_{P_1} \leqslant 5 \geqslant n_{Q_1}$ and $f \geqslant 28 - (4 + 5 + 5) = 14$. So there are at least four points $P \in \{P_2, \ldots, P_6\}$ with $n_P = 3$. For these points $a_2(P) \neq 0$. As $a_2(Q_1) \leqslant 1$ we get $\sum_{i=2}^{6} |\mathscr{L}_2(l_3^*, P_i)| \geqslant 3$ and the l_3^*-argument leads to a contradiction. So we can choose $a_5(Q_1) = 3$. Set $\mathscr{L}_5(Q_1) = \{l_3, l_4, l_5\}$, $l_i \cap l_1 = P_{i-2}$, $i = 3, 4, 5$. Set $\{X, Y\} = \mathfrak{S} - (l_1 \cup \ldots \cup l_5)$.

Choose $M \in l_5 \cap \mathscr{M}$ such that $XM \cap \{P_0, P_2\} = \emptyset$. Then $XM \in \mathscr{L}_5$ and thus $P_1 \in XM$. It follows $\{P_1X, P_1Y\} \subseteq \mathscr{L}_5$ and so $a_5(P_1) = 3$.

We have shown $a_5(X) \in \{0, 3\}$ for every $X \in \{P_1, \ldots, P_6, Q_1, \ldots, Q_6\}$.

Let $P \in \{P_4, P_5, P_6\}$. A (P, l_2)-argument and lemma 3 show, that $a_5(P) \neq 0$. For every $i \in \{1, 2, \ldots, 6\}$ we have $a_5(P_i) = 3$ and thus $a_2(P_i) \neq 0$. As $a_2(Q_1) \leqslant 2$, we get $\sum_{i=2}^{6} |\mathscr{L}_2(l_3^*, P_i)| \geqslant 3$ again and the l_3^*-argument provides us with a contradiction.

From now on we can assume, that Hyp. (2.2) doesn't hold, so that $\mathscr{L}_{m+2} = \mathscr{L}_{m+2}(P_0)$, $\mathscr{L}_{m+1} = \mathscr{L}_{m+1}(P_0)$. Clearly $m \geqslant 3$.

Assume $\mathscr{L}_2 \neq \mathscr{L}_2(P_0)$, choose $P \in l_1 \cap \mathfrak{S}$, $P \neq P_0$, $a_2(P) \neq 0$.

A (P, l_2)-argument yields the contradiction

$$m^2 + m + 4 \leqslant |\mathfrak{S}| \leqslant 2m + 5 + (m + 1)(m - 2) = m^2 + m + 3 .$$

Thus $\mathscr{L}_2 = \mathscr{L}_2(P_0)$.

Assume $\mathscr{L}_3 \neq \mathscr{L}_3(P_0)$, let $P_0 \notin l_3 \in \mathscr{L}_3$, $\{P_1, Q_1\} \subseteq l_3$. A (Q_1, l_1)-argument like above leads to equality and we get one of the following:

(i) $m = 3$, $a_3(Q_1) = m + 2$. (ii) $m \geqslant 4$, $a_3(Q_1) = 1$, $a_m(Q_1) = m + 1$.

Let first $m = 3$, $n = 10$. As $\mathscr{L}_5 = \mathscr{L}_5(P_0)$, $\mathscr{L}_4 = \mathscr{L}_4(P_0)$, we get $a_6 = 3$, $a_2 = a_4 = 0$. Let $l \in \mathscr{L}_3$. A l^*-argument leads to a contradiction. So $m \geqslant 4$. Clearly $a_m(P_1) = m + 1$. Assume $P_1 \in l \neq l_3$, st $(l) \geqslant 5$. Because of (ii) and lemma 3 then $l \in \mathscr{L}_{m+3}$, which is impossible.

Thus $m = 4$, $n = 17$. Let $Q_1 \in l \in \mathscr{L}_4$, $l_3 \cap \mathfrak{S} = \{P_1, Q_1, S\}$. Then there is $M \in l \cap \mathfrak{S}$, $M \notin \{Q_1, l \cap l_1\}$ such that $SM \not\ni P_0$. It follows $SM \in \mathscr{L}_4$. In this way we get the five lines of $\mathfrak{S}$ through S, which don't contain Q_1 or P_0. Necessarily then $l_0 = P_0S \in \mathscr{L}_7$. As $\mathscr{L}_5 = \mathscr{L}_5(P_0)$, the points in $\mathfrak{S} - (l_0 \cup l_1 \cup l_2)$ are collinear and so $P_0 \in k \in \mathscr{L}_6$. A k^*-argument yields the contradiction $\mathscr{L}_2 \neq \mathscr{L}_2(P_0)$.

Hence we can assume $\mathscr{L}_3 = \mathscr{L}_3(P_0)$. Assume $\mathscr{L}_m \neq \mathscr{L}_m(P_0)$, choose $P_0 \notin l_3 \in \mathscr{L}_m$, $\{P_1, Q_1\} \subseteq l_3$. As $\mathscr{L}_3 = \mathscr{L}_3(P_0)$, we have $m \geqslant 4$. Use a l_3^*-argument. We get

$$3(n + 1 - m) \leqslant n + m + 3 - (3m + 3) + 9(m + 1) ,$$

$$2n \leqslant 10m + 6 , \quad n \leqslant 5m + 3 .$$

Thus $m^2 + 1 \leqslant 5m + 3$, $m(m - 5) \leqslant 2$ and $m \in \{4, 5\}$. We have $\mathscr{L}(l_3^*, P_i) = \mathscr{L}_4(l_3^*, P_i)$, $i = 2, 3, \dots, m + 2$ because $\mathscr{L}_2 = \mathscr{L}_2(P_0)$, $\mathscr{L}_3 = \mathscr{L}_3(P_0)$. It follows $z(l_3^*) = 3(n + 1 - m) \geqslant 9(m + 1)$ and

$$(*) \qquad n \geqslant 4m + 2 .$$

First let $m = 5$, $i \in \{2, 3, \dots, 7\}$. Then a l_3^*-argument together with lemma 3 shows $a_4(P_i) \geqslant 3$. A (P_i, l_2)-argument shows $|\mathfrak{S}| \leqslant 15 + 3 \cdot 2 + 4 \cdot 3 = 33$, $n \leqslant 25$, which is impossible.

Finally let $m = 4$, so that $n \geqslant 18$ by $(*)$. A (Q_1, l_1)-argument shows $n \leqslant 18$, so that $n = 18$, $|\mathfrak{S}| = 25$. Let $X \in \mathfrak{S} - (l_1 \cup l_2)$. Then $XP_i \in \mathscr{L}_4$ for $i = 1, 2, \dots, 6$. Thus $P_0X \in \mathscr{L}_7$. It follows $a_7 = 4$. Set $\mathscr{L}_7 = \{l_1, l_2, l_R, l_S\}$, $l_R \cap \mathfrak{S} = \{P_0\} \cup \{R_i | i = 1, 2, \dots, 6\}$, $l_S \cap \mathfrak{S} = \{P_0\} \cup \{S_i | i = 1, 2, \dots, 6\}$ (see fig. 5).

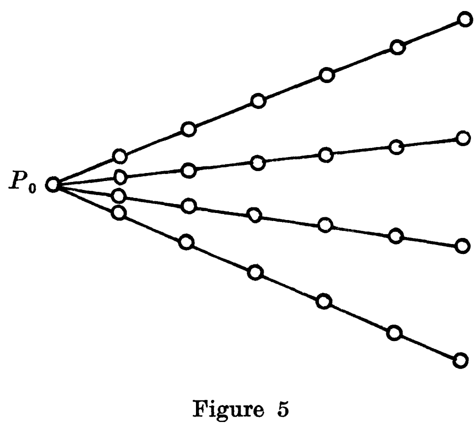

Figure 5

Consider the 6×6-matrix with entry (R_k, S_l) in the i-th row and j-th column whenever P_i, Q_j, R_k, S_l are collinear. It is immediate, that this has to be a pair of orthogonal 6×6-latin squares. However, such a pair doesn't exist (see [1], Chap. 5). Thus the proof of our theorem is complete.

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Manoscritto pervenuto in redazione il 12 agosto 1980.