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Finite groups with a standard component of type $L_3(4)$, - I

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1. Introduction.

Following Aschbacher, a quasi-simple subgroup A of a group G is called a standard subgroup of G, if $K = C_G(A)$ is tightly-embedded in G, $N_G(A) = N_G(K)$, and $[A, A^g] \neq 1$ for every $g \in G$. Here, K is tightly-embedded in G, if $|K| \equiv 0 \mod 2$ and $|K \cap K^g| \equiv 1 \mod 2$ hold for all $g \in G \setminus N_G(K)$. Assume that K is tightly-embedded in G and let x be an involution in K. If $y \in C_G(x)$, then $x = x^y \in K \cap K^y$, and so, $y \in N(K)$; it follows $C_G(x) \subseteq N_G(K)$ for every involution $x \in K$. If y is another involution of K and $x^g = y$ for some $g \in G$, then $x^g = y \in K^g \cap K$, and so, $g \in N(K)$, since K is tightly-embedded in G. This implies that the fusion of the 2-elements of K takes place only in $N(K)$.

The objective of this series of papers is to prove the following result:

THEOREM. Let G be a nonabelian, finite, simple group which possesses a standard subgroup A such that $A/Z(A) \simeq L_3(4)$. Then the following two assertions hold:

(1) If $|Z(A)|_2 = 1$, then $2^{11} \mid |G|$;

(2) If $|Z(A)|_2 > 1$, then G is isomorphic to He or $O'N$.

REMARK. From the work done by Cheng, Held, and Reifart it seems very likely that in case (1) we have $G \simeq Sz$.

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In this first paper we prove the following

THEOREM 1. Let G be a nonabelian, finite, simple group which possesses a standard subgroup A such that $A/\boldsymbol{Z}(A) \simeq L_3(4)$ and that the 2-rank of $\boldsymbol{Z}(A)$ is greater than 1. Then G is isomorphic to the sporadic simple group He.

2. Some facts about $L_3(4)$.

We shall state here some facts about $L_3(4)$ which are required in later sections. Throughout this section we set $L = L_3(4)$.

(2.1) Denote by « $\Rightarrow$ » the canonical homomorphism from $SL_3(4)$ onto L. Let $\alpha \neq 1$ be an element of $GF(4)^*$. Following the notation introduced by A. Reifart [10], we define the elements $\pi, \tau, \mu, \lambda, \zeta, \xi$ of L as follows:

$$\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \Rightarrow \pi, \qquad \begin{bmatrix}1&0&0\\0&\alpha&\alpha^2\\0&\alpha^2&\alpha\end{bmatrix} \Rightarrow \tau, \qquad \begin{bmatrix}1&0&0\\\alpha&1&0\\\alpha&0&1\end{bmatrix} \Rightarrow \mu,$$

$$\begin{bmatrix}1&0&0\\\alpha^2&1&0\\\alpha^2&0&1\end{bmatrix} \Rightarrow \lambda, \qquad \begin{bmatrix}1&\alpha^2&\alpha^2\\0&1&0\\0&0&1\end{bmatrix} \Rightarrow \zeta, \qquad \begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix} \Rightarrow \xi .$$

Then

$$\pi^2 = \tau^2 = \mu^2 = \lambda^2 = \zeta^2 = \xi^2 = 1 ,$$

$$[\pi, \tau] = [\mu, \lambda] = [\zeta, \xi] = [\mu, \pi] = [\mu, \tau] = [\lambda, \pi] = \\ = [\lambda, \tau] = [\zeta, \pi] = [\zeta, \tau] = [\xi, \pi] = [\xi, \tau] = 1 ,$$

$$[\mu, \zeta] = \pi , \qquad [\mu, \xi] = \pi\tau , \qquad [\lambda, \zeta] = \pi\tau , \qquad [\lambda, \xi] = \tau .$$

Set $P = \langle \pi, \tau, \mu, \lambda, \zeta, \xi \rangle$, $E_1 = \langle \pi, \tau, \mu, \lambda \rangle$, and $E_2 = \langle \pi, \tau, \zeta, \xi \rangle$. Then, $P \in \mathrm{Syl}_2(L)$, and E_1 and E_2 are the only elementary abelian subgroups of order 16 of P. We have $P = E_1E_2$ and $\boldsymbol{Z}(P) = P' = \boldsymbol{D}(P) = \langle \pi, \tau \rangle$. Every involution of P is contained in E_1 or E_2. The subgroup P possesses precisely three subgroups of type (4, 4), namely, $\langle \mu\lambda\xi, \lambda\zeta \rangle$, $\langle \mu\zeta, \lambda\xi \rangle$, and $\langle \mu\xi, \lambda\zeta\xi \rangle$; these subgroups are self-centralizing in P. There are precisely 27 involutions in P. Thus, every element of order 4 lies in one of the subgroups of type (4, 4) of P.

The element g of L with

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 1 & \alpha^2 \end{bmatrix} \Rightarrow g$$

has order 3 and operates on P in the following way:

$$g\colon \ \pi \to \pi\tau \to \tau\,, \ \mu \to \mu\lambda \to \lambda\,, \ \zeta \to \zeta\xi \to \xi\,.$$

One has $N_L(P) = P\langle g\rangle$. The subgroups $E_i \ldots,\ i \in \{1, 2\}, \ldots$ are self-centralizing in L and $N_L(E_i)$ is a transitive splitting extension of E_i by a subgroup isomorphic to A_5. The group L possesses precisely one class of involutions and we have $C_L(\pi) = P$.

(2.2) It is well known that

$$\textbf{\textit{Aut}}\,(L) = \textbf{\textit{Inn}}\,(L)\cdot\Sigma \quad \text{with} \quad \Sigma \cap \textbf{\textit{Inn}}\,(L) = 1 \quad \text{and} \quad \Sigma \cong \Sigma_3 \times Z_2\,.$$

As a complement Σ of L in $\textbf{\textit{Aut}}\,(L)$ we may choose $\langle \varphi, \varkappa, r\rangle$, where φ is induced by the field automorphism of $GL(3, 4)$, $\varkappa$ is induced by the transpose-inverse automorphism of $GL(3, 4)$, and r is induced by the element

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of $GL(3, 4)$.

Then, $\varphi^2 = \varkappa^2 = r^3 = 1$, $[\varphi, \varkappa] = 1$, $\langle r, \varphi\rangle \cong \Sigma_3$, and $[r, \varphi\varkappa] = 1$. Thus, $\Sigma = \langle r, \varphi\rangle \times \langle \varphi\varkappa\rangle \cong \Sigma_3 \times Z_2$.

The subgroup P of L as described in (2.1) is Σ-invariant. The operations of φ, $\varkappa$, and r on P are as follows:

$$\varphi\colon \ \pi \to \pi\,, \ \tau \to \pi\tau\,, \ \mu \to \lambda\,, \ \lambda \to \mu\,, \ \zeta \to \zeta\xi\,, \ \xi \to \xi;$$

$$\varkappa\colon \ \pi \to \pi\,, \ \tau \to \tau\,, \ \mu \to \zeta\xi\,, \ \lambda \to \zeta\,, \ \zeta \to \lambda\,, \ \xi \to \mu\lambda;$$

$$r\colon \ \pi \to \pi\,, \ \tau \to \tau\,, \ \mu \to \lambda \to \mu\lambda\,, \ \zeta \to \zeta\xi \to \xi\,.$$

The subgroups E_1 and E_2 are normalized by $\langle r, \varphi\rangle$ and permuted by the action of $\varkappa$.

(2.3) Let U be a subgroup of $\boldsymbol{Aut}(L)$ with $U \supseteq \boldsymbol{Inn}(L)$. Then, obviously, a S_2-subgroup of U is isomorphic to P, $P\langle\varphi\rangle$, $P\langle\varkappa\rangle$, $P\langle\varphi\varkappa\rangle$ or $P\langle\varphi, \varkappa\rangle$.

Set $U = \boldsymbol{Inn}(L)\cdot\Sigma_0$ with $\Sigma_0 = \Sigma \cap U$. We identify L with $\boldsymbol{Inn}(L)$. We have $P = E_1E_2$, where $E_1 \cong E_{2^4} \cong E_2$, and $\boldsymbol{C}_U(E_i) = E_i$, for $i \in \{1, 2\}$. Let $N = \boldsymbol{N}_U(E_i)/E_i$. Then, depending on Σ_0, one of the following cases arises:

(1) $N \cong A_5$, if $\Sigma_0 \cap \langle\varphi, r\rangle = 1$;

(2) $N \cong \Sigma_5$, if $r \notin \Sigma_0$ and φ, φr, or $\varphi r^2 \in \Sigma_0$;

(3) $N \cong A_5 \times Z_3$ if $r \in \Sigma_0$ and $\varphi \notin \Sigma_0$;

(4) $N \cong (A_5 \times Z_3)Z_2$ with $A_5Z_2 \cong \Sigma_5$ and $Z_3Z_2 \cong \Sigma_3$ if $\langle r, \varphi\rangle \subseteq \Sigma_0$.

(2.4) By direct computation one obtains

$$\begin{aligned}
\boldsymbol{C}_P(\varphi) &= \langle\pi, \mu\lambda, \xi\rangle \cong D_8\,,\\
\boldsymbol{C}_P(\varkappa) &= \langle\pi, \tau\rangle \cong E_{2^3}\,,\\
\boldsymbol{C}_P(\varphi\varkappa) &= \langle\pi, \mu\lambda\xi\tau, \mu\zeta\tau\rangle \cong Q_8\,.
\end{aligned}$$

Furthermore, for each $y \in \{\varphi, \varkappa, \varphi\varkappa\}$, all the involutions from the coset Py are conjugate to y under the action of P.

(2.5) The rank of $P\langle\varphi, \varkappa\rangle$ is equal to 4. Further, E_1 and E_2 are the only elementary abelian subgroups of order 16 of $P\langle\varphi, \varkappa\rangle$.

3. Some notations.

Throughout this paper let G denote a fixed group which satisfies the assumptions of Theorem 1. Let $a, b \in G$. Then $a\colon b \to c$ means $a^{-1}ba = c$. We shall write $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A)$ for $\boldsymbol{N}_G(A)$ and $\boldsymbol{C}_G(A)$, respectively.

Let A denote a fixed standard subgroup of G where $A/\boldsymbol{Z}(A) \cong$ $\cong L_3(4)$. Then, A is isomorphic to a homomorphic image of the representation group of $L_3(4)$. Note that the representation group of a simple group is uniquely determined up to isomorphism.

Set $K = \boldsymbol{C}(A)$. Then $AK/K \simeq L_3(4) \simeq A/\boldsymbol{Z}(A)$. Note that $[x, A] \subseteq \subseteq \boldsymbol{Z}(A)$ for some $x \in G$ implies $x \in K$. Let $Q \in \mathrm{Syl}_2(K)$. Since $\boldsymbol{Aut}(L)$ splits over L, and since $\boldsymbol{N}(A)/K$ is isomorphic to a subgroup of $\boldsymbol{Aut}(L)$, we obtain the existence of a subgroup C of $\boldsymbol{N}(A)$ such that $\boldsymbol{N}(A)/K = = C/K \cdot AK/K$ with $C \cap AK = K$. By Frattini's argument, we get $\boldsymbol{N}_C(Q)K = C$. Clearly, $\boldsymbol{N}_C(Q)/\boldsymbol{N}_K(Q) \simeq C/K$. We may choose a S_2-subgroup S of A such that $\boldsymbol{N}_C(Q)/\boldsymbol{N}_K(Q)$ operates on $S/\boldsymbol{Z}(A)_2$ in the same way as Σ_0 on P (in the former notation). Using the isomorphism $P \simeq S/\boldsymbol{Z}(A)_2$, we may put $S = \boldsymbol{Z}(A)_2\langle \pi, \tau, \mu, \lambda, \zeta, \xi \rangle$, where the generators are all 2-elements satisfying the relations modulo $\boldsymbol{Z}(A)_2$ which we had derived for P above. We may «identify» $\boldsymbol{N}_C(Q)/\boldsymbol{N}_K(Q)$ with $\Sigma_0 \subseteq \Sigma = \langle \varphi, r, \varkappa \rangle$, thus a S_2-subgroup of $\boldsymbol{N}_C(Q)$ could be Q, $Q\langle\varphi\rangle$, $Q\langle\varkappa\rangle$, $Q\langle\varphi\varkappa\rangle$, or $Q\langle\varphi, \varkappa\rangle$, and if r is present, we have $r^3 \in N_K(Q)$. We get that a S_2-subgroup of $\boldsymbol{N}(A)$ is one of the following types: QS, $QS\langle\varphi\rangle$, $QS\langle\varkappa\rangle$, $QS\langle\varphi\varkappa\rangle$, $QS\langle\varphi, \varkappa\rangle$.

In what follows, we denote by X a fixed S_2-subgroup of $\boldsymbol{N}(A)$. Furthermore, X is of type QS, $QS\langle\varphi\rangle$, $QS\langle\varkappa\rangle$, $QS\langle\varphi\varkappa\rangle$, $QS\langle\varphi, \varkappa\rangle$. Hence, $X \cap A = S$ and $X \cap K = Q$. One notes that X/Q is isomorphic to P, $P\langle\varphi\rangle$, $P\langle\varkappa\rangle$, $P\langle\varphi\varkappa\rangle$, or $P\langle\varphi, \varkappa\rangle$, where $P \in \mathrm{Syl}_2(L_3(4))$ as given in (2.1).

(3.1) Lemma. Let $y \in X \setminus QS$ with $y \in \{\varphi, \varkappa, \varphi\varkappa\}$. Let $z = uvy$ be an involution with $u \in Q$ and $v \in S$. Then $\boldsymbol{C}_S(z) \underset{S}{\sim} \boldsymbol{C}_S(y)$.

Proof. We put $z = uvy$, $u \in Q$, $v \in S$. Note that $[Q, S] = 1$. We get

$$1 = z^2 = uvyuvy = uvyvyy^{-1}uy = u(vy)^2 u^y$$

and

$$u^{-1}u^{-y} = (vy)^2 = vy^2 v^y = y^2(vv^y)\,,$$

since $y^2 \in Q$. It follows

$$v(v^y) = y^{-2}u^{-1}u^{-y} \in Q \cap S\,.$$

We look now at $S\langle z\rangle/Q \cap S = S\langle uvy\rangle/Q \cap S$. Since $[u, S] = 1$, the element vy induces the same automorphism of S as z. Thus,

$$S\langle z\rangle/Q \cap S \simeq S\langle vy\rangle/\langle Q \cap S, (vy)^2\rangle = S\langle y\rangle/\langle Q \cap S, (vy)^2\rangle\,,$$

where $\langle (vy)^2, y^2\rangle \subseteq \langle Q\cap S, (vy)^2\rangle \subseteq Q$; note that we have $y^2(vv^v) = u^{-1}u^{-v} = (vy)^2$ with $vv^v \in Q\cap S$. We see that $S\langle z\rangle/Q\cap S$ is isomorphic to $P\langle\varphi\rangle$, $P\langle\varkappa\rangle$, or $P\langle\varphi\varkappa\rangle$. From (2.4) we get that $\langle Q\cap S, (vy)^2\rangle y$ is conjugate to $\langle Q\cap S, (vy)^2\rangle vy$ under the action of $S\langle (vy)^2\rangle/ /\langle S\cap Q, (vy)^2\rangle$; this means that there is $s\in S$ such that $(vy)^s = yx$, $x\in\langle Q\cap S, (vy)^2\rangle$. Hence, $\boldsymbol{C}_S((vy)^s) = \boldsymbol{C}_S(yx) = \boldsymbol{C}_S(y)$, since $x\in Q$, and so, $\boldsymbol{C}_S(z) = \boldsymbol{C}_S(uvy) = \boldsymbol{C}_S(vy) \underset{S}{\sim} \boldsymbol{C}_S(y)$.

4. The structure of $\boldsymbol{N}(A)$.

We follow the notation of the last section and look at $X\in \mathrm{Syl}_2(\boldsymbol{N}(A))$ with $X\cap K = Q\in\mathrm{Syl}_2(K)$ and $X\cap A = S\in \mathrm{Syl}_2(A)$.

(4.1) Lemma. Q is elementary abelian and $Q\cap S$ is a four-group.

Proof. By assumption we have $m(Q\cap S)\geqslant 2$. A result of Aschbacher [1] yields that Q is elementary abelian. This implies that $Q\cap S$ is a four-group, as the Schur-multiplier of $L_3(4)$ is isomorphic to $Z_4\times Z_4\times Z_3$.

Throughout this paper we set $|Q| = 2^n$ and $Q\cap S = \langle q_1, q_2\rangle$.

(4.2) The structure of S.

It is well known that the sporadic simple group He contains a standard subgroup B, where $\boldsymbol{C}(B) = \boldsymbol{Z}(B) \cong Z_2\times Z_2$, and $B/\boldsymbol{Z}(B) \cong L_3(4)$. Hence, the S_2-subgroup S of A in our case is isomorphic to a S_2-subgroup H of B. The subgroup H contains exactly two elementary abelian subgroups H_1 and H_2 of order 2^6; we have $H = H_1H_2$ and $H_1\cap H_2 = \boldsymbol{Z}(H) = H' \cong E_{2^4}$.

Set $S = R_1R_2$ with $R_1 \cong E_{2^6} \cong R_2$ and $Z(S) = R_1\cap R_2 \cong E_{2^4} \cong S'$. Let $\bar{R}_i$ for $i\in\{1, 2\}$ be the image of R_i under the homomorphism from S onto $S/\langle q_1, q_2\rangle = \bar{S}$. The group $\bar{S}$ is isomorphic to a S_2-subgroup of $L_3(4)$. It is clear that R_1 and R_2 both contain $\langle q_1, q_2\rangle = Q\cap S$. Thus, $\bar{R}_1$ and $\bar{R}_2$ are the only elementary abelian subgroups of order 2^4 of $\bar{S}$. We may put

$$R_1 = \langle q_1, q_2, \pi, \tau, \mu, \lambda\rangle \quad \text{and} \quad R_2 = \langle q_1, q_2, \pi, \tau, \zeta, \xi\rangle\ ,$$

where the nontrivial commutator relations mod $\langle q_1, q_2\rangle$ are given by $[\mu, \zeta] = \pi$, $[\mu, \xi] = \pi\tau$, $[\lambda, \zeta] = \pi\tau$, $[\lambda, \xi] = \tau$. Since A is the epimorphic image of the full covering group of $L_3(4)$ modulo a characteristic subgroup of the full covering group, we see that A possesses automorphisms which are « lifted-induced » by the field-, transpose-inverse, and field times transpose-inverse automorphism of $L_3(4)$; we use here the fact that every automorphism of a perfect group can be lifted to an automorphism of the full covering group (see Griess [7]).

We want to determine the multiplication table of $S = R_1R_2$. Since the cosets $\langle q_1, q_2\rangle\pi$ and $\langle q_1, q_2\rangle\tau$ consist of involutions only, we may put without loss of generality $[\mu, \xi] = \pi\tau$ and $[\lambda, \xi] = \tau$. Since $S' = \langle q_1, q_2, \pi, \tau\rangle$, we have $[\mu, \zeta] = q\pi$ and $[\lambda, \zeta] = p\pi\tau$, where $q, p \in \in \langle q_1, q_2\rangle$. Note that $[\mu\lambda, \xi] = [\mu\lambda, \xi]^\varphi = \pi^\varphi = \pi$ where φ comes from the field automorphism of $L_3(4)$. If $[q, \varphi] = 1$, then we get

$$q\pi = [\mu, \zeta] \xrightarrow{\varphi} [\lambda, \zeta\xi] = q\pi = \lambda\xi\zeta\lambda\zeta\xi = \lambda\xi\lambda(\lambda\zeta\lambda\zeta)\xi = \tau p\pi\tau = p\pi ;$$

thus $p = q$. Hence, in this case, $[\mu, \zeta] = q\pi$ and $[\lambda, \zeta] = q\pi\tau$ which implies $|S'| = 8$ against $|S'| = 16$. Hence, $[q, \varphi] \neq 1$. In particular, $q \neq 1$ and $\langle q, q^\varphi\rangle = \boldsymbol{Z}(A)_2$.

Computing $[\mu, \zeta] = q\pi \xrightarrow{\varphi} [\lambda, \zeta\xi] = q^\varphi\pi = \lambda\xi\lambda(\lambda\zeta\lambda\zeta)\xi = p\pi$, we get $p = q^\varphi$. Put $q_2 = q$ and $q_1q_2 = q^\varphi$. Then $[\mu, \zeta] = q_2\pi$ and $[\lambda, \zeta] = = q_1q_2\pi\tau$. In this way we have obtained the multiplication table of S:

$$R_1 = \langle q_1, q_2, \pi, \tau, \mu, \lambda\rangle \cong R_2 = \langle q_1, q_2, \pi, \tau, \zeta, \xi\rangle \cong E_{2^6} ;$$

$$R_1 \cap R_2 = \langle q_1, q_2, \pi, \tau\rangle = \boldsymbol{Z}(R_1R_2) = S' = \boldsymbol{D}(S) ,$$

$$[\mu, \xi] = \pi\tau , \quad [\lambda, \xi] = \tau , \quad [\mu, \zeta] = q_2\pi , \quad [\lambda, \zeta] = q_1q_2\pi .$$

We know already that $[q_1, \varphi] = 1$, $q_2^\varphi = q_1q_2$, and $[\pi, \varphi] = 1$. Since $[\mu, \xi] = \pi\tau$ is mapped under φ onto $[\lambda, \xi] = \tau$, we get $\tau^\varphi = \pi\tau$. Hence, the action of φ on $\langle q_1, q_2, \pi, \tau\rangle$ is known.

We have $\pi^{\varphi\varkappa} = [\mu\lambda, \xi]^{\varphi\varkappa} = [\xi, \mu\lambda] = \pi$; $\tau^{\varphi\varkappa} = [\lambda, \xi]^{\varphi\varkappa} = [\zeta\xi, \mu\lambda] = = q_1\pi\tau$; $[\mu\lambda, \zeta]^{\varphi\varkappa} = (q_1\tau)^{\varphi\varkappa} = [\xi, \mu] = \pi\tau$ and $q_1^{\varphi\varkappa}\tau^{\varphi\varkappa} = \pi\tau = q_1^{\varphi\varkappa}q_1\pi\tau$ which implies $q_1^{\varphi\varkappa} = q_1$. Further, $[\mu, \zeta]^{\varphi\varkappa} = (q_2\pi)^{\varphi\varkappa} = [\zeta, \mu] = q_2\pi$ which implies $q_2^{\varphi\varkappa} = q_2$. Hence, the action of $\varphi\varkappa$ on $\boldsymbol{Z}(R_1R_2)$ is known.

Acting with $\varkappa$ on appropriate commutators we get $\pi^\varkappa = \pi$, $q_1^\varkappa = q_1$, $\tau^\varkappa = q_1\tau$, $q_2^\varkappa = q_1q_2$. Hence, the action of $\varkappa$ on $\boldsymbol{Z}(R_1R_2)$ is known.

We get $[\mu\lambda\xi, \lambda\zeta] = q_1 \neq 1$, $[\mu\zeta, \lambda\xi] = q_1q_2 \neq 1$ and $[\mu\xi, \lambda\zeta\xi] = = q_2 \neq 1$. In particular, we note that S does not contain any subgroup of type (4, 4). Clearly

$$\langle q_1, q_2\rangle \cap \boldsymbol{Z}(X) \supseteq \langle q_1\rangle .$$

(4.3) LEMMA. We have—depending on X—

$$\begin{array}{lllll} \varphi: & q_1 \to q_1, & q_2 \to q_1 q_2, & \pi \to \pi, & \tau \to \pi\tau; \\ \varkappa: & q_1 \to q_1, & q_2 \to q_1 q_2, & \pi \to \pi, & \tau \to q_1\tau; \\ \varphi\varkappa: & q_1 \to q_1, & q_2 \to q_2, & \pi \to \pi, & \tau \to q_1\pi\tau. \end{array}$$

If $3 \,\big|\, |N(A)/AK|$, then there is an element $r \in N(A) \setminus AK$ which induces an outer automorphism of order 3 of $A/Z(A) \cong L_3(4)$ and of A. The element r operates fixed-point-free on $\langle q_1, q_2\rangle$ and $r: \pi \to \to q_2\pi \to q_1 q_2 \pi$.

PROOF. This follows immediately from the operations of φ, $\varkappa$, $\varphi\varkappa$, and r on $S/Q \cap S \cong P$.

(4.4) LEMMA. Set $S_i = QR_i$ for $i \in \{1, 2\}$. Then, S_1 and S_2 are the only elementary abelian subgroups of order 2^{n+4} of X.

PROOF. This follows from (4.1) and (4.2).

(4.5) LEMMA. The involutions in $\langle q_1, q_2\rangle$ have no roots in S. If i is an involution of QS, then i lies in S_1 or S_2. Further, i is conjugate to an involution in $Q\langle\pi\rangle$ under A.

PROOF. Let $s \in S^\#$ with $s^2 \in \langle q_1, q_2\rangle$. Then Qs is an involution of QS/Q. Since QS/Q is of type $L_3(4)$, the involution Qs lies in QR_1/Q or QR_2/Q. Thus, s lies in QR_1 or QR_2. But QR_i is elementary abelian. Hence $s^2 = 1$.

Let $i = qs$ with $q \in Q$ and $s \in S$ be an involution of QS. Then, $1 = i^2 = q^2 s^2$. Since Q is elementary abelian, we get $q^2 = s^2 = 1$. It follows that $s \in R_1$ or $s \in R_2$, so that $i = qs$ lies in QR_1 or QR_2. Assume that i does not lie in Q. Then Qi is an involution of AQ/Q, and so, Qi is conjugate to $Q\pi$ under AQ/Q. The lemma is proved.

(4.6) LEMMA. Let $y \in X \setminus QS$ with $y \in \{\varphi, \varkappa, \varphi\varkappa\}$. We have $C_S(\varphi) \cong \cong Z_2 \times D_8$ with $\mho^1(C_S(\varphi)) = \langle\pi\rangle$ and $C_S(\varphi) \subseteq \langle q_1, q_2, \pi, \mu\lambda, \xi\rangle$; $C_S(\varkappa) = = C_{Z(S)}(\varkappa) = \langle q_1, \pi, q_2\tau\rangle \cong E_{2^3}$; and $C_S(\varphi\varkappa) = C_{Z(S)}(\varphi\varkappa) = \langle q_1, q_2, \pi\rangle \cong \cong E_{2^3}$. Let z be an involution from QSy. Then $\mho^1(C_S(z)) = \langle\pi\rangle$, if $y = \varphi$; $C_S(z) = C_S(\varkappa)$, if $y = \varkappa$; and $C_S(z) = C_S(\varphi\varkappa)$, if $y = \varphi\varkappa$. For each $y \in \{\varphi, \varkappa, \varphi\varkappa\}$, all involutions from Sz are conjugate to z under the action of S.

PROOF. Since the field automorphism of $L_3(4)$ centralizes $L_3(2)$, we get $C_S(\varphi) \simeq Z_2 \times D_8$, and, obviously, we must have $C_S(\varphi) \subseteq \langle q_1, q_2, \pi, \mu\lambda, \xi\rangle$. The latter group has exponent 4 and $(\mu\lambda\xi)^2 = \pi$. The assertion about $C_S(\varkappa)$ is clear as the contragradient automorphism of $L_3(4)$ centralizes A_5. We have $C_S(\varphi\varkappa) \subseteq \langle q_1, q_2, \pi, \mu\lambda\xi\tau, \mu\zeta\tau\rangle$. Compute

$$(\mu\zeta\tau)^{\varphi\varkappa} = q\zeta q^{\varphi\varkappa}\mu q_1\pi\tau = qq^{\varphi\varkappa}\mu\zeta[\zeta, \mu]q_1\pi\tau = q_1 q_2\mu\zeta\tau ,$$

and also

$$(\mu\lambda\xi\tau)^{\varphi\varkappa} = \xi q\mu\lambda q^{\varphi\varkappa}q_1\pi\tau = qq^{\varphi\varkappa}q_1\mu\lambda\xi[\xi, \mu\lambda]\pi\tau = q_1\mu\lambda\xi\tau ,$$

where $q \in \langle q_1, q_2\rangle$; and similarly one sees that $\mu\lambda\xi\tau\mu\zeta\tau$ is not centralized by $\varphi\varkappa$. All other assertions follow from counting the involutions in Sz using our knowledge about the involutions of $P\varphi$, $P\varkappa$, $P\varphi\varkappa$ and the action of the outer automorphisms of A on $\langle q_1, q_2\rangle$. The lemma is proved.

(4.7) LEMMA. The subgroup $N_A(R_i)$ for each $i \in \{1, 2\}$ is a splitting extension of $C_A(R_i)$ by a subgroup U_i isomorphic to A_5. Furthermore, U_i operates transitively on $R_i/\langle q_1, q_2\rangle$.

PROOF. This is a direct consequence of the structure of $L_3(4)$ and A.

5. The case $|X| = |G|_2$.

In this section all results will be proved under the title assumption.

(5.1) LEMMA. The group $\langle q_1, q_2\rangle$ is strongly closed in QS with respect to G. If i is an involution of S and $i^x \in QS$, then $i^x \in S$ for any $x \in G$.

PROOF. We know that every involution of QS is conjugate to an involution of $Q\langle\pi\rangle$ under the action of $N(A)$. Note that $Q \times \langle\pi, \tau\rangle = Z(QS) = S_1 \cap S_2$, where S_1 and S_2 are the only elementary abelian subgroups of order 2^{n+4} of X; clearly $S_1 S_2 = QS$.

We prove first that $\langle q_1, q_2\rangle$ is strongly closed in QS with respect to G. Assume that this is not the case. Then, there is $u \in \langle q_1, q_2\rangle^\#$ and $g \in G$ such that $u^g \in QS$ but $u^g \notin \langle q_1, q_2\rangle$. Hence, $u^g = ws$, $w \in Q$,

$s \in S$, $w^2 = s^2 = 1$. If $s \in Q$, then $u^g = ws \in Q \cap Q^g \neq 1$ implying $g \in \boldsymbol{N}(A)$, against $\langle q_1, q_2 \rangle \lhd \boldsymbol{N}(A)$. Thus, we must have $s \notin Q$. Under the action of A, we have that Qs is conjugate to $Q\pi$. Thus, we may put $u^g = w\pi$, $w \in Q$. Obviously, $g \notin \boldsymbol{N}(A)$. We have $\langle u, u^g \rangle \subseteq \boldsymbol{Z}(QS)$ char QS char X, and $QS \subseteq \boldsymbol{C}(u) \cap \boldsymbol{C}(u^g)$; note that QS is generated by the maximal elementary abelian subgroups of X. Clearly, $(QS)^g \subseteq \subseteq \boldsymbol{C}(u^g)$. Hence, there is $x \in \boldsymbol{C}(u^g)$ such that $QS = (QS)^{gx}$ and, obviously, $u^g = u^{gx}$. Thus, we may assume $g \in \boldsymbol{N}(QS)$. But $g \notin \boldsymbol{N}(A) = = \boldsymbol{N}(K)$. Put $V = \boldsymbol{Z}(QS) = Q \times \langle \pi, \tau \rangle$. We have $V^g = V$ and $Q^g \cap \cap Q \subseteq K \cap K^g = \langle 1 \rangle$. Thus ,$2^{2n} = |Q \times Q^g| \leqslant |V| = 2^{n+2}$ implying $|Q| = 4$. Hence, $Q = \langle q_1, q_2 \rangle$, and so $S_i = R_i$ and $QS = R_1R_2 = S$.

We know that $V = \langle q_1, q_2, \pi, \tau \rangle$ has order 16 and that $u^g = w\pi$ for some $g \in \boldsymbol{N}(QS) \setminus \boldsymbol{N}(A)$. We also know that $\boldsymbol{N}(QS)$ controls fusion of the elements of V. Since $q_1 \in \boldsymbol{Z}(X)$ and $X \subseteq \boldsymbol{N}(QS)$, we see that q_1 has an odd number of conjugates under $\boldsymbol{N}(QS)$ and $\boldsymbol{N}(V)$, and since $\boldsymbol{C}(q_1) \subseteq \boldsymbol{N}(A)$, $g \notin \boldsymbol{N}(A)$, we see that this number is not 1. Clearly, $q_1^g \notin Q = \langle q_1, q_2 \rangle$ as otherwise $g \in \boldsymbol{N}(A)$ because of $q_1^g \in Q \cap Q^g \subseteq K \cap K^g$ and $|K \cap K^g| \equiv 1 \bmod 2$. In a similar way one sees that each element of $Q^\#$ is conjugate under $\boldsymbol{N}(QS)$ and $\boldsymbol{N}(V)$ to an element of $V \setminus Q$. Note that no element of $\langle q_1, q_2 \rangle^\#$ has a root in QS, however, π, $q_2\pi$, and $q_1q_2\pi$ have the roots $\mu\lambda\xi$, $\mu\zeta$, and $\lambda\zeta\xi$ respectively, in QS; note that $q_1\pi$ has no root in QS, because if $(\mu^m \lambda^l \zeta^z \xi^x)^2 = q_1\pi$, then we would get $l = z = 1 = m = x$, but $(\mu\lambda\zeta\xi)^2 = q_1\pi\tau$. Assume first that q_1 has no conjugate other than itself in Q. By the above remarks, we see that $q_1 \sim q_1\pi$ holds in $\boldsymbol{N}(QS)$ and that q_1 is not conjugate to another element of $Q\pi$; note that $Q\pi \sim Q\tau \sim Q\pi\tau$ holds in A/Q. Thus, q_1 would have precisely four G-conjugates in V which is not possible. Hence, there is a conjugate of q_1 in Q other than q_1. As the number of conjugates of q_1 under the action of $\boldsymbol{N}(A)$ is odd, we get that all three elements of $Q^\#$ are conjugate; note that fusion of the elements of Q takes place precisely in $\boldsymbol{N}(K) = \boldsymbol{N}(A)$. Thus, q_1 has precisely 6 G-conjugates in V as $q_1 \sim q_2 \sim q_1q_2 \sim q_1\pi$ holds and $Q\pi \sim Q\tau \sim Q\pi\tau$ under A/Q. Since $6 \not\equiv 1 \bmod 2$, we have obtained a contradiction which proves that $\langle q_1, q_2 \rangle$ is strongly closed in QS with respect to G.

Finally, let i be an involution of $S \setminus Q$ and assume that there is $x \in G$ such that $i^x \in QS$ but $i^x \notin S$. Conjugating with elements of A, we may and shall assume that i and i^x lie in $\boldsymbol{Z}(QS) = V$. Since $N(QS)$ controls fusion of the involutions of V, there is $y \in \boldsymbol{N}(V)$ such that $i = i^{xy}$. Clearly $y \in \boldsymbol{N}(V) \setminus \boldsymbol{N}(A)$. Thus, $Q \times Q^y \subseteq \boldsymbol{Z}(QS)$ and $Q \cap Q^y = 1$; and so, we get $|Q| = 4$. This means that $Q = \langle q_1, q_2 \rangle \subseteq S = QS$. Thus, $i^x \in S$ which is a contradiction to our assumption. The lemma is proved.

(5.2) LEMMA. The assumption of this section is not possible.

PROOF. Assume false. We have $|X| = |G|_2$, and $\langle q_1, q_2\rangle$ is strongly closed in QS with respect to G. Application of a result of Goldschmidt [5] yields that there is a conjugate of some element of $\langle q_1, q_2\rangle$ in $X\setminus QS$; in particular, we get $X \supset QS$. Glauberman's $\boldsymbol{Z}^*$-theorem [4] yields that q_1 has a conjugate in $X\setminus\langle q_1\rangle$. We want to show that q_1 is conjugate to an element of $X\setminus QS$. Assume that this is false. Then q_1 is conjugate to an element of $\langle q_1, q_2\rangle$ different from q_1, say q. The conjugation $q_1 \sim q$ is performed in $\boldsymbol{N}(A)$, and as q_1 is 2-central, we see that all involutions of $\langle q_1, q_2\rangle$ are conjugate. Thus, q_1 must be conjugate to an element of $X\setminus QS$.

Let $z \in QSy$, where $z \sim q_1$ and $y \in \{\varphi, \varkappa, \varphi\varkappa\}$. Let $\tilde{X} \in \mathrm{Syl}_2(\boldsymbol{C}(z))$ such that $\tilde{X} \supseteq \boldsymbol{C}_X(z)$. Denote by A_z the unique standard subgroup in $\boldsymbol{C}(z)$ isomorphic to A. Put $\tilde{Q} = \tilde{X} \cap \boldsymbol{C}(A_z)$ and $\tilde{S} = \tilde{X} \cap A_z$. Since $z \sim q_1$, we have $Q \sim \tilde{Q}$, $S \sim \tilde{S}$, and $z \in \tilde{Q} \cap \tilde{S}$. Clearly, $\boldsymbol{C}(z) \cap \boldsymbol{N}(A)$ does not possess a subgroup isomorphic to QS, since such a subgroup lies in KA, and so, z would centralize a S_2-subgroup of A which is not the case. If there were a $q \in Q^\#$ such that $q \in \tilde{Q}\tilde{S}$, then $q \in \tilde{Q}$ and $\boldsymbol{C}(q) \cap \boldsymbol{C}(z) \supseteq \tilde{Q}\tilde{S}$ which is against $\boldsymbol{C}(q) \subseteq \boldsymbol{N}(A)$, namely: If $q \in Q$ would be conjugate to an element of $Q\pi\setminus Q$, then the conjugation $q \sim w\pi$, $w \in Q$, would be performed in $\boldsymbol{N}(QS) \subseteq \boldsymbol{N}(V)$; obviously, $w\pi \notin K$, and so, $\boldsymbol{N}(QS) \not\subseteq \boldsymbol{N}(A)$; this implies $Q = \langle q_1, q_2\rangle$; but $\langle q_1, q_2\rangle$ is strongly closed in QS with respect to G; hence q is not conjugate to an element of $Q\pi\setminus Q$. We have shown that $q \notin \tilde{Q}\tilde{S}$ for each $q \in Q^\#$.

Let $z \in QS\varphi$. Then, we have $\mho^1(\boldsymbol{C}_S(z)) = \langle\pi\rangle$. Because of $\mho^1(\tilde{X}/\tilde{Q}\tilde{S}) = \langle 1\rangle$, we get $\pi \in \tilde{Q}\tilde{S}$, and this yields $\pi \in \tilde{S}$, since $\pi \in S$ and $S \sim \tilde{S}$. By an above remark, we have $q_1 \notin \tilde{Q}\tilde{S}$. Consider the coset $\tilde{Q}\tilde{S}q_1$. Obviously, $\tilde{Q}\tilde{S}q_1$ is conjugate in G to QSy for some $y \in \{\varphi, \varkappa, \varphi\varkappa\}$; note that $\tilde{X}$ contains $\tilde{Q}\tilde{S}\langle q_1\rangle$ as a subgroup of index 1 or 2. An earlier result yields that all involutions of $\tilde{S}q_1$ are conjugate to q_1 under the action of $\tilde{S}$. Thus, $q_1 \underset{\tilde{S}}{\sim} q_1\pi$. But $\langle q_1, q_2\rangle$ is strongly closed in QS with respect to G, and we have obtained a contradiction.

Let $z \in QS\varkappa$. We have $\boldsymbol{C}_S(z) = \langle q_1, \pi, q_2\tau\rangle$. Since $|\tilde{X}/\tilde{Q}\tilde{S}| \leqslant 4$, we get $\boldsymbol{C}_S(z) \cap \tilde{Q}\tilde{S} \neq \langle 1\rangle$. We know that $q_1 \notin \tilde{Q}\tilde{S}$. Hence, there exists s in $\langle q_1, \pi, q_2\tau\rangle\setminus\langle q_1\rangle$ such that $s \in \tilde{Q}\tilde{S}$. Because of $s \in S \sim \tilde{S}$, we get $s \in \tilde{S}$. As q_1 and $q_1 s$ are involutions of $\tilde{S}q_1$, we get $q_1 \underset{\tilde{S}}{\sim} q_1 s$. This is a contradiction, since $q_1 s \in QS\setminus\langle q_1, q_2\rangle$ and $\langle q_1, q_2\rangle$ is strongly closed in QS with respect to G.

Let finally $z \in QS\varphi\varkappa$. Then, $\boldsymbol{C}_s(z) = \langle q_1, q_2, \pi\rangle$. If $|\tilde{X}/\tilde{Q}\tilde{S}| = 2$, then a four-subgroup of $C_S(z)$ lies in $\tilde{Q}\tilde{S}$, and so, $\tilde{Q}\tilde{S}$ would contain an element of $Q^{\#}$ which is not possible. Suppose that $|\tilde{X}/\tilde{Q}\tilde{S}| = 4$. Then, $\tilde{X} = \tilde{Q}\tilde{S}\langle q_1, q_2\rangle$ and $\tilde{Q}\tilde{S} \cap \pi\langle q_1, q_2\rangle \neq \emptyset$. Let $q\pi$ be an element of $\tilde{Q}\tilde{S}$ with $q \in \langle q_1, q_2\rangle$. Since $q\pi$ is an involution of S, we conclude $q\pi \in \tilde{S}$. Clearly, $\tilde{Q}\tilde{S}q_1$ is conjugate to QSy for some $y \in \{\varphi, \varkappa, \varphi\varkappa\}$. Thus, $q_1 \sim q_1 q\pi$ under the action of $\tilde{S}$; note that $q_1 q\pi \in QS \setminus \langle q_1, q_2\rangle$. This contradicts the strong closure of $\langle q_1, q_2\rangle$ in QS with respect to G. The lemma is proved.

6. The identification of G with He.

From the result of section 5 we know that X is not a S_2-subgroup of G.

(6.1) LEMMA. We have $Q = \langle q_1, q_2\rangle$.

PROOF. Let T be a subgroup of G containing X as a subgroup of index 2. Let $t \in T \setminus X$. Clearly, $t \notin N(A)$ and t normalizes QS and $Z(QS)$ which is equal to $Q \times \langle \pi, \tau\rangle$. This gives $Q \cap Q^t = \langle 1\rangle$ and $Q \times Q^t \subseteq Z(QS)$, and so, $|Q| = 4$. The lemma is proved.

(6.2) LEMMA. The case $X = S\langle\varkappa\rangle$ does not arise.

PROOF. Assume that $X = S\langle\varkappa\rangle$. One computes $X'' = \langle[\mu\lambda\xi, \lambda\zeta]\rangle = \langle q_1\rangle$. This implies $N(X) \subseteq C(q_1) \subseteq N(A)$. This, however, is not the case.

(6.3) LEMMA. The case $X = S\langle\varphi\varkappa\rangle$ does not arise.

PROOF. Assume $X = S\langle\varphi\varkappa\rangle$. Then $Z(X) = \langle q_1, q_2, \pi\rangle$. Denote by T a subgroup of G which contains X as a subgroup of index 2. Let $t \in T \setminus X$. Then, $Q \cap Q^t = \langle 1\rangle$ and $Q \times Q^t \subseteq Z(X)$ which is against $|Q| = 4$ and $|Z(X)| = 8$.

(6.4) LEMMA. We have $3 \,\big|\, |N(A)/KA|$. In particular, $q_1 \sim q_2 \sim q_1q_2$, and for $i \in \{1, 2\}$ we have

$$N_{N(A)}(R_i)/C(R_i) \cong \begin{cases} A_5 \times Z_3, & \text{if } X = S \\ (A_5 \times Z_3)Z_2, & \text{if } X \in \{S\langle\varphi\rangle, S\langle\varphi, \varkappa\rangle\}, \end{cases}$$

here $A_5 \cdot Z_2 \cong \Sigma_5$ and $Z_3 \cdot Z_2 \cong \Sigma_3$.

PROOF. By way of contradiction assume that $3\nmid|N(A)/KA|$. Then, $N(A) = AKX$ and $q_1 \in Z(N(A))$. Since K is tightly-embedded in G, we get $q_2 \not\sim q_1 \not\sim q_1q_2$ in G. Under the action of $N_A(S)$ the set $\langle q_1, q_2, \pi, \tau\rangle^\#$ splits into 7 conjugate classes with representatives $q_1, q_2, q_1q_2, \pi, q_1\pi, q_2\pi, q_1q_2\pi$. We know that π, $q_2\pi$, and $q_1q_2\pi$ have roots in S, whereas q_1, q_2, q_1q_2, and $q_1\pi$ have no roots in S.

Let T be a subgroup of G containing X as a subgroup of index 2. Let $t \in T\setminus X$. Then, t normalizes $Z(S)$ and maps q_1 onto an element of $V\setminus Q$, where $V = Z(S) = \langle q_1, q_2, \pi, \tau\rangle$. It follows $q_1 \sim q_1\pi$ in G. Also, one gets that q_2 is mapped under t onto an element of $V\setminus Q$ which implies $q_2 \sim q_1\pi$. Hence, $q_1 \sim q_2$ which is not possible. Thus, we have that 3 divides the order of $N(A)/KA$.

From (2.3) and (4.3) we get $q_1 \sim q_2 \sim q_1q_2$ and $N_{N(A)}(R_i)/C(R_i)$ has the stated structure.

(6.5) LEMMA. The S_2-subgroup X of $N(A)$ splits over S.

PROOF. By (6.1), (6.2), and (6.3) we have $X \in \{S, S\langle\varphi\rangle, S\langle\varphi, \varkappa\rangle\}$. We may assume $X \supset S$. If $X = S\langle\varphi\rangle$, then $\langle q_1, q_2, \varphi\rangle \cong D_8$, and there is an involution in $X\setminus S$.

Let $X = S\langle\varphi, \varkappa\rangle$. Clearly, $K = H \times Q$, where $H = O(N(A))$. We know that $N(A)/K = C/K\cdot KA/K \cong \mathbf{Aut}\,(L_3(4))$ and that $C/K \cong \cong \mathbf{Out}(L_3(4))$. We look at C/H and determine the normalizer in C/H of an element r of order 3 in C/H. We get $|N(\langle r\rangle) \cap C/H| = 3\cdot 4$, since r acts fixed-point-free on QH/H. Since $\langle r\rangle \in \mathrm{Syl}_3\,(C/H)$ and $HQ\langle r\rangle/H \lhd C/H$, we get that a S_2-subgroup of $N(\langle r\rangle) \cap C/H$ is a four-group. This means that a S_2-subgroup of C splits over Q. Thus, $Q\langle\varphi, \varkappa\rangle$ splits over Q, and so, $S\langle\varphi, \varkappa\rangle$ splits over Q. The lemma is proved.

(6.6) LEMMA. The involution π is not conjugate to q_1 in G.

PROOF. We know that $X \in \mathrm{Syl}_2\,(C(q_1))$, and we know that S is the Thompson-subgroup of X, i.e., S is generated by the elementary abelian subgroups of X of greatest possible order. Assume that $q_1^g = \pi$ for some $g \in G$. Then, $\langle S, S^g\rangle \subseteq C(q_1^g) = C(\pi)$, and so there is $x \in C(\pi)$ such that $S = S^{gx}$ and $q_1^{gx} = \pi$. It follows that q_1 and π are conjugate in $N(S)$. But q_1 has no root in S, whereas π has a root in S. Thus, $q_1 \not\sim \pi$.

(6.7) LEMMA. The case $X = S$ does not arise.

PROOF. Assume $X = S$. We have $\boldsymbol{Z}(X) = \boldsymbol{Z}(S) = \langle q_1, q_2, \pi, \tau\rangle$. We know that under the action of $\boldsymbol{N}(S) \cap \boldsymbol{N}(A)$ the set $\boldsymbol{Z}(S)^{\#}$ splits into conjugate classes in the following way: $3q_1$, $3q_1\pi$, 9π; q_1 and $q_1\pi$ have no roots in S and a 3-element acts non-trivially on the coset $\langle q_1, q_2\rangle\pi$. Clearly, $\boldsymbol{N}(X) \supset \boldsymbol{N}_{\boldsymbol{N}(A)}(X)$, as $|X| < |G|_2$. Since $q_1 \not\sim \pi$ in G, we get $q_1 \sim q_1\pi$ in $\boldsymbol{N}(X)$, as $q_1 \sim q_1 q_2 \sim q_2$ takes place only in $\boldsymbol{N}(A)$. Thus, q_1 has precisely 6 conjugates under the action of $\boldsymbol{N}(X)$. Thus, $|\boldsymbol{N}_{\boldsymbol{N}(A)}(X) : \boldsymbol{N}(X) \cap \boldsymbol{C}(q_1)| = 3$ and $|\boldsymbol{N}(X) : \boldsymbol{N}_{\boldsymbol{N}(A)}(X)| = 2$.

Recall that R_1 and R_2 are the only elementary abelian subgroups of order 2^6 of X. We shall determine $\boldsymbol{N}_G(R_i)$ for $i \in \{1, 2\}$. We have $\boldsymbol{N}_{\boldsymbol{N}(A)}(R_i)/\boldsymbol{C}(R_i) \cong A_5 \times Z_3$, where $\boldsymbol{N}_A(R_i)/\boldsymbol{C}_A(R_i) \cong A_5$ operates transitively on R_i/Q. From (4.3) and (6.4) we get that under $\boldsymbol{N}_{\boldsymbol{N}(A)}(R_i)$ the involutions of R_i split into exactly three classes: $3q_1$, $15q_1\pi$, 45π.

We have shown earlier that $q_1 \sim q_1\pi$ in $\boldsymbol{N}(X)$ and $\pi \underset{G}{\not\sim} q_1$. Since $X \notin \mathrm{Syl}_2(G)$, there is a 2-element t in $\boldsymbol{N}(X) \setminus X$. Since $t \notin \boldsymbol{N}(A)$, we get $q_1^t \in \langle q_1, q_2, \pi, \tau\rangle \setminus \langle q_1, q_2\rangle$. If $R_1^t = R_2$ and $q_1 \not\sim q_1\pi$ in $\boldsymbol{N}(R_i)$, a class of three elements would be mapped onto a class of 15 elements by t which is not possible. Hence, $q_1 \sim q_1\pi$ in $\boldsymbol{N}(R_i)$ in any case for $i \in \{1, 2\}$. Thus, q_1 has precisely 18 conjugates and π precisely 45 conjugates in $\boldsymbol{N}(R_i)$, $i \in \{1, 2\}$, and so, $\boldsymbol{N}(R_i) \supset \boldsymbol{N}(R_i) \cap \boldsymbol{N}(A)$. Since

$$|\boldsymbol{N}_{\boldsymbol{N}(A)}(R_i) : \boldsymbol{C}_{\boldsymbol{N}(R_i)}(q_1)| = 3 \quad \text{and} \quad |\boldsymbol{N}(R_i) : \boldsymbol{C}_{\boldsymbol{N}(R_i)}(q_1)| = 18 ,$$

we get $|\boldsymbol{N}(R_i) : \boldsymbol{N}_{\boldsymbol{N}(A)}(R_i)| = 6$. Since $X \subseteq \boldsymbol{N}(R_i)$, we get from an above result that $|\boldsymbol{N}(X)|_2 = 2|X| = |\boldsymbol{N}(R_i)|_2$.

Set $N_0 = \boldsymbol{N}_{\boldsymbol{N}(A)}(R_1)$, $N = \boldsymbol{N}(R_1)$, $C = \boldsymbol{C}(R_1)$ and $O = \boldsymbol{O}(N(R_1))$. Then, $O \subseteq C$. Hence, $C = O \times R_1 \subseteq \boldsymbol{C}(q_1) \subseteq \boldsymbol{N}(A)$ and $K = O \times Q$. We recall that $N_0/C \cong A_5 \times Z_3$. Denote by $\bar{x}, \bar{H}$ the images of $x \in N$, $H \subseteq N$, respectively, under the epimorphism $N \to N/O$. Let w be an element of order 5 of $A \cap \boldsymbol{N}(R_1)$. Then, $\bar{w} \in \bar{N}_0$, $o(\bar{w}) = 5$. Note that $|\bar{N}/\bar{R}_1| = 2^3 \cdot 3^3 \cdot 5$. We have $\boldsymbol{C}_{\bar{R}_1}(\bar{w}) = \bar{Q}$ and $\bar{Q} \in \mathrm{Syl}_2(\boldsymbol{C}(\bar{w}) \cap \bar{N}_0)$. Since $|\boldsymbol{C}(w) \cap \boldsymbol{C}(q_1)|_2 = 4$, and since the fusion of involutions of $\langle q_1, q_2\rangle$ is controlled by $\boldsymbol{N}(A)$, we see that $Q \in \mathrm{Syl}_2(C_G(w))$. We want to show that $|\boldsymbol{C}(\bar{w}) \cap \bar{N}|_2 = 4$. Note that $O \lhd \boldsymbol{N}(R_1)$. Assume that there is a subgroup Q_1 of $\boldsymbol{N}(R_1)$ which contains Q as a subgroup of index 2 such that $[w, Q_1] \subseteq O$. As $|Q_1 : Q| = 2$, we get $OQ_1\langle w\rangle \subseteq \boldsymbol{N}(A)$. But in $\boldsymbol{N}(A)$, an element of order 5 of A does not centralize a subgroup of order 8. Thus, $|\boldsymbol{C}(\bar{w}) \cap \bar{N}|_2 = 4$. From the structure of $\boldsymbol{N}(A)$ and $\bar{N}_0/\bar{C}$ we get $\boldsymbol{C}_{\bar{N}_0}(\bar{w}) \cong A_4 \times Z_5$. Obviously, $\bar{Q} \lhd C_{\bar{N}}(\bar{w})$, and so,

$$O \times Q \lhd \boldsymbol{C}_N(w \bmod O) \subseteq \boldsymbol{N}(A) ,$$

since $\boldsymbol{N}(Q) \subseteq \boldsymbol{N}(A)$. Hence,

$$\boldsymbol{C}_{\overline{N}}(\overline{w}) \subseteq \overline{\boldsymbol{N}(A)} \cap \boldsymbol{C}(\overline{w})$$

which implies $|\boldsymbol{C}_{\overline{N}}(\overline{w})| = 2^2 \cdot 3 \cdot 5$.

Let $\mathfrak{X}$ be a minimal normal subgroup of $\mathfrak{N} = N/C$; note that $|\mathfrak{N}| = 2^3 \cdot 3^3 \cdot 5$. If $\mathfrak{X}$ is not solvable, then $\mathfrak{X} \cong A_5$ or $\mathfrak{X} \cong A_6$. The fact that 3^2 does not divide $|\boldsymbol{C}_{\overline{N}}(\overline{w})|$ yields $\mathfrak{X} \cong A_6$. Thus, $\mathfrak{X}$ is centralized by a group of order 3. This gives $\boldsymbol{O}(\mathfrak{N}) \neq \langle 1 \rangle$. If $\mathfrak{X}$ is solvable, then $\mathfrak{X}$ is not a 2-group because of $|\boldsymbol{C}_{\overline{N}}(\overline{w})|_2 = 4$. So, in any case, we must have $\boldsymbol{O}(\mathfrak{N}) \neq \langle 1 \rangle$. The structure of $\boldsymbol{C}_{\overline{N}}(\overline{w})$ yields $|\boldsymbol{O}(\mathfrak{N})| = 3$. In particular, $\mathfrak{N}$ contains a chief factor isomorphic to A_6.

Let $T \in \mathrm{Syl}_2(\boldsymbol{N}(R_1))$ with $T \supset X$. We have $|T : X| = 2$ and by assumption $R_1R_2 = S = X$.

Hence, $T = S\langle t \rangle$ for any $t \in T \setminus S$. Since $\boldsymbol{N}(R_1)/\boldsymbol{C}(R_1)$ contains a chief factor isomorphic to A_6 and $|\boldsymbol{N}(R_1)/\boldsymbol{C}(R_1)|_2 = 2^3$, we get that T/R_1 is isomorphic to a dihedral group of order 8. We may therefore choose the element t so that $t^2 \in R_1$. Consider now the action of t on R_1. We have $|\boldsymbol{C}_{R_1}(\zeta)| = 2^4 = |\boldsymbol{C}_{R_1}(\xi)|$, where $\zeta, \xi \in T \setminus R_1$. Since A_6 has only one class of involutions, we have $|C_{R_1}(t)| = 2^4$. Since K is tightly-embedded in G, we have $\boldsymbol{C}(t) \cap Q = \langle 1 \rangle$ as $t \notin \boldsymbol{N}(A)$. But t acts on $\boldsymbol{Z}(S) = \langle Q, \pi, \tau \rangle$ as an involution, and so, by the Jordan-canonical-form, we must have $|\boldsymbol{C}_{\boldsymbol{Z}(S)}(t)| = 2^2$. Since $t \in \boldsymbol{N}(R_1)$, we have $t \in \boldsymbol{N}(R_2)$; note that $\boldsymbol{Z}(S) = R_1 \cap R_2$. It follows

$$R_1 = Q \times \boldsymbol{C}_{R_1}(t)\ ,\ R_1R_2 = R_2\boldsymbol{C}_{R_1}(t) \text{ and } T = R_1R_2\langle t \rangle = R_2\boldsymbol{C}_{R_1}(t)\langle t \rangle\ .$$

Thus, T/R_2 is abelian of order 8. Working with $\boldsymbol{N}(R_2)$ in the same way as we did with $\boldsymbol{N}(R_1)$, we get $T/R_2 \cong D_8$. This is a contradiction proving the lemma.

(6.8) Lemma. The case $X = S\langle \varphi, \varkappa \rangle$ is not possible.

Proof. Assume that $X = S\langle \varphi, \varkappa \rangle$. From (6.5) we get that X splits over S. Hence, there are elements $\varphi' \in S\varphi$ and $\varkappa' \in S\varkappa$ such that $\langle \varphi', \varkappa' \rangle$ is a four-group. Note that $S = R_1R_2$ char X, $\boldsymbol{Z}(X) = \langle q_1, \pi \rangle$ and $\boldsymbol{N}(X) \supset \boldsymbol{N}_{\boldsymbol{N}(A)}(X)$. We know that under $\boldsymbol{N}_{\boldsymbol{N}(A)}(S)$ the set $\boldsymbol{Z}(S)^\#$ splits into exactly three conjugate classes with three conjugates of q_1 and $q_1\pi$, each, and nine conjugates of π; here $q_1\pi$ is the only involution from $\langle q_1, q_2 \rangle \pi$ which has no roots in S. Since $q_1 \nsim \pi$ in G, we get $q_1 \sim q_1\pi$ in $\boldsymbol{N}(X)$; since $\boldsymbol{N}(X)$ normalizes S, we get $q_1 \sim$

$\sim q_1\pi$ in $\boldsymbol{N}(S)$. It follows $|\boldsymbol{N}(X): \boldsymbol{N}_{\boldsymbol{N}(A)}(X)| = 2$; note that $\boldsymbol{N}_{\boldsymbol{N}(A)}(X) \subseteq \subseteq \boldsymbol{C}(\langle q_1, \pi\rangle)$. Also, we get $|\boldsymbol{N}(S): \boldsymbol{N}_{\boldsymbol{N}(A)}(S)| = 2$. As in the proof of (6.7) and by the presence of $\varkappa$, we get $|\boldsymbol{N}(R_i): \boldsymbol{N}_{\boldsymbol{N}(A)}(R_i)| = 6$ for $i \in \{1, 2\}$. We remark that $S\langle\varphi\rangle \in \mathrm{Syl}_2\,(\boldsymbol{N}_{\boldsymbol{N}(A)}(R_i))$ and

$$\boldsymbol{N}_{\boldsymbol{N}(A)}(R_i)/\boldsymbol{C}(R_i) \cong (A_5 \times Z_3)\cdot Z_2\,.$$

Let $T \in \mathrm{Syl}_2\,(\boldsymbol{N}(X))$. Then, $T \in \mathrm{Syl}_2\,(\boldsymbol{N}(S))$, and obviously, $X \subset T$ with $|T: X| = 2$. Since $R_1^{\varkappa} = R_2$, we have $|T: \boldsymbol{N}_T(R_i)| = 2$. Hence, $|X| = |\boldsymbol{N}_T(R_i)| = 2^{10} = |\boldsymbol{N}(R_i)|_2$ and $\boldsymbol{N}_T(R_i) \in \mathrm{Syl}_2\,(\boldsymbol{N}(R_i))$.

Consider $\boldsymbol{N}(R_1)$. Set $\mathfrak{N} = \boldsymbol{N}(R_1)/\boldsymbol{C}(R_1)$. As in the proof of (6.7), we can show that $\mathfrak{N}$ possesses a chief factor isomorphic to A_6 and that $|\boldsymbol{O}(\mathfrak{N})| = 3$. It follows from the structure of $\boldsymbol{N}(A)$ that a generator for $\boldsymbol{O}(\mathfrak{N})$ acts fixed-point-free on R_1, and that $\boldsymbol{O}(\mathfrak{N})$ acts on Q. It is now possible to show that $\mathfrak{N}'$ is isomorphic to the tripple cover of A_6 and that $\mathfrak{N}/\boldsymbol{O}(\mathfrak{N}) \cong \Sigma_6$. Further, we get that a S_2-subgroup of $\boldsymbol{N}(R_1)$ is of type M_{24}. Let $Y \in \mathrm{Syl}_2\,(\boldsymbol{N}(R_1))$ with $Y \supset S\langle\varphi\rangle$. There is an involution $t \in Y$ such that $Y = S\langle\varphi, t\rangle$, since Y is generated by involutions. We have $X = S\langle\varphi, \varkappa\rangle$ and we may put $T = S\langle\varphi, \varkappa, t\rangle$. We show that $T \in \mathrm{Syl}_2\,(G)$. Let R be an elementary abelian subgroup of T of order 2^6 such that $T = XR$. Then, $|X \cap R| = 2^5$. From the action of φ and $\varkappa$ on S we get that $|S \cap R| = 2^4$ is not possible. From the structure of X we get that there is no four-group in X intersecting S in $\langle 1\rangle$ which centralizes an elementary abelian subgroup of order 8 of S. It follows $S = R_1R_2$ char T, and since $T \in \mathrm{Syl}_2\,(\boldsymbol{N}(S))$, we get that $T \in \mathrm{Syl}_2\,(G)$. Remember that $|\boldsymbol{N}(S): \boldsymbol{N}_{\boldsymbol{N}(A)}(S)| = 2$, and so, $\boldsymbol{N}_{\boldsymbol{N}(A)}(S) \lhd \boldsymbol{N}(S)$. We know that mod H, $H = \boldsymbol{O}(\boldsymbol{N}(A))$, the group $\boldsymbol{N}(S) \cap \boldsymbol{N}(A)$ is an extension of S by a group of type (3,3) and by a four-group. Thus, $\boldsymbol{N}(S) = \boldsymbol{O}_{2',2,3,2}(\boldsymbol{N}(S))$ and $\boldsymbol{N}(S)/\boldsymbol{O}_{2',2,3}(\boldsymbol{N}(S))$ is a group of order 8 containing a four-subgroup. Clearly, $\boldsymbol{N}(S)/\boldsymbol{O}_{2',2}(\boldsymbol{N}(S))$ is faithful extension of an elementary abelian group of order 9 by a dihedral group of order 8, as otherwise we would get $T \subseteq \boldsymbol{N}(A)$ which is not the case. By the fixed-point-free action of the 3-layer on $\boldsymbol{O}S/\boldsymbol{O}$, we see that T splits over S. Thus, there is a dihedral subgroup $\langle\varphi', t'\rangle\langle\varkappa'\rangle$ of order 8 in T such that $S\langle\varphi', t'\rangle\langle\varkappa'\rangle = T$; $\varphi' \in S\varphi$, $\varkappa' \in S\varkappa$, $t' \in St$; the elements $\varphi', t', \varkappa'$ are involutions, $S\langle\varphi', t'\rangle \in \mathrm{Syl}_2$ $(\boldsymbol{N}(R_1))$, and $Z(\langle\varphi', t', \varkappa'\rangle) = \langle\varphi'\rangle$; note that φ acts invertingly on $\boldsymbol{O}_{2',2,3}(\boldsymbol{N}(S))/\boldsymbol{O}_{2',2}(\boldsymbol{N}(S))$. Obviously, $\varkappa' \sim \varphi'\varkappa'$ by t'. We have that $\boldsymbol{C}_T(\varphi'\varkappa') = \langle q_1, q_2, \pi, \varphi'\varkappa', \varphi'\rangle = W$ by (4.6). Now, $W' = \langle q_1\rangle$ and this implies that W is a S_2-subgroup of $\boldsymbol{C}_G(\varphi'\varkappa')$. However, by a transfer lemma of J. Thompson, we get that in G the involution $\varphi'\varkappa'$ must be conjugate to an involution of $S\langle\varkappa' t'\rangle$, and so, $\varphi'\varkappa'$ must be conjugate to an element of $S\langle\varphi'\rangle = S\langle\varphi\rangle$. Representatives for the G-classes of involutions of $S\langle\varphi\rangle$ are π, q_1, and φ. But the centralizers of these elements have larger S_2-subgroups. This contradiction proves the lemma.

(6.9) LEMMA. The group G is isomorphic to He.

PROOF. From the preceding results, we have to consider finally the case in which $X = S\langle\varphi\rangle$, $\varphi^2 = 1$ and $C_G(q_1) = AKX$. Clearly, $H = O(C(q_1))$. Set $\mathfrak{C} = C(q_1)/H$. Then, $\mathfrak{C}$ is isomorphic to the centralizer of a non-central involution of He. A characterization of Deckers and Held leads to $G \simeq He$.

The theorem is proved.

REFERENCES

[1] M. ASCHBACHER, *On finite groups of component type*, Ill. J. M., **19** (1975), pp. 87-116.

[2] B. BEISIEGEL, *Über einfache, endliche Gruppen mit Sylow-2-Gruppen der Ordung höchstens* 2^{10}, Dissertation, Mainz University, 1975.

[3] M. DECKERS, *On groups related to Held's simple group*, Arch. der Math., **25** (1974), pp. 23-28.

[4] G. GLAUBERMAN, *Central elements in core-free groups*, J. Alg., **4** (1966), pp. 403-420.

[5] D. M. GOLDSCHMIDT, *2-Fusion in finite groups*, Ann. of Math., **99** (1974), pp. 70-117.

[6] D. GORENSTEIN, *Finite Groups*, Harper and Row, New York, 1968.

[7] R. GRIESS, *Schur multipliers of finite simple groups of Lie type*, Trans. A.M.S., **183** (1973), pp. 355-421.

[8] D. HELD, *The simple groups related to* M_{24}, J. Alg., **13** (1969), pp. 253-296.

[9] B. HUPPERT, *Endliche Gruppen, I*, Springer, Berlin, New York, 1967.

[10] A. REIFART, *A characterization of the sporadic simple group of Suzuki*, J. Alg., **33** (1975), pp. 288-305.

[11] I. SCHUR, *Untersuchung über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionem*, J. Math., **132** (1907), pp. 85-137.

[12] R. STEINBERG, *Automorphisms of finite linear groups*, Can. J. Math., **12** (1960), pp. 606-615.

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